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WITH CENTRES ALL OVER AUSTRALIA

NQT EDUCATION

YEAR 12

VCE MATHS METHODS
UNIT 3 TERM 2 SOLUTIONS

WEEKS 10 - 18

NOTES TO YEAR 12 MATHS METHODS TUTORS:
This document is for staff use only. No student is to view the contents within. The purpose of this solutions book is for tutors to FACILITATE the learning of students. It is not to be regarded as a means to "Spoon-feed" answers. You may, provide solutions after students have attempted the questions, as a means of giving feedback to their responses.

NQT EDUCATION

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HOW TO USE THIS BOOKLET

WELCOME TO VCE STUDIES AT NQT EDUCATION. Let us tell you a little about our classes and what you can do to maximise your learning with us.

NQT Education currently offers classes in the following VCE subjects:
- VCE English for years 11 & 12
- VCE Mathematical Methods for years 11 & 12

NQT Education's VCE curriculum follows closely in line with the Victorian Curriculum and Assessment Authority (VCAA's) Study Designs so that what you are learning topics in line with what you are studying at school. However, given that each school is different and it is likely you may be covering Areas of Study different to that of your peers, the material covered in NQT classes may be pre-taught or revisional in nature.

The work is divided into weeks and each cover sheet outlines clearly the Area of Study you will be undertaking as well as the key Outcomes for the different Areas of Study. It is important that you stick to the allocated weeks in this book and you are encouraged to complete all activities for homework if unable to complete all tasks in lesson.

VCE English at NQT Education

It is highly likely that your classmates are studying different text(s) from you. It is also likely your tutor may not be necessarily familiar with the texts you are studying. HOWEVER, the focus of VCE English classes at NQT is about gaining essential skills that will help you prepare for your SACs, assessment tasks and / or exam(s).

At NQT Education, we understand that in order to achieve your very best at VCE English, you will need to develop and hone your writing and analytical skills and with the help of our worksheets and your tutor's expertise, you should be able to achieve your very best. Ensure that you bring in any relevant work, texts, notes, assessment tasks, draft SACs, sample exams, etc. to supplement your studies. You are also strongly encouraged to bring in any drafts or writing tasks for your tutor to look over as they will also be able to provide invaluable advice and feedback.

VCE Mathematical Methods at NQT

It is essential that you bring in your CAS calculator each week as well as your notebook as there will be substantial workings out that will need to be completed in addition to the work within this book. Each week, there is clearly explained theory, definitions of key terms as well as worked examples. This is then followed up by series of activities that progress in difficulty to allow you ample practice in new topics and concepts. Again, your tutor is there to help should y0u also require assistance with your own VCE Mathematical Methods coursework.

NQT EDUCATION

YEAR 12

VCE MATHS METHODS
UNIT 3 TERM 2 WORKBOOK

WEEK 10

Introduction to Circular Functions

Revision on Dilation of sine and cosine graph
Translation applied to sine and cosine graph
General rule for circular function
Addition of ordinates

Revision to transformation of sine and cosine graphs

Here is some revision of last topic work.

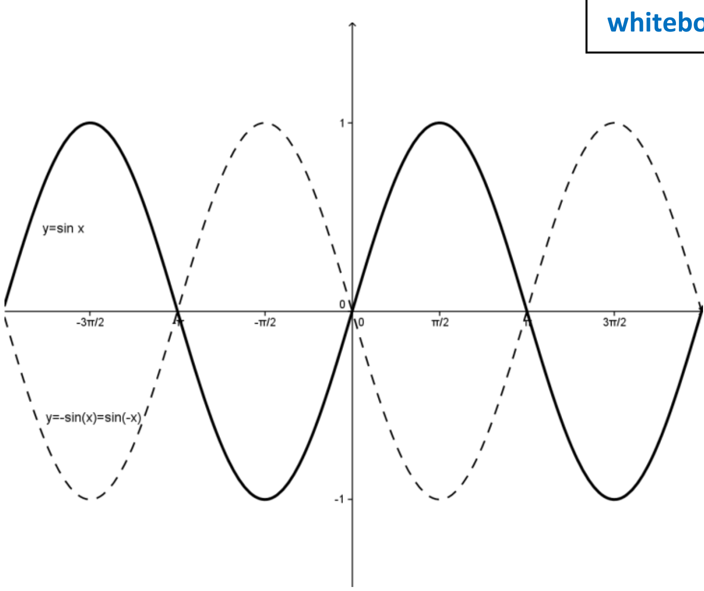
$f: R \to R, f(x) = a\sin(nx)$	$f: R \to R, f(x) = a\cos(nx)$
Period = $\frac{2\pi}{\lvert n \rvert}$	Period = $\frac{2\pi}{\lvert n \rvert}$
Amplitude = |a|	Amplitude = |a|
Range = [-|a|, |a|]	Range = [-|a|,|a|]

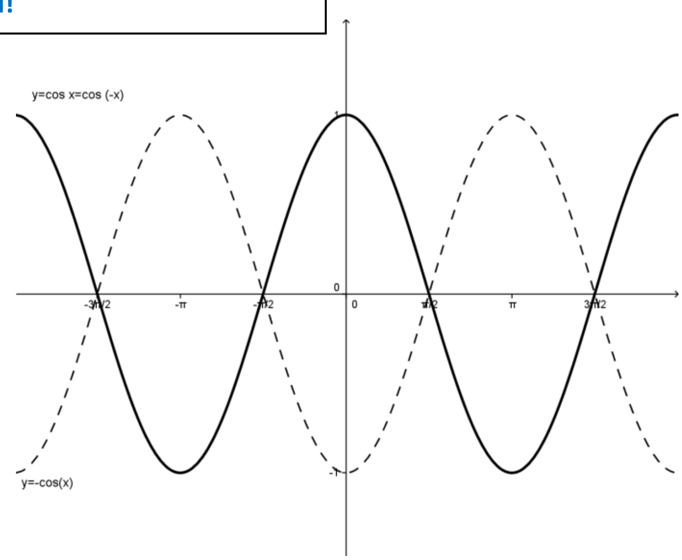
- The dilation of the '1/n' in the y axis affects the period of the function. i.e. larger 'n' value, the smaller the period.
- The dilation of 'a' in the x axis affects the amplitude of the function. i.e. the range is [-|a|,|a|]
- The reflection on the x axis flips the graph along the x axis.

There is one more reflection we need to introduce. It is the reflection on the y axis.

- The function $f(x) = \sin x$ is an odd function, which means $f(-x) = -f(x)$. The reflection in the y axis for an odd function will give you the same result as reflection in the x axis.
  i.e $-\sin x = \sin -x$
- The function $f(x) = \cos x$ is an even function, which means $f(-x) = f(x)$. The reflection in the y axis for an even function will give you the same result as the original function.
  i.e $\cos -x = \cos x$

Tutor should demonstrate the reflection on the y axis on the whiteboard!





Translation in the direction of the y axis

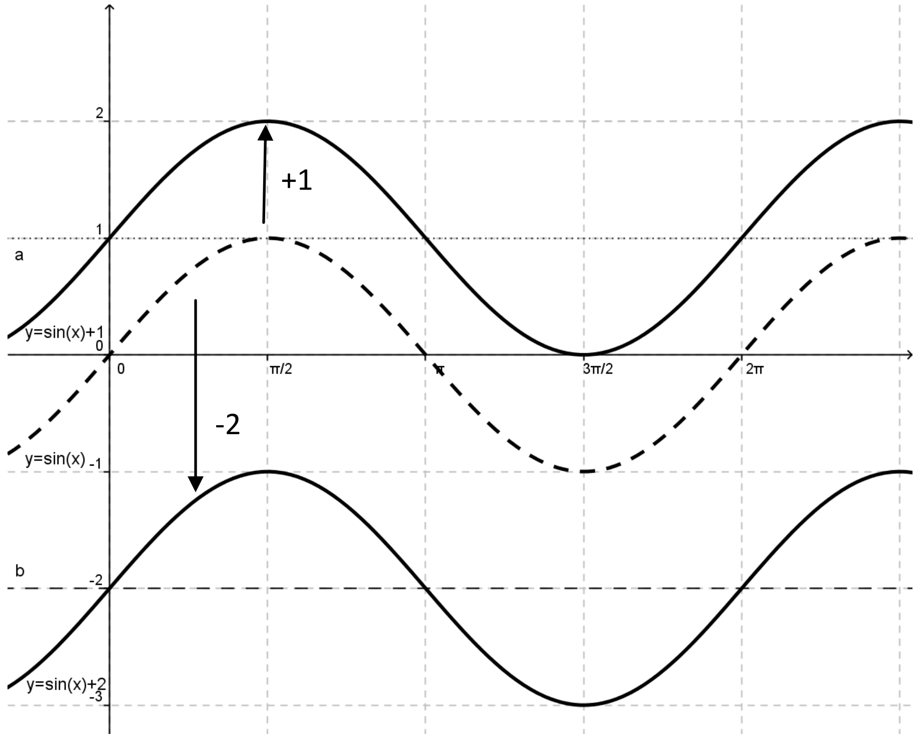
The graph of $y = \sin x + 1$ is the translation of 1 unit in the positive direction of the y-axis of $y = \sin x$.

For $y = \sin x - 2$, it is the translation of 2 unit in the negative direction of the y-axis of $y = \sin x$.

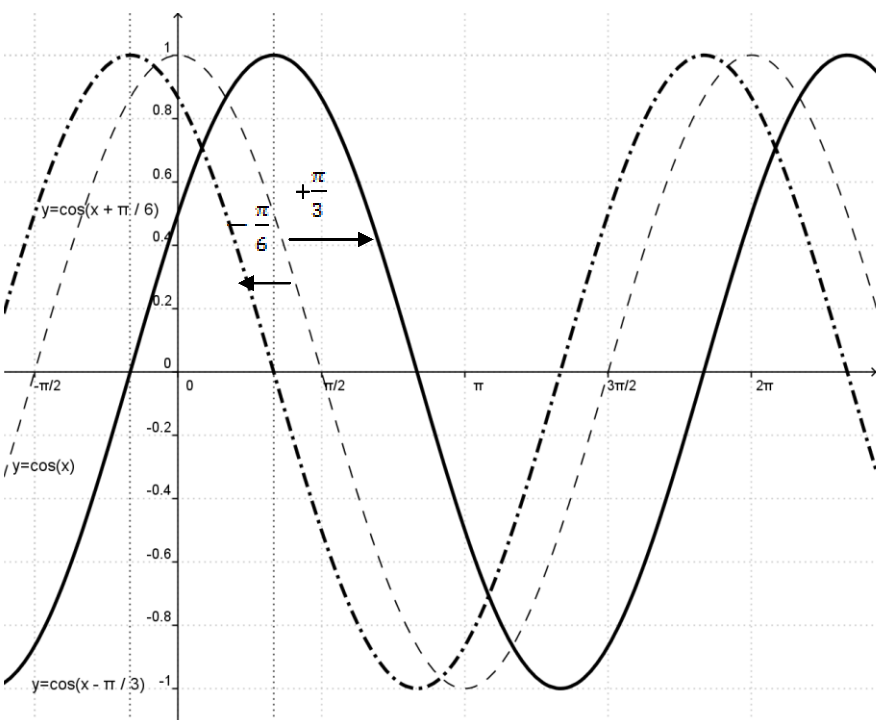
In general

$y = \sin x + c \; and \; y = \cos x + c$,

where c is the unit translated on the y axis.



Translation in the direction of the x axis



Consider the graph $y = \cos\left(x - \frac{\pi}{3}\right)$. The function is the translation of $\frac{\pi}{3}$ units in the positive direction of x-axis. In order words the graph is shifted to the right

The graph $y = \cos\left(x + \frac{\pi}{6}\right)$ is the translation of $\frac{\pi}{6}$ units in the negative direction of x-axis; shifted to the left.

In general $y = \sin(x - c) \; and$

$y = \cos(x - c)$, where c is the unit translated on the x axis.

The General Form of Sine and Cosine Graphing

From previous topic of transformation of sine and cosine graphs, we can now determine the general form of the sine and cosine functions.

$$f: R \to R, f(x) = a \sin n(x - b) + c \quad \text{and} \quad f: R \to R, f(x) = a \cos n(x - b) + c$$

Period = $\frac{2\pi}{n}$ Amplitude = |a| Range = [-|a|+b,|a|+b]

If b is positive then the graph shifts to the **right** and if negative then the graph shifts to the **left.**
If c is positive then the graph move **upwards** and if negative then the graph moves **downwards.**

Example

Sketch the graphs of $y = 3\sin 2\left(x - \frac{\pi}{4}\right) + 1$ over one period.

