MATHEMATICAL METHODS (CAS) UNITS 3 & 4

2014 Trial **EXAMINATION 1**

July 2014

Solutions

Section B: No CAS or reference book allowed

There is a total of 40 marks available for this section.

Writing time: 1 hour

Instructions to students

This section consists of 12 questions.
All questions should be answered in the spaces provided.
The marks allocated to each of the questions are indicated throughout.
An exact answer is required for all questions unless specified otherwise.
Where more than one mark is allocated to a question, appropriate working must be shown.
Diagrams in this trial exam are not drawn to scale.

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Question 1

a. Let $y=\sqrt{2x^2-1}$. Find $\frac{dy}{dx}$.

Method 1	Method 2
$\frac{dy}{dx}=\frac{1}{2}(2x^2-1)^{-\frac{1}{2}}\times 4x$ $=\frac{2x}{(2x^2-1)^{\frac{1}{2}}}$ $=\frac{2x}{\sqrt{2x^2-1}}$	$y=(2x^2-1)^{\frac{1}{2}}$ let $u=2x^2-1$ $y=u^{\frac{1}{2}}$ $\frac{du}{dx}=4x$ $\frac{dy}{du}=\frac{1}{2}u^{-\frac{1}{2}}$ $=\frac{1}{2u^{\frac{1}{2}}}=\frac{1}{2\sqrt{u}}$ $\frac{dy}{dx}=\frac{dy}{du}\cdot\frac{du}{dx}$ (chain rule) $=\frac{1}{2\sqrt{u}}\times 4x=\frac{4x}{2\sqrt{2x^2-1}}$ $=\frac{2x}{\sqrt{2x^2-1}}$

2 marks

b. Find the derivative of $\log_e(\sin(x))$.

$$\frac{d}{dx}(\log_e(\sin(x)))=\frac{\cos(x)}{\sin(x)}$$

1 mark

c. Let $f(x)=\frac{x}{e^{3x}}$. Find $f'(1)$.

$$f'(x)=\frac{e^{3x}\times 1-3e^{3x}\times x}{(e^{3x})^2} \quad \text{(quotient rule)}$$

$$=\frac{e^{3x}-3xe^{3x}}{e^{6x}}$$

$$f'(1)=\frac{e^3-3e^3}{e^6}$$

$$=\frac{-2e^3}{e^6}$$

$$=\frac{-2}{e^3}$$

3 marks

Question 2

Solve $\log_e(3)+2\log_e(x)=\log_e(4x)$ for x.

$$\log_e(3)+\log_e(x^2)=\log_e(4x)$$
$$\log_e(3x^2)=\log_e(4x)$$
$$3x^2=4x$$
$$3x^2-4x=0$$
$$x(3x-4)=0$$
$x=0$ or $x=\frac{4}{3}$ but $\log_e(x)$ is not defined for $x=0$ so $x=\frac{4}{3}$

3 marks

Question 3

Let $g:(2,\infty)\to R,\ g(x)=3\log_e(x-2)$. Find g^{-1}, the inverse function of g.

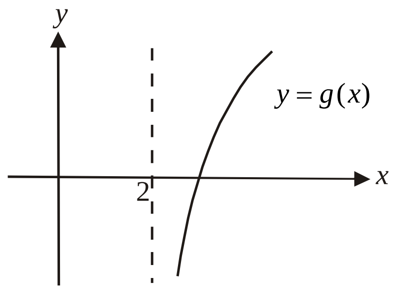
Let $y=3\log_e(x-2)$

Swap x and y for inverse.

$$x=3\log_e(y-2)$$
$$\frac{x}{3}=\log_e(y-2)$$
$$e^{\frac{x}{3}}=y-2$$
$$y=2+e^{\frac{x}{3}}$$
$$d_g=(2,\ \infty)\quad r_g=R$$

So $d_{g^{-1}}=R\quad r_{g^{-1}}=(2,\ \infty)$

So $g^{-1}:R\to R,\ g^{-1}(x)=2+e^{\frac{x}{3}}$



3 marks

Question 4

Let $g: R\setminus\{0\} \to R,\ g(x) = 1 + \frac{1}{x}$.

Show that $4g(2u) - g(-u) = 3g(u)$.