In order to sketch the function, we must follow a procedure that guide us to the solution. Here is a method use to help you sketch the graph!

Step 1. First Identify the transformation!

- Dilation of factor 3 from the x axis
- Dilation of factor 1/2 from the y-axis
- A translation of $\frac{\pi}{4}$ in the positive direction of x-axis

Be careful! The period and shift of circular function is obtain from this form n(x-c) not (nx-c)! If there is a multiple factor in front of the x you must take the factor out by factorising it.

i.e (3x-6) =3(x-2)
n= 3 and c=2. Not n=3 and c=6!
This is a common mistake!

Step 2. State the properties.
Writing the properties down will help you through your working out

Period = π	Minimum Position=-2
Amplitude =3	Mean position=1
Range =[-2,4]	Maximum Position =4

Step 3. Solve for the X coordinates

Find the coordinates if it was $3\sin 2x$

$(0,0), \left(\frac{\pi}{4}, 3\right), \left(\frac{\pi}{2}, 0\right), \left(\frac{3\pi}{4}, -3\right), (\pi, 0)$

Then add $\frac{\pi}{4}$ to the x coordinates

$\left(\frac{\pi}{4}, 0\right), \left(\frac{\pi}{2}, 3\right), \left(\frac{3\pi}{4}, 0\right), (\pi, -3), \left(\frac{5\pi}{4}, 0\right)$

Step 4. Y coordinates

With the new coordinates you found add +1 to the y coordinates

$\left(\frac{\pi}{4}, 0\right), \left(\frac{\pi}{2}, 3\right), \left(\frac{3\pi}{4}, 0\right), (\pi, -3), \left(\frac{5\pi}{4}, 0\right)$

Then add $\frac{\pi}{4}$ to the x coordinates

$\left(\frac{\pi}{4}, 1\right), \left(\frac{\pi}{2}, 4\right), \left(\frac{3\pi}{4}, 1\right), (\pi, -2), \left(\frac{5\pi}{4}, 1\right)$

In a sketch, you must provide the common coordinates, which includes the x and y intercepts!

Step 5. Determine the intercepts

Solve for Y intercept

$y_{int} = 3\sin 2\left(0 - \frac{\pi}{4}\right) + 1$

$= 3\sin\left(-\frac{\pi}{2}\right) + 1 = 3 \times -1 + 1 = -2$ So **(0,-2)**

Solve for X intercept s

$0 = 3\sin 2\left(x_{int} - \frac{\pi}{4}\right) + 1$

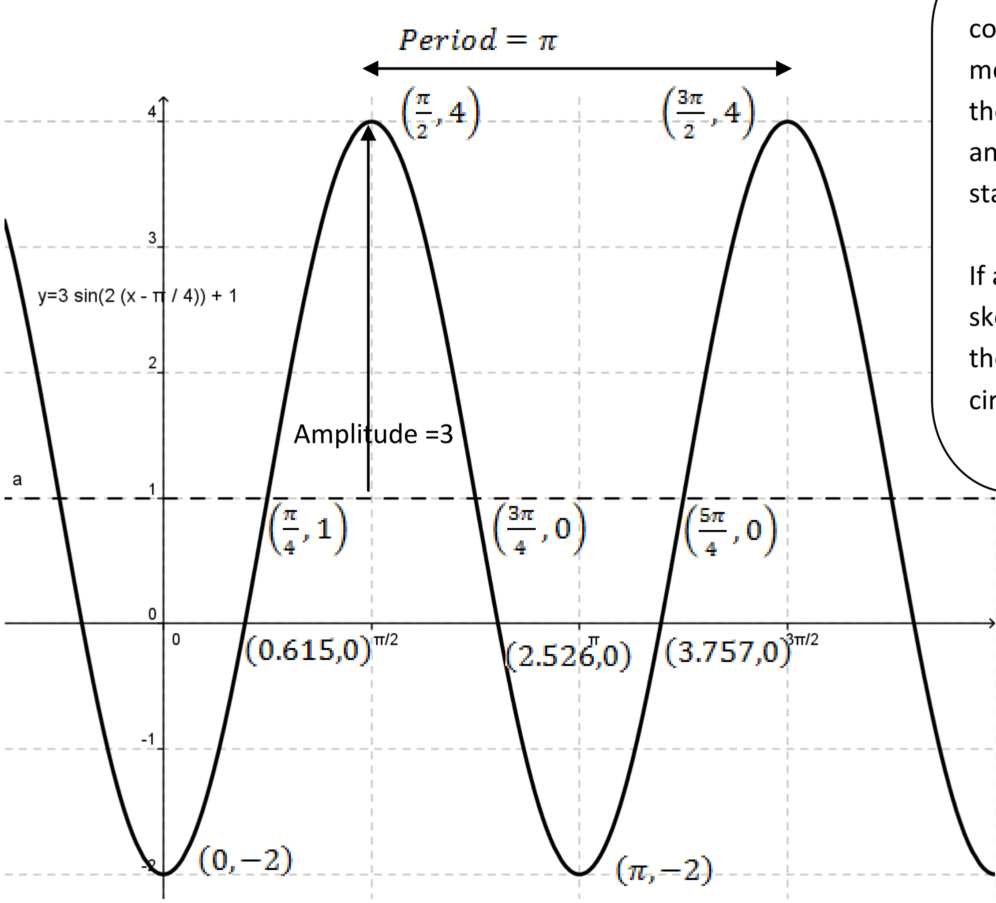
$\frac{-1}{3} = \sin 2\left(x_{int} - \frac{\pi}{4}\right)$

$2(x_{int} - \frac{\pi}{4}) = \sin^{-1}\left(-\frac{1}{3}\right), -\sin^{-1}\left(\frac{1}{3}\right), \pi + \sin^{-1}\left(\frac{1}{3}\right), 2\pi - \sin^{-1}\left(\frac{1}{3}\right), etc$

$x_{int} - \frac{\pi}{4} = \frac{1}{2}\sin^{-1}\left(-\frac{1}{3}\right) = -\frac{1}{2}\sin^{-1}\left(\frac{1}{3}\right), \frac{1}{2}(\pi + \sin^{-1}\left(\frac{1}{3}\right)), \frac{1}{2}(2\pi - \sin^{-1}\left(\frac{1}{3}\right)), etc$

$x \cong -0.17 + \frac{\pi}{4}, 1.74 + \frac{\pi}{4}, 2.97 + \frac{\pi}{4}$, etc = 0.615, 2.526, 3.757, etc

(0.615,0) , (2.526,0), (3.757,0), etc NOTE: depending on your domain you may have more x-intercepts!



Make sure you label the coordinates of each maximum, mean and minimum position and the x and y intercepts! Also start and end coordinates if domain is stated.

If a question requires you to sketch the graph without using the calculator, expected some circular function exact values!

Testing Understanding

1. State the amplitude, period, range and the sequence transformation of the following function.

a. $y = \sin 2\left(x - \frac{\pi}{4}\right)$

A,P,R 1,pi,[-1,1]

Dilate 1/2 in y axis

translate pi/4 in + x direction

b. $y = -\cos\left(x + \frac{\pi}{6}\right)$

1,2pi,[-1,1]

reflect on x axis

translate pi/6 in - x direction

c. $y = 4\sin\left(-x - \frac{\pi}{9}\right) + 1$

4,2pi,[-3,5]

dilate 4 in the x axis

reflect on y axis

**translate pi/9 in - x direction

translate 1 in +x direction

2. Write the equation of the image of graph $\sin(x)$ *under*

d. Dilation of factor 2 from the x-axis, dilation of factor 3 from the y-axis and translation of $\frac{\pi}{4}$ units in the positive direction of x-axis. 2sin(1/3(x-pi/4)

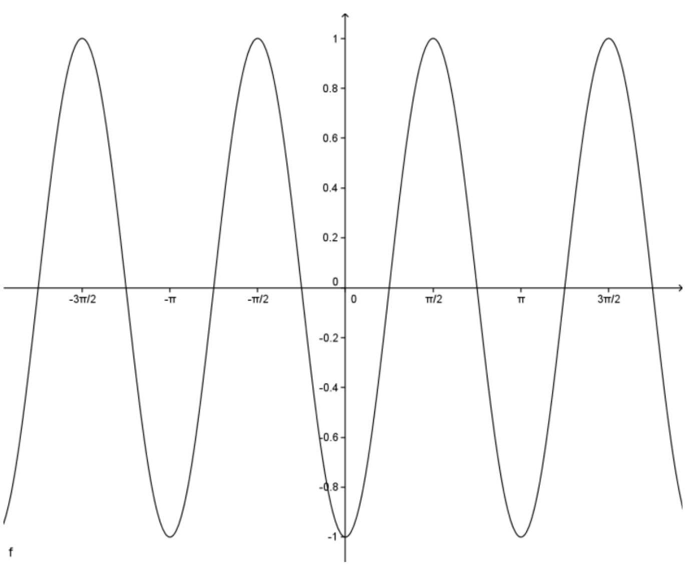
e. Reflection on the y axis, dilation of factor 3 parallel to the y-axis, dilation of factor 1/2 from the y-axis, translation of $\frac{\pi}{7}$ units in the negative direction of x-axis and translation of 4 in the positive direction of y-axis

3sin(-2(x+pi/7))+4

3. Sketch the following function over one period - Provide all working out

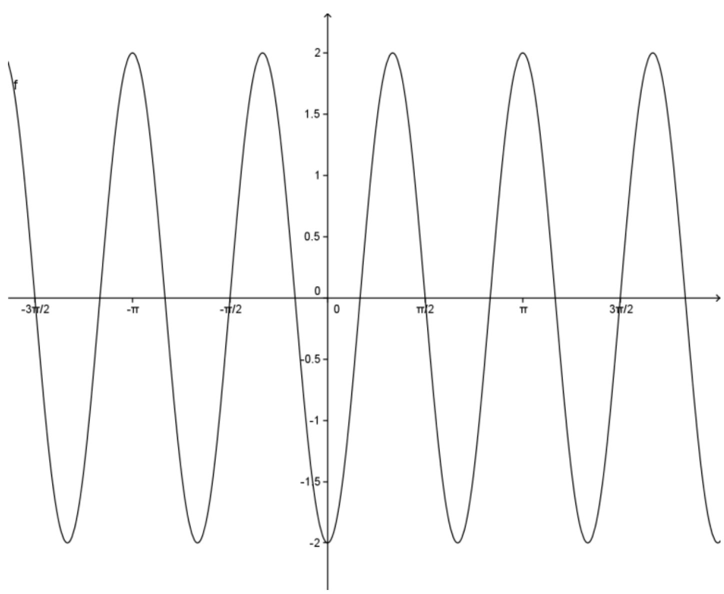
a. $y = \sin 2\left(x - \frac{\pi}{4}\right)$

Period = pi Amp=1



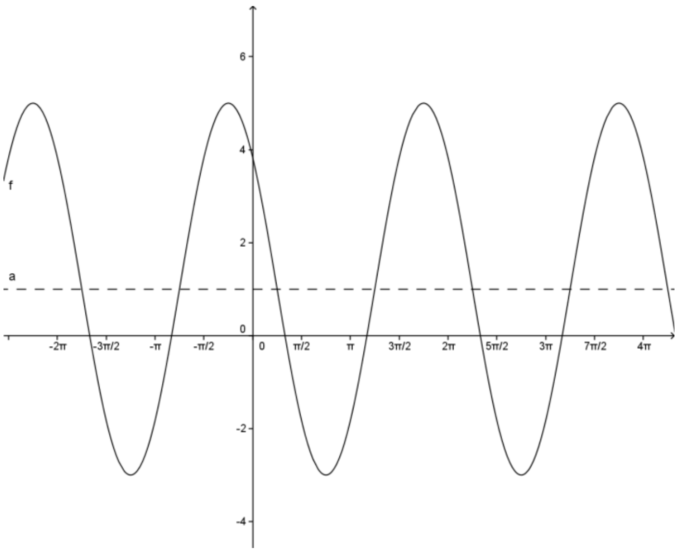
b. $y = 2\cos 3\left(x + \frac{\pi}{3}\right)$

Period = 2pi/3 Amp=2



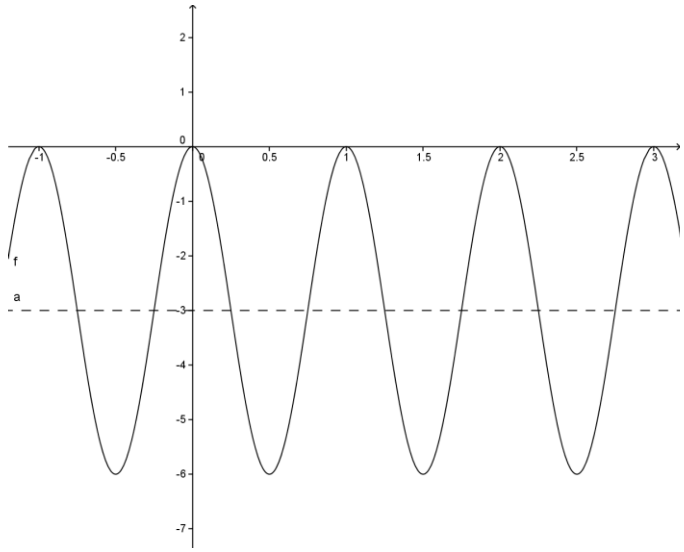
c. $y = -4\sin\left(x - \frac{\pi}{4}\right) + 1$

Period =2pi Amp=4



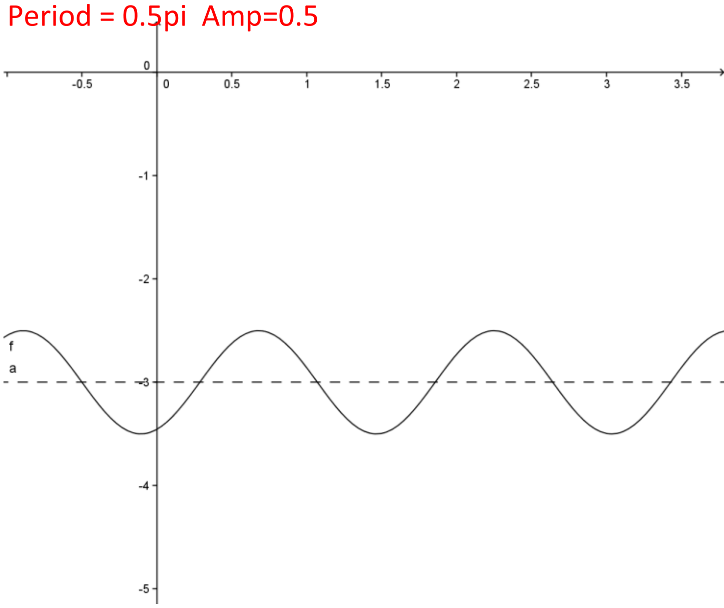
d. $y = 3\cos 2\pi(x - 2) - 3$

Period =1 Amp =3



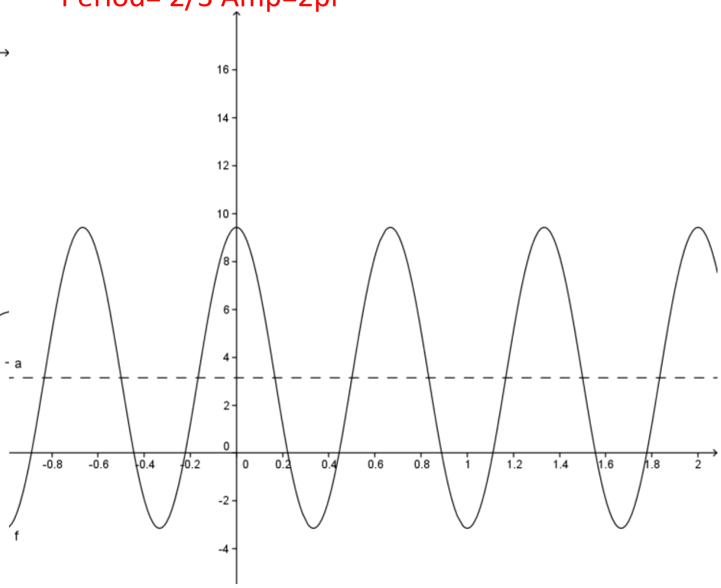
e. $y = \frac{1}{2}\sin(-4x - 2) - 3$

Period = 0.5pi Amp=0.5



f. $y = 2\pi\cos(-3\pi x - 18\pi) + \pi$

Period= 2/3 Amp=2pi



Addition of Ordinates

Basically you can graph additional circular function graphs together. i.e $y = 2\cos(x) + \sin\,(2x)$.
Subtraction of ordinate exactly the same as addition of ordinate. Avoid subtracting coordinate as it may get complicated. The best thing to do is convert negative across to a function and use addition of ordinate method.

i.e $y = 2\cos(x) - \sin(2x) \rightarrow \quad y = 2\cos(x) + (-\sin(2x))$

Method for Addition of Ordinates

First you will need to sketch $y_1 = 2\cos(x)$ and $y_2 = \sin\,(2x)$ on the same graph.

Then find the appropriate coordinates for both functions.

Coordinates for $\mathbf{y_1 = 2\cos(x)}$

(0,2), (π/4,$\frac{\sqrt{2}}{2}$), (π/2,0), (3π/4,$\frac{-\sqrt{2}}{2}$), (π,-2), (5π/4,$\frac{-\sqrt{2}}{2}$), (3π/2, 0), (7π/4,$\frac{\sqrt{2}}{2}$), and (2π,2)

Coordinates for $\mathbf{y_2 = \sin(2x)}$

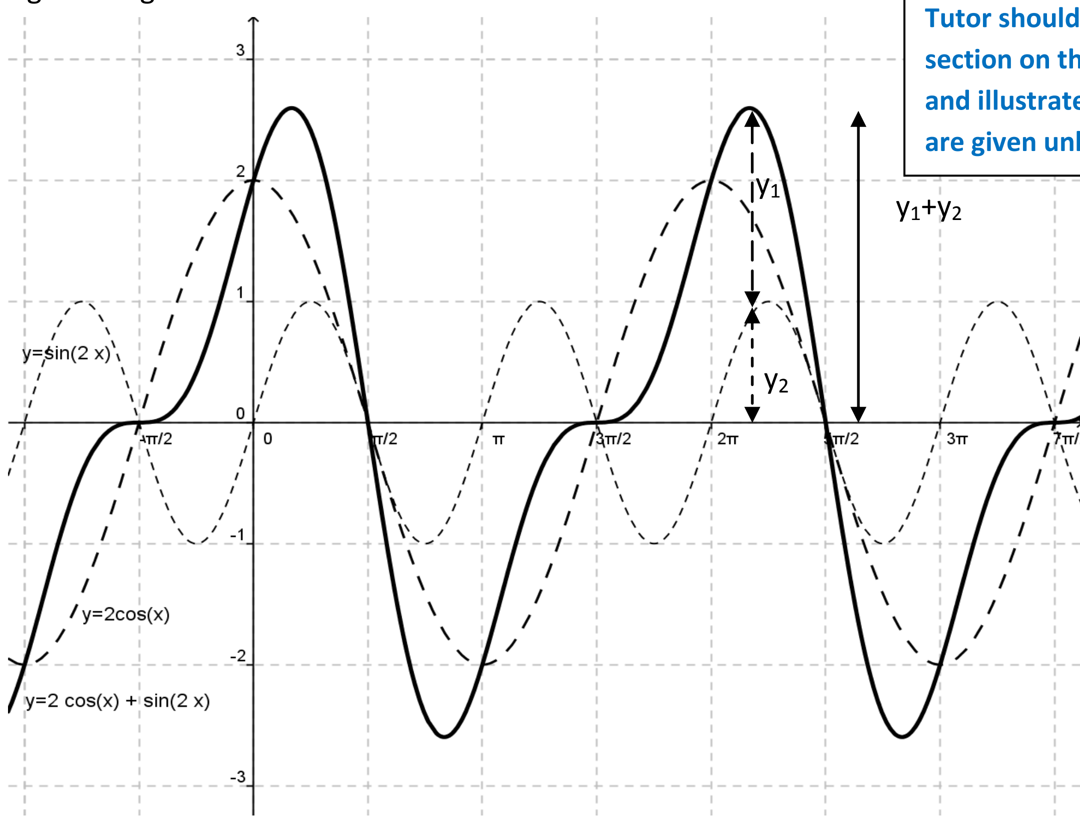
(0,0), (π/4,1), (π/2,0), (3π/4,-1), (π,0), (5π/4,1), (3π/2,0), (7π/4,-1), (2π,0)

For every known x coordinate, sum the y coordinates together!

(0,2), (π/4,$1 + \frac{\sqrt{2}}{2}$), (π/2,0), (3π/4,$-1 - \frac{\sqrt{2}}{2}$), (π,-2), (5π/4,$1 - \frac{\sqrt{2}}{2}$), (3π/2, 0), (7π/4,$-1 + \frac{\sqrt{2}}{2}$), and (2π,2)

If the function is unknown and you are given two graphs, you can produce the additional of ordinate graph by summing the heights on a certain x value.

Tutor should go through this section on the whiteboard and illustrate this if function are given unknown.

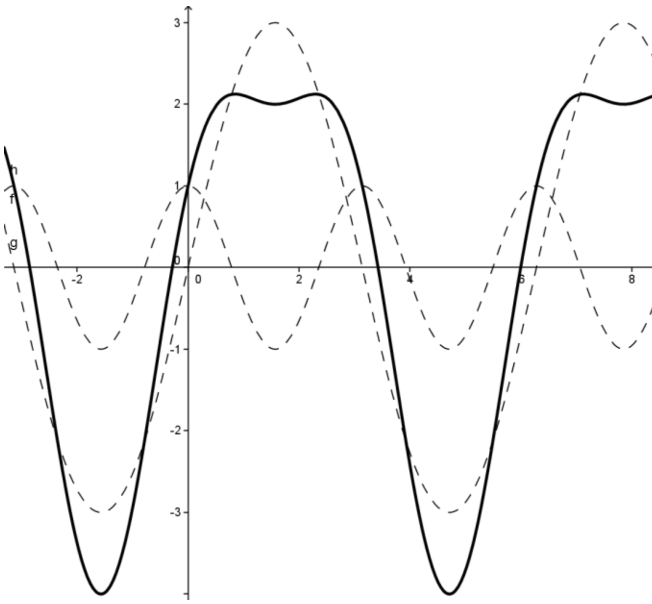


Checking Understanding

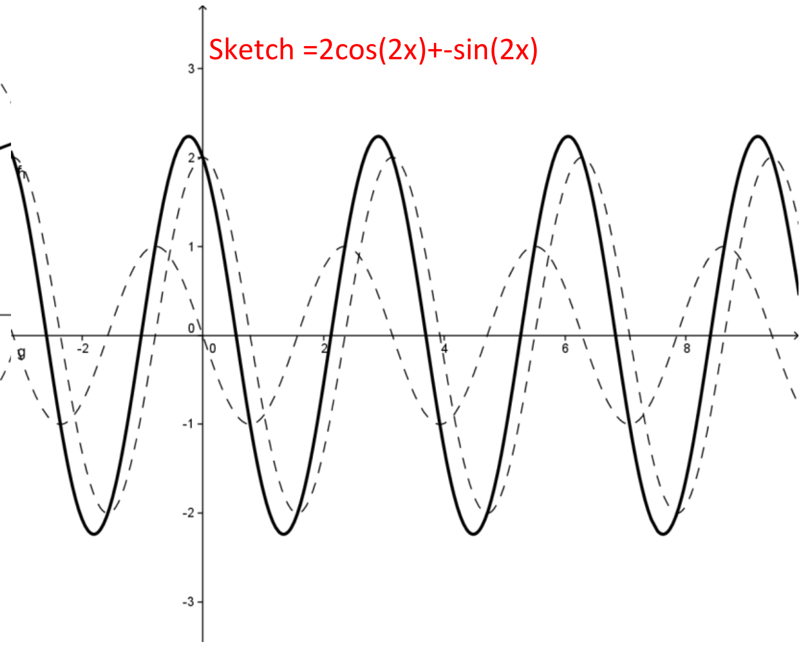
4. Sketch the following expression by using addition of ordinate method

a. $\cos(2x) + 3\sin(x)$

b. $2\cos(2x) - \sin(2x)$



Sketch =2cos(2x)+-sin(2x)



Extension

A meat slicer device slices meat at a periodic pattern. Initially, the slicer starts at its maximum position and can only deviate by 30cm. It takes 2 seconds per slice and its minimum peak is 2cm. The device must stop before 10 seconds or the machine might jam.

a. What is the amplitude, period and mean position of this device?

b. Find the function of the device, time domain and sketch the function.

c. How many cycles are there before the device stops?

The butcher realised that most meat width is 20cm and the meat travels at 2m/min in the slicer device.

d. In over what time domains the device slice does not slice anything? Illustrate the domains on the same graph.

e. What is the largest length of the meat when sliced?

f. If the butcher wants the meat no more than 2cm in length, what adjustment to the meat should be implemented? And solve for the changes.