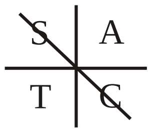
$$
\begin{aligned}
LHS &= 4g(2u) - g(-u) \\
&= 4\left(1 + \frac{1}{2u}\right) - \left(1 - \frac{1}{u}\right) \\
&= 4 + \frac{4}{2u} - 1 + \frac{1}{u} \\
&= 3 + \frac{2}{u} + \frac{1}{u} \\
&= 3 + \frac{3}{u} \\
&= 3\left(1 + \frac{1}{u}\right) \\
&= 3g(u) \\
&= RHS
\end{aligned}
$$

2 marks

Question 5

Solve the equation $\sin\left(\frac{x}{2}\right) + \frac{1}{\sqrt{3}}\cos\left(\frac{x}{2}\right) = 0$ for $x \in R$.

$$
\begin{aligned}
\sin\left(\frac{x}{2}\right) + \frac{1}{\sqrt{3}}\cos\left(\frac{x}{2}\right) &= 0 \\
\sin\left(\frac{x}{2}\right) &= -\frac{1}{\sqrt{3}}\cos\left(\frac{x}{2}\right) \\
\frac{\sin\left(\frac{x}{2}\right)}{\cos\left(\frac{x}{2}\right)} &= -\frac{1}{\sqrt{3}} \\
\tan\left(\frac{x}{2}\right) &= -\frac{1}{\sqrt{3}} \\
\frac{x}{2} &= \frac{5\pi}{6} + n\pi, \quad n \in Z \\
x &= 2\left(\frac{5\pi}{6} + n\pi\right), \quad n \in Z \\
x &= \frac{5\pi}{3} + 2n\pi, \quad n \in Z
\end{aligned}
$$



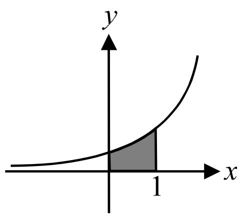
(alternative answer $\frac{x}{2} = \frac{-\pi}{6} + n\pi,\ n \in Z$ so $x = \frac{-\pi}{3} + 2n\pi,\ n \in Z$)

3 marks

Question 6

Find the exact area enclosed by the graph of $y = e^{\frac{x}{2}}$, the line $x = 1$ and the positive *x* and *y* axes.

Sketch the graph.



$$\text{Area} = \int_0^1 e^{\frac{x}{2}}\,dx$$

$$= \left[2e^{\frac{x}{2}}\right]_0^1$$

$$= 2e^{\frac{1}{2}} - 2e^0$$

$$= 2\sqrt{e} - 2$$

$$= 2\left(\sqrt{e} - 1\right)\text{square units}$$

3 marks

Question 7

A spherical balloon is being inflated. Its volume is increasing at the rate of 2cm^3 per second. Find the rate in cm/sec, at which the radius of the balloon is increasing when the radius is 4cm.

Now, $\frac{dr}{dt} = \frac{dr}{dV} \cdot \frac{dV}{dt}$ (chain rule)

Now, $V = \frac{4}{3}\pi r^3$ (formula sheet)

$$\frac{dV}{dr} = 4\pi r^2$$

$$\frac{dr}{dV} = \frac{1}{4\pi r^2}$$

Also $\frac{dV}{dt} = 2$ (given)

So $\frac{dr}{dt} = \frac{dr}{dV} \times \frac{dV}{dt}$

becomes $\frac{dr}{dt} = \frac{1}{4\pi r^2} \times 2$

$$= \frac{1}{2\pi r^2}$$

When $r = 4$, $\frac{dr}{dt} = \frac{1}{32\pi}$

The radius of the balloon is increasing at the rate of $\frac{1}{32\pi}$ cm/sec.

3 marks

Question 8

A transformation is described by the equation

$$\begin{bmatrix} x' \\ y' \end{bmatrix} = \begin{bmatrix} -2 & 0 \\ 0 & 3 \end{bmatrix}\left(\begin{bmatrix} x \\ y \end{bmatrix} + \begin{bmatrix} 1 \\ -1 \end{bmatrix}\right).$$

Find the image of the curve with equation $y = \frac{2}{x+1} - 1$ under this transformation. Give your answer in the form $y = \frac{a}{x} + b$ where a and b are real constants.

$x' = -2(x+1),\ y' = 3(y-1)$

$x = -\frac{x'}{2} - 1,\ y = \frac{y'}{3} + 1$

Substitute into $y = \frac{2}{x+1} - 1$

$\frac{y'}{3} + 1 = -\frac{4}{x'} - 1$

$y' = -\frac{12}{x'} - 6$

The equation of the image is $y = -\frac{12}{x} - 6$

3 marks

Question 9

Given that $f(x+h) \approx f(x) + hf'(x)$, where h is small, find an approximate value of $\sqrt{9.03}$.