Solution to Extension.

a. Amplitude deviation is 30cm from max to min position so 30/2 = 15cm

Period: One slice is half period. Therefore 2*2= 4 seconds

Mean position: Since minimum is 2cm therefore 32 is maximum position.

(Maximum position - Mean Position)= Amplitude -> Mean position= 17cm

b.

Period =2pi/N= 4 therefore N=0.5pi

y=A cos N(x) +C

y= 15 cos0.5pi(x)+17

Domain - [0,10) or 0<=t<10

Graph is on the last page!

c. 10/4 = 2.5 cycles

d. 20cm width so from the mean position +10 and -10. Maximum slice = 27 and Minimum slice = 7

so y> 27 and y< 7

15 cos0.5pi(x)+17= 27

cos0.5pi(x)=2/3

0.5pi*x = cos^-1(2/3)=0.84106 =0.84106, 2pi-0.84106, 2pi+0.84106, 4pi-0.84106, 4pi+0.84106

x=2*cos^-1(2/3) / pi=0.535, 3.465, 4.535, 7.464,and 8.535

15 cos0.5pi(x)+17= 7

cos0.5pi(x)=-2/3

0.5pi*x = cos^-1(-2/3) =pi-0.84106, pi+0.84106, 3pi-0.84106, 3pi+0.84106 and 5pi-0.84106

x=2*cos^-1(-2/3) / pi=1.465,2.535, 5.465,6.535 and 9.465

so Domain are [0, 0.535] , [1.46,2.535], [3.465, 4.535], [5.465,6.535], [7.464, 8.535] and [9.465, 10]

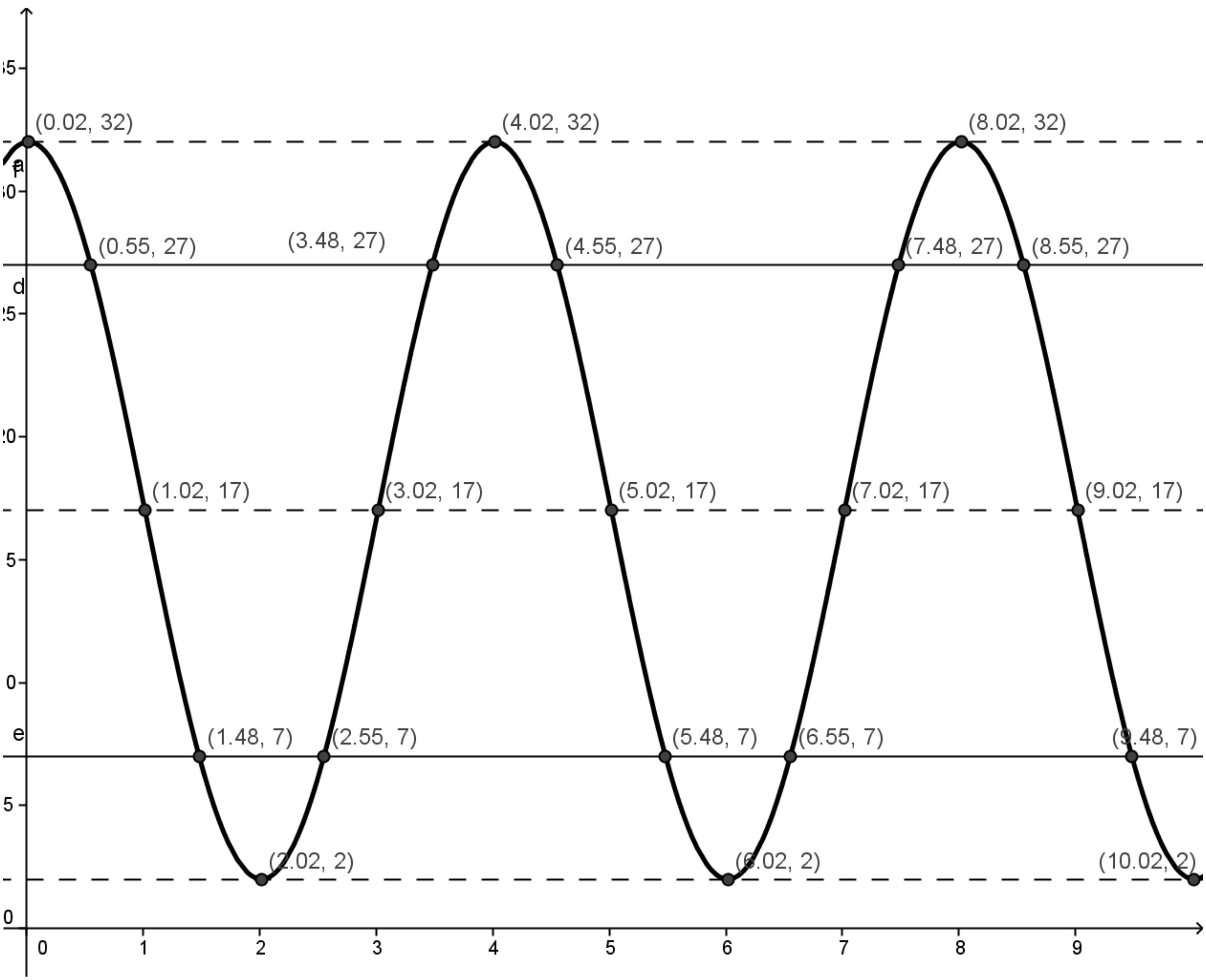
e. Find the domain difference i.e. , [3.465, 4.535], = >4.535 - 3.465=1.07 seconds

speed =2m/min = 200cm/60 seconds =10/3 cm/second

speed=distance/time = 10/3 * 1.07=3.567cms

f. Change the speed of the meat travelling.

speed = 2/1.07 = 1.869cm/s =1.12metres/min



NQT EDUCATION

YEAR 12

VCE MATHS METHODS
UNIT 3 TERM 2 WORKBOOK

WEEK 11

Introduction to Circular Functions

The tangent function

Solving trigonometric equation

Modeling circular function in application

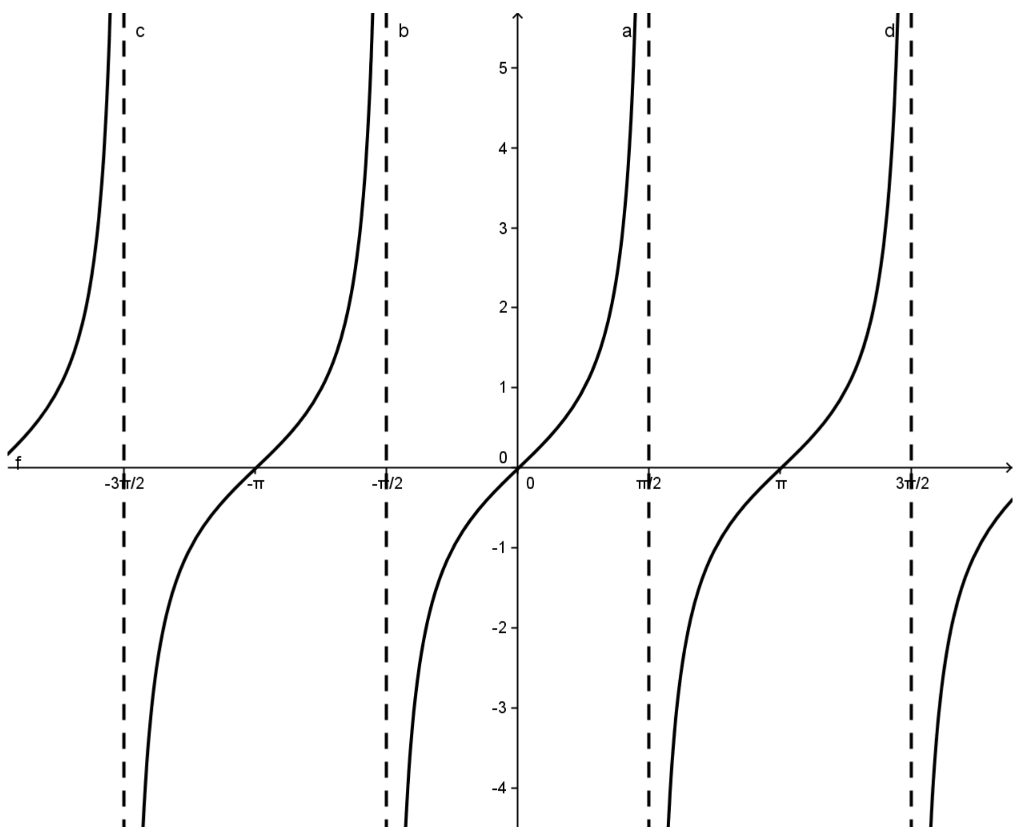
The tangent function

The formula of tangent can be recognised as $\tan x = \frac{\sin x}{\cos x}$ where $\cos x \neq 0$. Where $\cos x = 0$ here exist vertical asymptotes of the graph $y = \tan x$. i.e $x = \cdots -\frac{3\pi}{2}, -\frac{\pi}{2}, \frac{\pi}{2}, \frac{3\pi}{2}, \ldots$ The vertical asymptotes values can be expressed as $x = (2k+1)\frac{\pi}{2}$ where k *is a natural number*.

Here is a table of values for $y = \tan x$.

x	$-\pi$	$-\frac{3\pi}{4}$	$-\frac{\pi}{2}$	$-\frac{\pi}{4}$	0	$\frac{\pi}{4}$	$\frac{\pi}{2}$	$\frac{3\pi}{4}$	π
y	0	1	undefined	-1	0	1	undefined	-1	0

Note: $x = -\frac{\pi}{2}$ *and* $\frac{\pi}{2}$ are asymptotes



Through observation from the graph

- The graph repeats itself every π units.
- The range of tan is R, all real numbers.
- The vertical asymptotes have the equation $x = (2k+1)\frac{\pi}{2}$ where k *is a natural number*.

Example

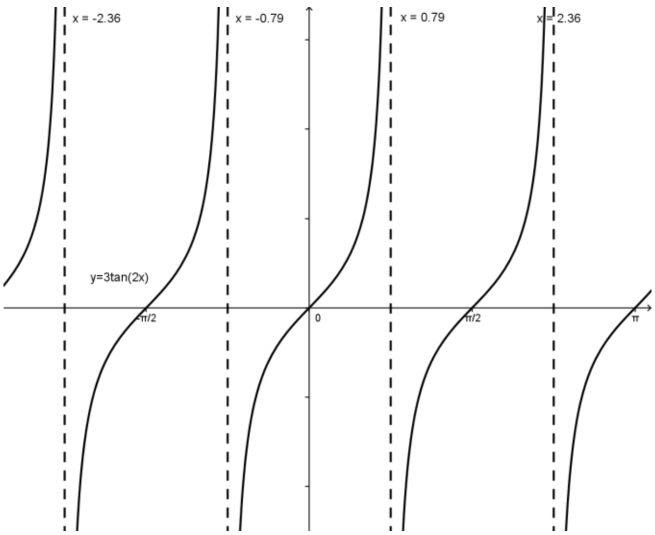
Sketch $y_1 = 3\tan 2x$ and $y_2 = -2\tan 3x$

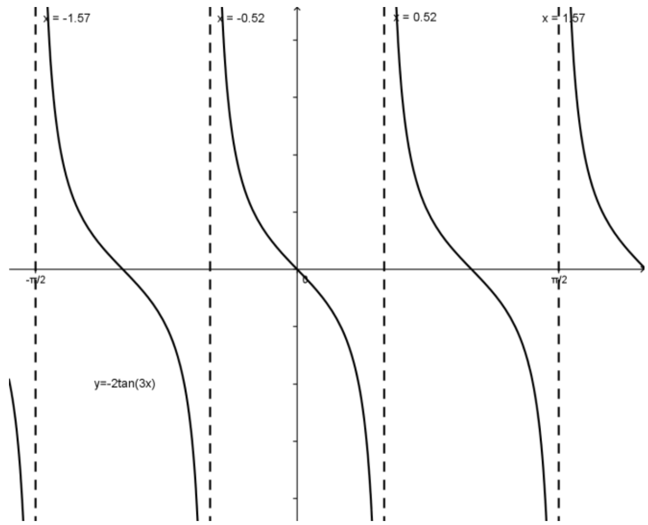
First determine the period of the function. Note where k is a natural number.

Period $n_1 = \frac{\pi}{2}$ $n_2 = \frac{\pi}{3}$

Asymptotes $x_1 = \frac{(2k+1)}{2}\frac{\pi}{2} = \frac{(2k+1)\pi}{4}$ $x_2 = \frac{(2k+1)}{3}\frac{\pi}{2} = \frac{(2k+1)\pi}{6}$

Axes intercepts $x_1 = k\frac{\pi}{2}$ $x_2 = k\frac{\pi}{3}$





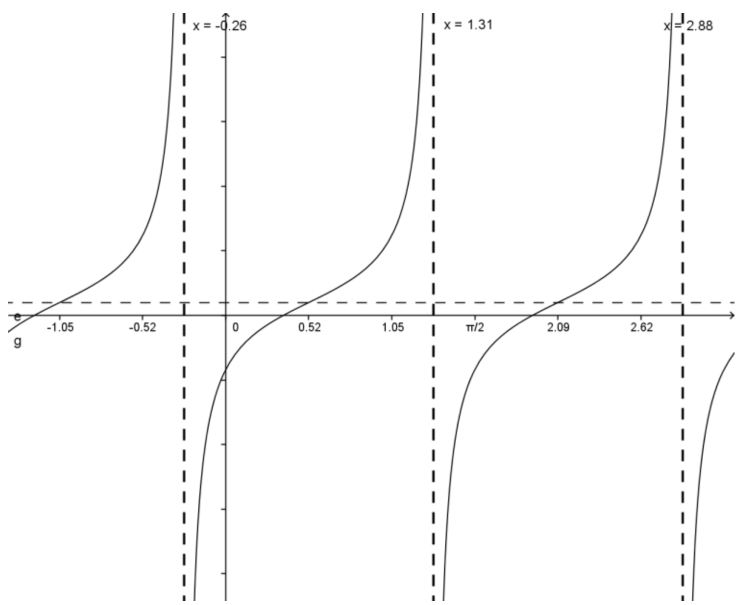
Sketch $y = 3\tan(2x - \frac{\pi}{3}) + 1$

$y = 3\tan(2x - \frac{\pi}{3}) = 3\tan\left(2\left(x - \frac{\pi}{6}\right)\right) + 1$

The transformation are:

- dilation of factor 3 from the x-axis
- dilation of factor 1/2 from the y-axis
- translation of $\frac{\pi}{6}$ units in the positive direction of the x-axis
- translation of 1 units in the positive direction of the y-axis

  Period $=\frac{\pi}{2}$ Range is R



In general:

$$f: R \setminus \left\{x : x = \frac{(2k+1)}{n}\frac{\pi}{2} + b, k \in Z\right\} \to R, f(x) = a\tan\big(n(x-b)\big) + c$$

Period $= \frac{\boldsymbol{\pi}}{|\boldsymbol{n}|}$ Range = R

Testing Understanding

1. State the transformation sequence, period and asymptote expression then graph the following:

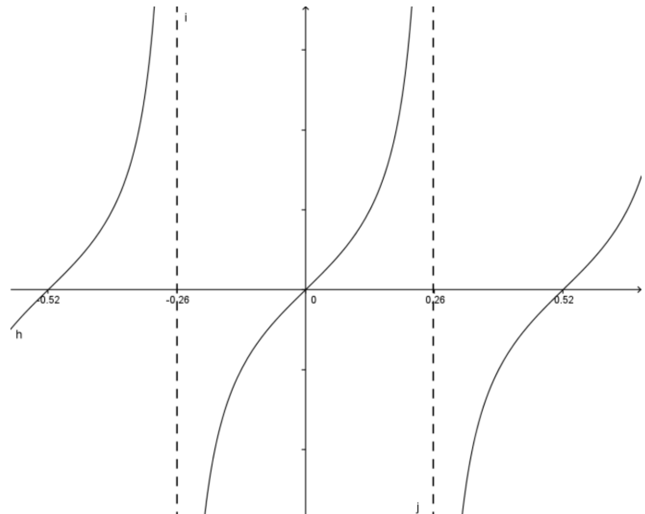
a. $y = 2\tan(6x)$

Dilation of 2 in the x axis

Dilation of 1/6 in the y axis

Period pi/6

asy (2k+1)pi/12



b. $y = 7\tan\left(\frac{1}{2}x\right) - 1$

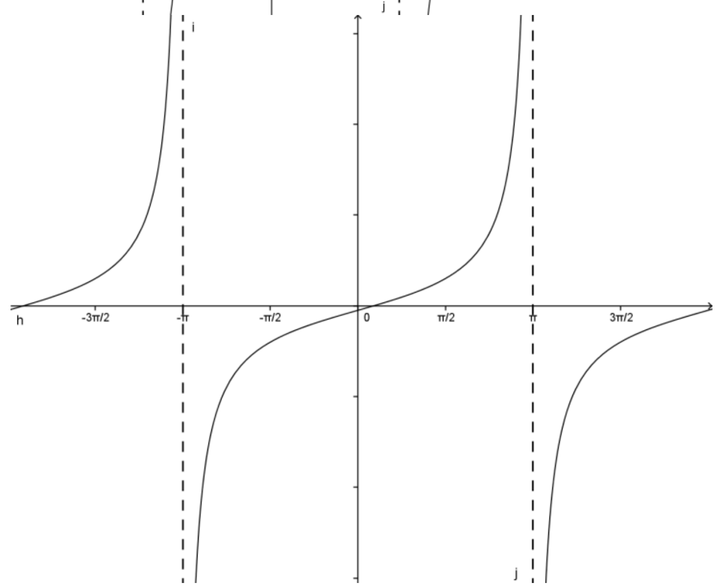
Dilation of 7 in the x axis

Dilation of 2 in the y axis0

translate 1 in the neg direction on the y axis

Period 2pi

asy (2k+1)pi



c. $y = -\tan(\pi x - \pi)$

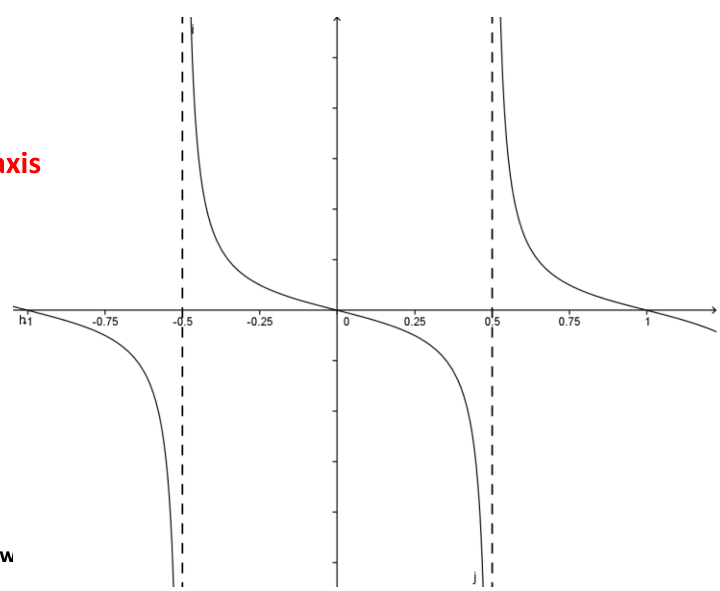
reflection on the x axis

dilation of 1/pi in the x axis

translate 1 unit in the positive direction on the x axis

Period 1

asy = (2k+1)/2



Solving trigonometric equation

There are some forms that you will need to know how to solve as it will appear most often.
In general, $\sin nx = k \cos nx$ form can be expressed as $\tan nx = k$

Examples

1. *Solve the equation* $tan\left(\frac{1}{2}\left(x-\frac{\pi}{4}\right)\right) = -1\ for\ \ x \in [-2\pi, 2\pi]$

2. Find the coordinates of the points of intersection for graphs $y = sin\ x\ and\ y = cos\ x$.

SOLUTION

$tan\left(\frac{1}{2}\left(x-\frac{\pi}{4}\right)\right) = -1$

$\frac{1}{2}\left(x-\frac{\pi}{4}\right) = \tan^{-1}(-1) = -\frac{5\pi}{4}, -\frac{\pi}{4}, \frac{3\pi}{4}\ or\ \frac{7\pi}{4}$

$x - \frac{\pi}{4} = -\frac{5\pi}{2}, -\frac{\pi}{2}, \frac{3\pi}{2}\ or\ \frac{7\pi}{2}$

$x = -\frac{9\pi}{4}, -\frac{\pi}{2}, \frac{7\pi}{4}\ or\ \frac{15\pi}{4}$

The x values that satisfy the domains are $-\frac{\pi}{2}\ and\ \frac{7\pi}{4}$

SOLUTION

- For point of intersection, it is when the two equations equal each other.

$\sin x = cos\ x$

- Simplify the equation by dividing cosine

$\tan x = 1$

$x = \tan^{-1}(1) = \frac{\pi}{4}\ or\ \frac{5\pi}{4}$

So substitute the x values into either *sin x or cos x*

There will be questions that might require you to algebraically solve or with the use of calculators.
Here are some examples for solving trigonometric equations.

Example

Solve each of the following equations for x:

a. $\cos(2x) + \cos^2 x = 2\sin^2 x$

b. $x^2 = cos\ x$

SOLUTION

Consider some trigonometric formulae

$$\cos(2x) = 1 - 2\sin^2 x = 2\cos^2 x - 1$$

Convert circular functions to one common function

$1 - \cos(2x) = 2\sin^2 x$

$\cos(2x) + 1 = 2\cos^2 x$

Substitute into the equation

$\cos(2x) + \frac{1}{2}(\cos(2x) + 1) = 1 - \cos(2x)$

$\frac{5}{2}\cos(2x) = \frac{1}{2}$

$\cos(2x) = \frac{1}{5}$

$2x = \cos^{-1}\left(\frac{1}{5}\right) \approx 1.369$

$x = \frac{1}{2}\cos^{-1}\left(\frac{1}{5}\right) \approx 0.685$

There maybe times where you cannot solve it by hand.

$\cos(2x) = 1 - 2\sin^2 x = 2\cos^2 x - 1$

- PLOT Y1=X^2 and Y2=cos(X)
- On your graphics calculator use the **INTERSECTION** function to solve for the x value. x≅0.824
- You can always trial and error by guess a range of x values then repeat with smaller range.

x	0.820	0.825	0.830	0.835
$\mathbf{x^2}$	**0.6724**	**0.6806**	0.6889	0.6972
Cos(x)	**0.6822**	**0.6786**	0.6749	0.6712

Thus solution is 0.820<x<0.825 as difference of solution is the smallest.

Checking Understanding

2. Solve the following equations for x by hand.

a. $\tan(2x) = 1$ over [0,2π]

pi/8, 5pi/8, 9pi/8, 13pi/8,

b.$\tan(x + \pi/6) = \sqrt{3}$ over [-2π,2π]

-11pi/6, -5pi/6,pi/6, 7pi/6,

c.$\sin 3x = \frac{1}{\sqrt{3}}\cos 3x$ over [0,2π]

pi/18,7pi/18,13pi/18,19pi/18,25pi/18,31pi/18

d. $\cos^2 x - \sin^2 x = -\sqrt{3}\sin(2x)$ over [0,π]

5pi/12, 11pi/12

e. $\tan(x) + \cos^2 x = -\sin^2 x$ over [0,2π]

3pi/4, 7pi/4

f. $(\sin x + \cos x)^2 - x = \sin 2x$

x=1

3. Solve the following equations for x. Use calculator!

a. $\sin x = x$

x=0

b. $\cos x = e^{-x}$

x=0, 1.2927

c. $2\cos x = 2 - x^2$

x=0

d .$\sin 2x = x^3$

x=0.8028

Modelling circular function in application

In exam, SAC or assignment you will encounter problem solving questions that requires you to model and interpret the problem into mathematics. The best preparation for this is to do more problem solving questions! Eventually you will be more confident as well as know what to expect in a circular function application.

Here is one example for circular function applications!

In a tidal river, the time between high tides is 12hours. The average depth of water in the port is 5m; at high tide the depth is 8m. Assume that the depth of water is given by

$$h(t) = A\sin(nt + b) + c$$

where t is the number of hours after 12:00 noon. At 12:00 noon there is a high tide

a. Find the values of A,n, b and c.

b. At what times is the depth of the water 6m

SOLUTION

a. Average Depth of water is 5m, and highest is 8m. Given that 'max - mean' = amplitude. Therefore A=3

Period is defined as = $\frac{2\pi}{n} = 12 \quad n = \frac{\pi}{6}$

The mean position which is also c= 5

In a form $nt + b = n(t + \frac{b}{n})$, $\frac{b}{n}$ is the shift to the left! Since the nearest maximum for sine is at 3 $(\frac{Period}{4})$, $\frac{b}{n} = 3$.