$f(x) = \sqrt{x} = x^{\frac{1}{2}} \qquad h = 0.03$

$f'(x) = \frac{1}{2}x^{-\frac{1}{2}}$

$= \frac{1}{2\sqrt{x}}$

$f(x+h) \approx f(x) + h\,f'(x)$ becomes $f(x+0.03) \approx \sqrt{x} + \frac{0.03}{2\sqrt{x}}$

$f(9+0.03) \approx \sqrt{9} + \frac{0.03}{2\times\sqrt{9}}$

$= 3 + \frac{0.03}{6}$

$= 3 + 0.005 = 3.005$

3 marks

Question 10

The graph of $y=3x^2+a$; where a is a real constant, has a normal with equation $y=\frac{x}{3}+1$.
Find the value of a.

$$y=3x^2+a$$
$$\frac{dy}{dx}=6x$$

The gradient of a normal to $y=3x^2+a$ is $-\frac{1}{6x}$.

Also the gradient of the normal $y=\frac{x}{3}+1$ is $\frac{1}{3}$.

$$\text{When } -\frac{1}{6x}=\frac{1}{3}$$
$$-3=6x$$
$$x=-\frac{1}{2}$$

The x-coordinate of the point where the normal hits the curve is $-\frac{1}{2}$.

$$\text{So } y=-\frac{1}{2}\div 3+1$$
$$=-\frac{1}{2}\times\frac{1}{3}+1$$
$$=-\frac{1}{6}+1$$
$$=\frac{5}{6}$$

The curve and the normal both pass through the point $\left(-\frac{1}{2},\ \frac{5}{6}\right)$.

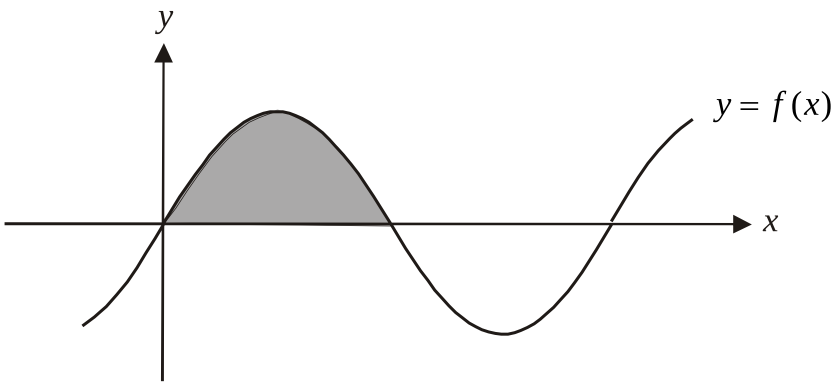
Substituting this point into

$$y=3x^2+a$$
$$\text{gives } \frac{5}{6}=3\times\left(-\frac{1}{2}\right)^2+a$$
$$\frac{5}{6}=3\times\frac{1}{4}+a$$
$$\text{So } a=\frac{5}{6}-\frac{3}{4}$$
$$=\frac{10-9}{12}$$
$$=\frac{1}{12}$$

4 marks

Question 11

Part of the graph of the function $f: R \to R,\ f(x) = a\sin(2x)$ where a is a positive constant is shown below.



The shaded region represents an area of 4 square units.   Find the value of a.

$$\int_0^{\frac{\pi}{2}} a\sin(2x)dx = 4$$

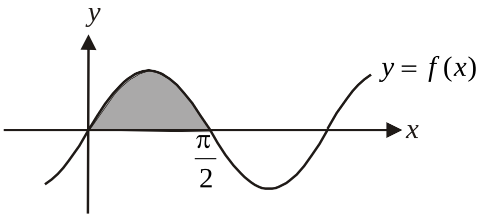
$$a\left[-\frac{1}{2}\cos(2x)\right]_0^{\frac{\pi}{2}} = 4$$

$$-\frac{a}{2}(\cos(\pi) - \cos(0)) = 4$$

$$-\frac{a}{2}(-1-1) = 4$$

$$-\frac{a}{2} \times -2 = 4$$

$$a = 4$$

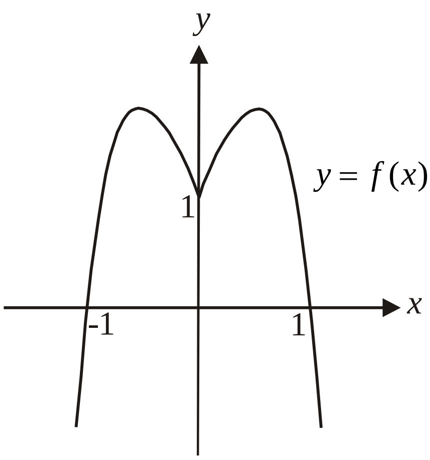


4 marks

Question 12

Let $f: R \to R,\ f(x) = 2|x| - 3x^4 + 1$.