Therefore b =$\frac{\pi}{2}$

b. $h(t) = 3sin\left(\frac{\pi}{6}t + \frac{\pi}{2}\right) + 5 = 6$

$$sin\left(\frac{\pi}{6}t + \frac{\pi}{2}\right) = \frac{1}{3}$$

$\frac{\pi}{6}t + \frac{\pi}{2} = \sin^{-1}\left(\frac{1}{3}\right) \approx 0.339 = \pi - \sin^{-1}\left(\frac{1}{3}\right), 2\pi + \sin^{-1}\left(\frac{1}{3}\right), 3\pi - \sin^{-1}\left(\frac{1}{3}\right),\ 4\pi + \sin^{-1}\left(\frac{1}{3}\right), 5\pi - \sin^{-1}\left(\frac{1}{3}\right)$

$t = \frac{6}{\pi}\left(\pi - 0.3398 - \frac{\pi}{2}\right), \frac{6}{\pi}\left(2\pi + 0.3398 - \frac{\pi}{2}\right), \frac{6}{\pi}\left(3\pi - 0.3398 - \frac{\pi}{2}\right), \frac{6}{\pi}\left(4\pi + 0.3398 - \frac{\pi}{2}\right), \frac{6}{\pi}\left(5\pi - 0.3398 - \frac{\pi}{2}\right)$

$$t = -2.3509, 2.3509, 9.649, 14.3509, 21.649, 26.350.$$

Note that time is between 0 to 24 hours. Thus the solutions are

2.21PM,9.39PM,2.21AM and 9.39AM

Checking Understanding

1. A ball is dropped from a height of 5 metres and bounces continuously in a regular motion for the next $\frac{8\pi}{5}$ seconds. The height of the ball above the ground is given by $H(t) = |5\cos(5t)|$ where H is the height in metres and t is the time in seconds after the ball is dropped.

a. Find when the ball first hits the ground.

pi/10=~0.314 seconds

b. Find when the ball first rebounds to maximum height.

pi/5=~0.6283

Another ball is released given by $H_2(t) = \left|5\sqrt{3}\sin(5t)\right|$.

c. Find when the two ball first meets.

tan5x=1/root3

x=1/5 (0.52359)=0.1047 seconds

Realistically, the ball cannot rebound to its previous height. Instead the rebound height on each bounce will diminish with each bounce. It is found that the height of a ball released from a height of 5metres can be more accurately modelled by the equation $H(t) = |5e^{-0.8t}\cos(5t)|$.

d. Find the first time, correct to 2 decimal places when the ball is exactly 3 metres above the ground.
GRAPHIC x=0.1635

e. How many time during the motions as described by the model is the ball exactly at a height of 2 metres above the ground.
Illustrate on GRAPHIC and check by intersection or zoom

2. The temperature A°C inside a house at t hours after 4a.m. is given by

$A(t) = 21 - 3\cos\left(\frac{\pi t}{12}\right)\ for\ 0 \le t \le 24$ and the temperature B°C outside the house at the same time is given by $B(t) = 22 - 5\cos\left(\frac{\pi t}{12}\right)\ for\ 0 \le t \le 24.$

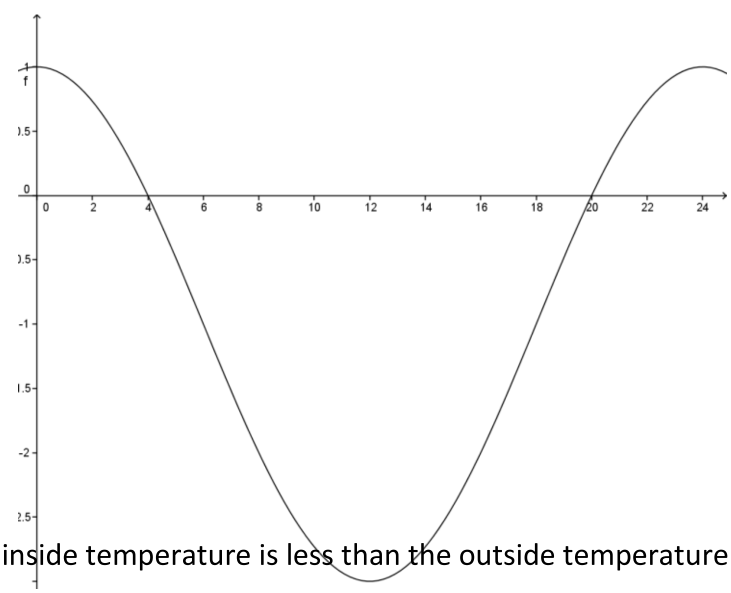
a. Find the temperature inside the house at 8 a.m.

19.5Degrees

b. Write down an expression for D=A-B, the difference between the inside and outside temperature

D=-1+2cos(pi t/12)

c. Sketch the graph of D for $0 \le t \le 24$



d. Determine when the inside temperature is less than the outside temperature.
4<t<20

e. Find when the absolute difference in temperature is exactly $(\sqrt{2}-1)$°C
3,21, 4.864,19.135

NQT EDUCATION

YEAR 12

VCE MATHS METHODS
UNIT 3 TERM 2 WORKBOOK

WEEK 12

Differentiation:

Continuity and Limits

The gradient function

Derivatives of polynomials and functions with rational powers

Introduction to Differentiation

Differentiation is a topic in mathematics that explores change of quantities. For example, the change in distance with respect to time is velocity. Thus, differentiation can be used to model an event that describes the rate of change of a variable with respect to the other. The derivative is known as $\frac{dy}{dx}$. Before we study differentiation we need to understand the conditions and other topics that strongly relates to differentiation.

Continuity

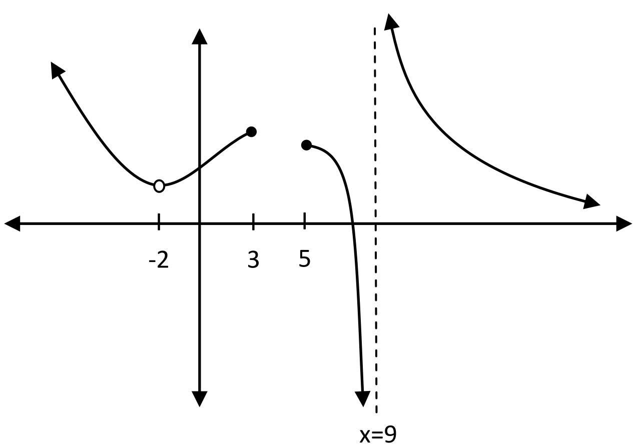
Continuity define as

1. $f(a)\ exist\ such\ that\ in\ a\ f(x) is\ defined\ at\ x = a$
2. $\lim_{x\to a} f(x)\ exist$
3. $\lim_{x\to a} f(x) = f(a)$

In other words the graph has no holes, breaks or jumps. Simply imagine a graph drawn without lifting our pen off the paper.

The typical discontinuity types are: **holes, jumps and asymptotes.**

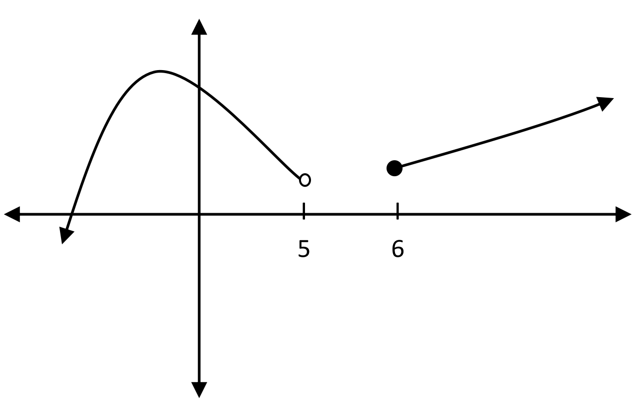
Where the f(x) is undefined the function is discontinuous at those points.



Continuous interval	Discontinuous interval
$-\infty < x < -2$	$x = -2$ (Hole)
$-2 < x \leq 3$	$3 < x < 5$ (Interval)
$5 \leq x < 9$	$x=9$ (Asymptotes)
$9 < x < \infty$	

Example

State the values of x across which f(x) is discontinuous and continuous.



SOLUTION

First you need to identify the holes, gaps and breaks. Gap between 5 and 6 where 5 is a hole.

Then use appropriate inequalities.

Discontinuous across $\{x: 5 \leq x < 6\}$

Continuous across $\{x: x < 5\} \cup \{x: x \geq 6\}$

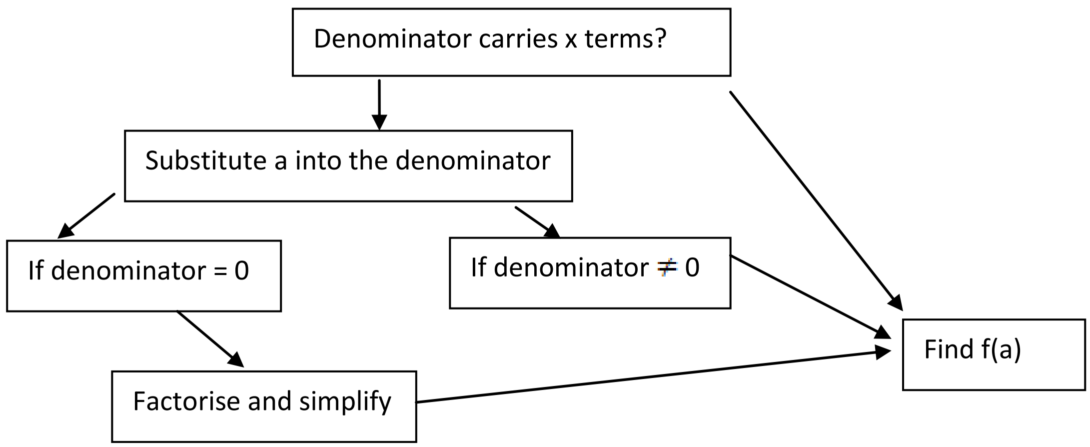
Limits

The behaviour of the function can be described by the use of limits. It also allows us to describe rates of change. Limit can be considered as a certain value or solution that curves approach as the independent value gets closer to a particular number.

$$if\ f(x) is\ continuous\ at\ x = a, then\ \lim_{\mathrm{x}\to\mathrm{a}} \mathrm{f(x)} = \mathrm{f(a)}$$

When evaluating the limits there are some problems that you might face.

Use this flow diagram to guide you to solve problems involving limits.



Note that when the function is in the simplified form, the limit of the function exists.

Example

Evaluate the following

a. $\lim_{x\to 4} x+3$

b. $\lim_{x\to 1} \frac{x^2+x-2}{x-1}$

Solution

Substitute $x = a$ into the f(x) equation to find f(a)

$\lim_{x\to 4} x+3$

$=4+3=7$

Solution

Check if the denominator =0 when a is substituted.

Since its so, factorise the given equation

Substitute $x = a$ into the f(x) equation to find f(a)

$\lim_{x\to 1} \frac{x^2+x-2}{x-1} = \lim_{x\to 1} \frac{\cancel{(x-1)}(x+2)}{\cancel{(x-1)}} = \lim_{x\to 1} x+2$

$=1+3=4$

Thus

$$\lim_{x\to 1} \frac{x^2+x-2}{x-1} = 4, x \neq 1$$

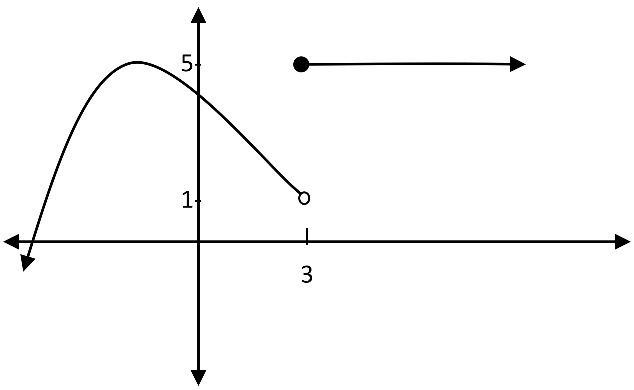
Be careful with the $x \to a$ (approaching to a). It doesn't necessary mean $x = a$

To test if the function is discontinuous, solve where the left-hand limit equals to the right hand limit as x→ a

If the limits are not equal then the function is discontinuous across an interval and the 'a' lies within this interval.

Consider the function

$$f(x) = \begin{cases} 5, & x \geq 3 \\ g(x), & x < 3 \end{cases}$$



If you start from the left hand side and trace to when x approaches to 3 the solution is 1. However when starting from the right hand side and approaching to 3 the solution is 5. The two limits doesn't equal, therefore function is **discontinuous.**

This is usually express as

$\boldsymbol{\lim_{x \to 3^+} f(x) = 5}$

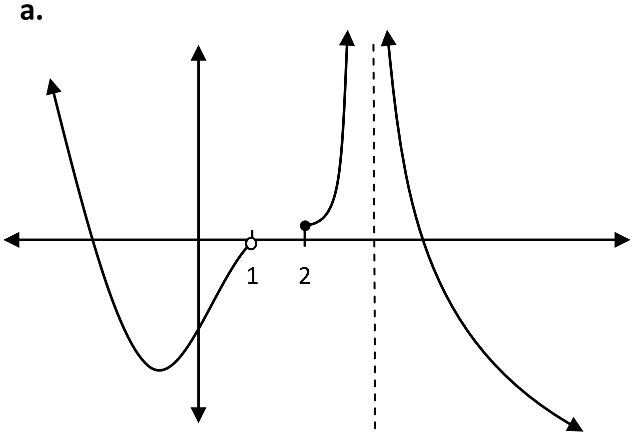
$\boldsymbol{\lim_{x \to 3^-} f(x) = 1}$

The function above is continuous when $x \in R\backslash\{3\}$, every real number except for 3.

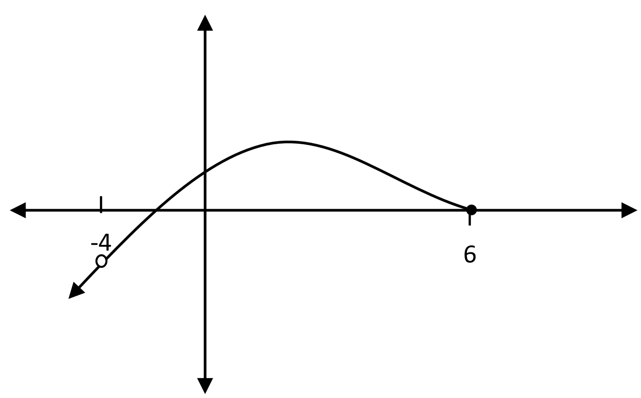
Testing Understanding

1. State the values of x across which f(x) is discontinuous and continuous for each graph.

a.



b.



2. Evaluate the following

a. $\lim_{x \to 4} 1$

=1

b. $\lim_{x \to 3} \frac{x+1}{x+3}$

=2/3

c. $\lim_{x \to 1} \frac{x^2}{x+5}$

=1/6

d. $\lim_{x \to -2} \frac{x^2-4}{x+2}$

=-4

e. $\lim_{x \to -2} \frac{x+2}{x^2+5x+6}$

=1

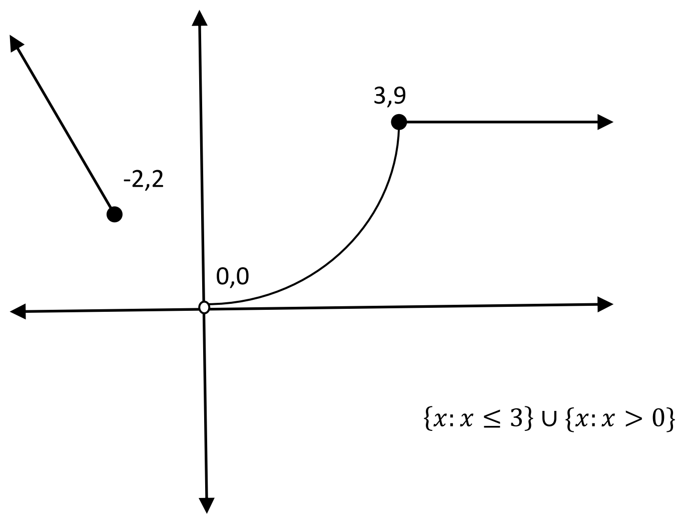
f. $\lim_{x \to 2} \frac{x^3-8}{x-2}$

=12

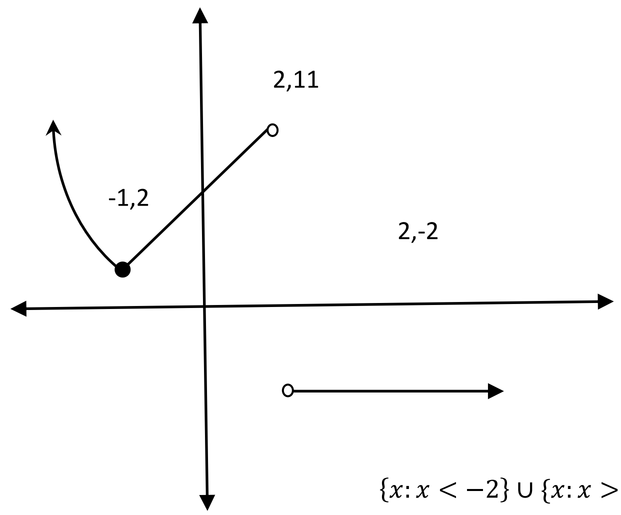
3. Sketch the following and state the values of x across f(x) is continuous

a. $f(x) = \begin{cases} -x, & x \le -2 \\ x^2, & 0 < x < 3 \\ 9, & x > 3 \end{cases}$

b. $f(x) = \begin{cases} 2x^2, & x \le -1 \\ 3x+5, & -1 < x < 2 \\ -2, & x > 2 \end{cases}$



$\{x: x \le 3\} \cup \{x: x > 0\}$

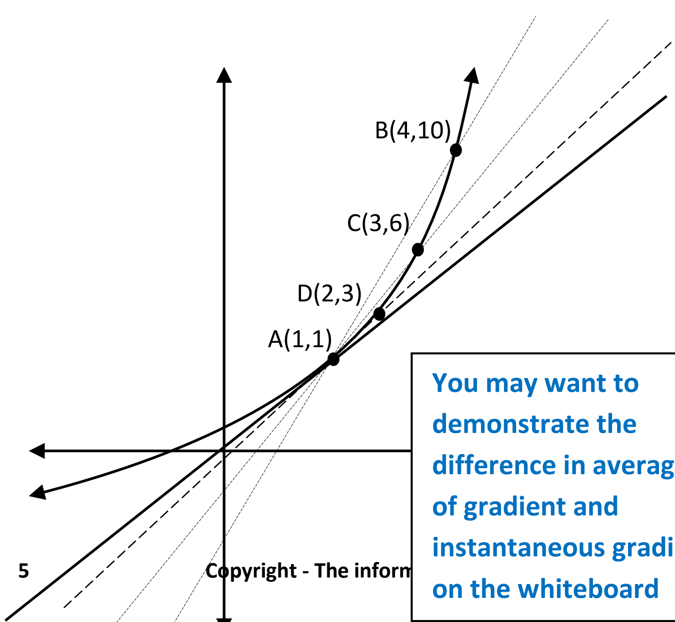


$\{x: x < -2\} \cup \{x: x > 2\}$

The gradient function

In the previous topic, the gradient is the measure of rates of change. The rate describes how the change of one quantity is compared with respect to another quantity. For example the distance covered over a certain time describes the speed. The gradient of a curve is the measurement of rate of change. From linear algebra, we were dealing with constant gradients, $m \text{ or } \frac{dy}{dx}$.

However, the gradient of a non linear curve is continuously changing with respect to 'x'. The rate at which y is changing with respect to x varies and is dependent upon the value of x at which the rate is being evaluated. The tangent is the line that touches the curve at one point only and can be used to evaluate the rate of change.



The gradient of AB is $\frac{10-1}{4-1} = 3$

The gradient of AB is $\frac{6-1}{3-1} = \frac{5}{2}$

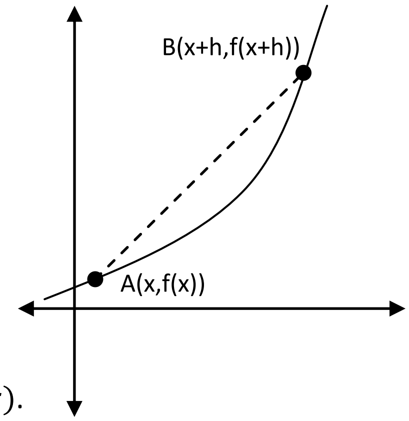
The gradient of AB is $\frac{3-1}{2-1} = 2$

As the difference between two points approaches to zero, the gradient convergences to an accurate value of the tangent, where it only hits one point!

You may want to demonstrate the difference in average of gradient and instantaneous gradient on the whiteboard

When approximating the gradient without the exact coordinates, you will need to extend the gradient line and use the furthest line values of x and y for an accurate result!

The gradient of the chord AB is



$$m = \frac{y_2 - y_1}{x_2 - x_1} = \frac{f(x+h) - f(x)}{(x+h) - x} = \frac{f(x+h) - f(x)}{h}$$

However if the step 'h' approaches to zero we can determine the tangent to the curve.

$\lim_{h \to 0} \frac{f(x+h) - f(x)}{h}$

This limit gives the gradient of the tangent at any point. It is known as gradient function, derivative function $f'(x)$ or derivate of function $f(x)$.

Other notations of tangent are $\frac{df}{dx}, \frac{d}{dx} f(x),\ y' and\ D_x(f)$

In general we express

$$f': R \to R \ and\ f'(x) = \lim_{h \to 0} \frac{f(x+h) - f(x)}{h}$$

$\frac{dy}{dx}$ *is not 'dy divide by dx',*

it is a notation that represent the

differentiation of a function.

This process of obtaining $f'(x)$ or $\frac{df}{dx}$ is called **differentiation** and finding the gradient of the tangent by evaluating the limit of gradient of the chord is known as **differentiation from first principle.** The function f is said to be differentiable at point $(a, f(a))$ $if\ \lim_{h \to 0} \frac{f(a+h) - f(a)}{h}$

Example

a. Use the first principle find the derivate of $f(x) = 4x^3 - 1$

Tutor should prepare for another example on the whiteboard.

b. Find the gradient of tangent at x=2

c. Find the coordinates of the points on the curve which the gradient of the tangent equals 12

a. First, use the rules for instantaneous rate of change. $f'(x) = \lim_{h \to 0} \frac{f(x+h) - f(x)}{h}$

$$Solve\ for\ f(x+h) = 4(x+h)^3 - 1$$
$$= 4(x^3 + 3x^2h + 3xh^2 + h^3) - 1$$
$$= 4x^3 + 12x^2h + 12xh^2 + 4h^3 - 1$$

Substitute into the rule and simplify

$$f'(x) = \lim_{h \to 0} \frac{(4x^3 + 12x^2h + 12xh^2 + 4h^3 - 1) - (4x^3 - 1)}{h}$$

Take out the common factor h the evaluate

$$\lim_{h \to 0} \frac{12x^2h + 12xh^2 + 4h^3}{h} = \lim_{h \to 0} 12x^2 + 12xh + 4h^2 = 12x^2$$

b. Now you know that $f'(x) = 12x^2$ so by substituting x =2
$f'(2) = 12x^2 = 12(2)^2 = 48$

c. Given that the gradient is $f'(x) = 12$, we will transpose the equation to solve for x
$12x^2 = 12$
$x^2 = 1$
$x = -1 \; or \; 1$ there are two cases were the gradient equals to 12!
To solve for y, substitute the value of x into f(x)
$f(\pm 1) = 4(\pm 1)^3 - 1 = \pm 4 - 1 = 3 \; or - 5$

Coordinates are $(1,3) and \; (-1,-5)$

Derivatives of polynomials and functions with rational powers

For now you will be dealing with differentiation of polynomial and rational functions. For polynomial functions the derived functions always exist. With further investigation we can deduce that

$For\ f(x) = 1, f'(x) = \ \ 0$

$For\ f(x) = x^2, f'(x) = \ \ 2x$

$For\ f(x) = x^3, f'^{(x)} = \ \ 3x^2$

$For\ f(x) = x^4, f'^{(x)} = \ \ 4x^3$

Can you find a pattern or relationship between $f(x)\ and\ f'(x)$?