The graph of $y = f(x)$ is shown below.



a. Write down the domain of the derivative function $f'(x)$.

Since the graph of $y = f(x)$ is not smooth at the point where $x = 0$, then $d_{f'} = R \setminus \{0\}$.

1 mark

b. Find the rule for $f'(x)$.

Method 1	Method 2
$f(x) = \begin{cases} 2x - 3x^4 + 1 & \text{if } x \geq 0 \\ -2x - 3x^4 + 1 & \text{if } x < 0 \end{cases}$ $f'(x) = \begin{cases} 2 - 12x^3 & \text{if } x > 0 \\ -2 - 12x^3 & \text{if } x < 0 \end{cases}$	$f'(x) = \frac{2\lvert x \rvert}{x} - 12x^3 \quad \text{for } x \in R \setminus \{0\}$

2 marks

End of Section B

Mathematical Methods (CAS) Formulas

Mensuration

area of a trapezium:	$\frac{1}{2}(a+b)h$	volume of a pyramid:	$\frac{1}{3}Ah$
curved surface area of a cylinder:	$2\pi rh$	volume of a sphere:	$\frac{4}{3}\pi r^3$
volume of a cylinder:	$\pi r^2 h$	area of a triangle:	$\frac{1}{2}bc\sin A$
volume of a cone:	$\frac{1}{3}\pi r^2 h$		

Calculus

$$\frac{d}{dx}\left(x^n\right) = nx^{n-1}$$

$$\int x^n\,dx = \frac{1}{n+1}x^{n+1} + c,\ n \neq -1$$

$$\frac{d}{dx}\left(e^{ax}\right) = ae^{ax}$$

$$\int e^{ax}\,dx = \frac{1}{a}e^{ax} + c$$

$$\frac{d}{dx}\left(\log_e(x)\right) = \frac{1}{x}$$

$$\int \frac{1}{x}\,dx = \log_e|x| + c$$

$$\frac{d}{dx}\left(\sin(ax)\right) = a\cos(ax)$$

$$\int \sin(ax)\,dx = -\frac{1}{a}\cos(ax) + c$$

$$\frac{d}{dx}\left(\cos(ax)\right) = -a\sin(ax)$$

$$\int \cos(ax)\,dx = \frac{1}{a}\sin(ax) + c$$

$$\frac{d}{dx}\left(\tan(ax)\right) = \frac{a}{\cos^2(ax)} = a\sec^2(ax)$$

product rule: $\frac{d}{dx}(uv) = u\frac{dv}{dx} + v\frac{du}{dx}$

quotient rule: $\frac{d}{dx}\left(\frac{u}{v}\right) = \frac{v\frac{du}{dx} - u\frac{dv}{dx}}{v^2}$

chain rule: $\frac{dy}{dx} = \frac{dy}{du}\frac{du}{dx}$

approximation: $f(x+h) \approx f(x) + hf'(x)$

Probability

$\Pr(A) = 1 - \Pr(A')$

$\Pr(A \cup B) = \Pr(A) + \Pr(B) - \Pr(A \cap B)$

$\Pr(A|B) = \frac{\Pr(A \cap B)}{\Pr(B)}$

transition matrices: $S_n = T^n \times S_0$

mean: $\mu = \mathrm{E}(X)$

variance: $\mathrm{var}(X) = \sigma^2 = \mathrm{E}((X-\mu)^2) = \mathrm{E}(X^2) - \mu^2$

probability distribution		**mean**	**variance**
discrete	$\Pr(X = x) = p(x)$	$\mu = \Sigma\, x\, p(x)$	$\sigma^2 = \Sigma\,(x-\mu)^2\, p(x)$
continuous	$\Pr(a < X < b) = \int_a^b f(x)\,dx$	$\mu = \int_{-\infty}^{\infty} x\, f(x)\,dx$	$\sigma^2 = \int_{-\infty}^{\infty} (x-\mu)^2\, f(x)\,dx$