Tutor should demonstrate the pattern by using the first principle

The general result:

$$For\ f(x) = x^n, f'(x) = \ nx^{n-1}, n = 1, 2, 3, \ldots \quad For\ f(x) = 1, f'(x) = 0$$

The **derivate of a number multiple is the multiple of derivative**

$$if\ f(x) = kx^n, then\ f'(x) = knx^{n-1}$$

In general it is expressed as

$$g(x) = k(fx), where\ k\ is\ a\ constant, then\ g'(x) = f'(x)$$

Another rule for differentiation is the **derivative of sum is the sum of derivatives**

$$if\ f(x) = g(x) + h(x), then\ f'(x) = g'(x) + h'(x)$$

Example

Find the derivative of the following

a. $5x^3$

b. $x^3 + x^2$

c. $4x^3 + x^{5/2}$

Solution

Consider Multiple of derivative

$f'(x) = 5(3x^2)$

Solution

Consider Sum of derivatives

$f'(x) = (3x^2) + (2x)$

Solution

Consider General rule, multiple and sum of derivatives

$f'(x) = 4(3x^2) + (\frac{5}{2}x^{\frac{3}{2}})$

Go through these examples together as a class!

Checking Understanding

4. Using the first principle of differentiation solve the following

a. $f(x) = 2x^2 - 5$

f'=4x

b . $f(x) = 6x^2 + x - 1$

f'=12x+1

c. $f(x) = (2x - 3)^2$

f'=8x-12

5. Find the derivative of the following

a. $f(x) = x^7$

f'=7x^6

b. $f(x) = 4x^2 + x - 5$

f'=8x+1

c. $f(x) = x + x^6 - x^3$

f'=1+6x^5-3x^2

d. $f(x) = 5x^{7/3}$

f'=35/3 x^4/3

e. $f(x) = (x - 2)^2$

f'=2x-4

f. $f(x) = 3x(x - x^3)$

f'=6x-12x^3

g. $f(x) = \frac{x+x^3}{x} + 5$

f'=2x

h. $f(x) = \frac{x^2+5x+6}{x+2}$

f'=1

i. $f(x) = x^{\frac{5}{2}} + 6x^{11/3}$

f'=5/2 x^3/2 + 66/3 x^8/3

NQT EDUCATION

YEAR 12

VCE MATHS METHODS
UNIT 3 TERM 2 WORKBOOK

WEEK 13

Differentiation:

Differentiating where powers are negative
Chain Rule
Product Rule
Quotient Rule

Differentiating where powers are negative

In the previous topic, we were dealing with derivatives of positive indexes. However, you will figure that for negative indexes, the form which we differentiate polynomial is the same!

For example

$f: R\backslash\{0\} \to R, f(x) = \frac{1}{x} = x^{-1}$

We can solve the above function by using first principle.

$$f'(x) = \lim_{h\to 0} \frac{f(x+h) - f(x)}{h}$$

So,

$$= \lim_{h\to 0} \frac{\frac{1}{x+h} - \frac{1}{x}}{h} = \lim_{h\to 0} \frac{x-(x+h)}{(x+h)x} \times \frac{1}{h} = \lim_{h\to 0} -\frac{1}{(x+h)x} = -\frac{1}{x^2}$$

Therefore $f'(x) = -x^{-2}$

For further investigation we get

$f: R\backslash\{0\} \to R, f(x) = \frac{1}{x^2} = x^{-2}, f'(x) = -2x^{-3}$

$f: R\backslash\{0\} \to R, f(x) = \frac{1}{x^3} = x^{-3}, f'(x) = -3x^{-4}$

In general

$For\ f: R\backslash\{0\} \to R, f(x) = 1/x^n = x^{-n}, f'(x) = -nx^{-(n+1)}$

Where n is a non-zero integer, note that $f(x) = 1, f'(x) = 0$

Note:
For $n \leq -1, the\ domain\ f\ can\ be\ taken\ to\ be\ R\backslash\{0\} and\ for\ n \geq 1\ we\ take\ the\ domain\ of\ f\ to\ be\ R$

For example

a. Find the gradient of the curve determined by the function $f: R\backslash\{0\} \rightarrow R, f(x) = x^3 + x - 1/x^2$

b. Solve for the gradient when $x = 2$

c. State the coordinates when the gradient of the curve is 1

Solution:

a. By following the differential rule:

$$f: R\backslash\{0\} \rightarrow R, f(x) = x^3 + x - 1/x^2$$
$$f': R\backslash\{0\} \rightarrow R, f'(x) = (3x^2) + 1 - (-2x^{-3})$$

Therefore

$$f'(x) = 3x^2 + 1 + 2x^{-3}$$

Solution:

b. Once we solve for the equation of the curve gradient we can substitute $x = 2$,

$$f'(x) = 3x^2 + 1 + 2x^{-3}$$

$$f'(2) = 3(2)^2 + 1 + 2(2)^{-3}$$

$$f'(2) = 12 + 1 + \frac{1}{4} = 13.25$$

Solution:

c. We substitute $f'(x) = 1 \ to\ the\ equation$

$f'(x) = 3x^2 + 1 + 2x^{-3} = 1$

Multiply x^3 both sides

$3x^5 + 1x^3 + 2 = 1x^3$

$3x^5 + 2 = 0$

$x = \left(-\frac{2}{3}\right)^{\frac{1}{5}}$

Substitute the x value into the function. Therefore (-0.922,-2.88)

Testing Understanding

1. Find the derivative of the following

a. $f: R\backslash\{0\} \to R, f(x) = x^{-4} - 1/x^2$

-4x^-5+2/x^3

b. $f: R\backslash\{0\} \to R, f(x) = \frac{-x^5+4x-x^{-2}}{x^4}$

-1-12x^-4+6x^-7

c. $f: R\backslash\{0\} \to R, f(x) = -\frac{1}{x^3} + \frac{3}{x} + 3$

3/x^4-3/x^2

d. $f: R\backslash\{0\} \to R, f(x) = \frac{6}{x} - 3x^7$

-6/x^2-21x^6

2. Find the gradient of the curve determined by the function

a. $f: R\backslash\{0\} \to R, f(x) = x + \frac{3}{x^2} - \frac{2}{x}$ and $g: R\backslash\{0\} \to R, g(x) = x - \frac{1}{x^4}$

b. Solve for the gradient when $x = 4$

c. For which coordinates when the gradient of the curve is 0

a.1-6/x^3+2/x^2 and 1+4/x^5

b.1-6/64+2/16=33/32 and 1+4/1024=257/256

c 1-6/x^3+2/x^2=0

1 x^3-6 +2x=0

use calculator thus x=1.4562 then f=1.4975

for 1+4/x^5=0

x^5+4=0, x^5=-4,x =(-4)^(1/5)=-1.3195 then y =-1.6493

Chain Rule

If we were given a function $y(x) = (x+2)^2$ to differentiate, we will simply expand then solve. However what happens if it is $y(x) = (x+2)^3$, $y(x) = (x+2)^4$ or even $y(x) = (x+2)^{10}$!
We can express $y(x) = (x+2)^2$ as the composition of two functions

$f(x) = g(x)^2 = u^2$ and $g(x) = u = (x+2)$

The variables are 'chained' together; x is linked with u, and u is linked with y.

This can be written as $y(x) = f\left(g(x)\right) = f \ o \ g(x), which\ you\ have\ seen\ before$!

In general

If $y(x) = f(g(x))$ and g is differentiable at x, and f is differentiable at g(x), then the derivative of y exists and

$$y'(x) = f'(g(x))g'(x)$$

Or in notation of Leibniz, which is more recognisable as

$$\frac{dy}{dx} = \frac{dy}{du} \times \frac{du}{dx}$$

This is called the **Chain Rule**

For example

$Differentiate\ y = (2x - 3)^3$

Solution

We let $u = 2x - 3$, then $y = u^3$.

Next we find $\frac{du}{dx}$ and $\frac{dy}{du}$

$\frac{du}{dx} = 2\ and\ \frac{dy}{dx} = 3u^2$

Use the Chain Rule

$$\frac{dy}{dx} = \frac{dy}{du} \times \frac{du}{dx} = 3u^2 \times 2 = 6u^2$$

The substituting u into the equation

$$\frac{dy}{dx} = 6u^2 = 6(2x - 3)^2$$

Product Rule

When an expression contains a product of two functions, we can use the product rule to differentiate these problems.

If $F(x) = f(x).g(x)$ such that $f'(x) and\ g'(x) exist,\ then$

$$F'(x) = f(x).g'(x) + g(x).f'(x)$$

It can be also written as

$$\frac{dy}{dx} = \frac{d}{dx}(uv) = u.\frac{du}{dx} + v.\frac{du}{dx}$$

Example

Find the derivative of the following

a. $(3x^2 - 1)(x^5 + x)$

b. $x^2(4x^2 - 1)^3$

Solution

First we express the function as two independent equations

$y = (3x^2 - 1)(x^5 + x)$

Let $u = (3x^2 - 1)$ *and* $v = (x^5 + x)$

Then solve the derivatives of each equation

$$\frac{du}{dx} = 6x \text{ and } \frac{dv}{dx} = 5x^4 + 1$$

So following the product rule

$$\frac{dy}{dx} = u.\frac{du}{dx} + v.\frac{du}{dx}$$

$= (3x^2 - 1).(5x^4 + 1) + (x^5 + x).6x$
$= 15x^6 + 3x^2 - 5x^4 - 1 + 6x^6 + 6x^2$
$= 16x^6 - 5x^4 + 9x^2 - 1$

Solution

First we express the function as two independent equations

$y = x^2(4x^2 - 1)^3$

Let $u = x^2$ *and* $v = (4x^2 - 1)^3$

Then solve the derivatives of each equation

$\frac{du}{dx} = 2x \text{ and } \frac{dv}{dx} = 24x(4x^2 - 1)^2$ *

So following the product rule

$$\frac{dy}{dx} = u.\frac{du}{dx} + v.\frac{du}{dx}$$

$= x^2.24x(4x^2 - 1)^2 + (4x^2 - 1)^3.2x$
$= (4x^2 - 1)^2(24x^3 + 2x(4x^2 - 1))$
$= (4x^2 - 1)^2(32x^3 - 2x)$
$= 2x(4x^2 - 1)^2(16x^2 - 1)$

Quotient Rule

Similar to product rule, if an expression has $F(x) = \frac{f(x)}{g(x)}$, *where* $g(x) \neq 0$ *and such that* $f'(x)$ *and* $g'(x)$ *exist*, then the expression can be differentiated.

If $F(x) = f(x)/g(x)$ *where* $g(x) \neq 0$ such that $f'(x)$ *and* $g'(x)$ *exist, then*

$$F'(x) = \frac{g(x).f'(x) - f(x).g'(x)}{[g(x)]^2}$$

It can be also written as

$$\frac{dy}{dx} = \frac{d}{dx}(u/v) = \frac{v.\frac{du}{dx} - u.\frac{du}{dx}}{v^2}$$

If you are having trouble memorising the quotient rule, you can still differentiate this quotient form by using product rule!

By simply transforming $y = \frac{u}{v}$ *to* $y = uv^{-1}$

Tutors should demonstrate how to solve a function with chain rule and derivative in a derivative. ie

$$\frac{2x^2+3}{(5x^3-2)^3\left(x^6-x^{\frac{1}{2}}\right)^7}$$

Example

Find the derivative of $\frac{x^2+3}{6x-1}$ with respect to x.

Solution

$We\ let\ y = \frac{x^2+3}{6x-1}$

Let $u = x^2 + 3$ and $v = 6x - 1$
Solve for the derivative of each function.

$\frac{du}{dx} = 2x \;\; and \;\; \frac{dv}{dx} = 6$

Then use the quotient rule

$$\frac{dy}{dx} = \frac{v.\frac{du}{dx} - u.\frac{du}{dx}}{v^2} = \frac{(6x-1).2x-(x^2+3).6}{(6x-1)^2} = \frac{12x^2-2x-6x^2-18}{(6x-1)^2} = \frac{6x^2-2x-18}{(6x-1)^2}$$

Testing Understanding

3. Find the derivative of the following using chain rule.

a. $y = (x+3)^2$ **b.** $y = (3x^2+7)^{-2}$ **c.** $y = \sqrt[4]{(x^6 - \frac{1}{x})}$

2(x+3) **-12x(3x^3+7)^-3** **1/4(6x^5+1/x^2)(x^6-1/x)^(-3/4)**

4. Find the derivative of the following

a. $y = x(x^2+1)$ **b.** $y = (x^2+1)(2x-3)$ **c.** $y = (7x^2+8x+1)^3$

2x^2+(x^2+1) **(x^2+1)(2)+(2x-3)(2x)** **3(14x+8)(7x^2+8x+1)^2**

d. $y=\frac{x^4+3}{x+1}$ **e.** $y=(x^2+x)\sqrt{(2x^4-\frac{1}{x})}$ **f.** $y=x^3\left(4x^3-x+\frac{1}{x^2}\right)^{-2}$

d ((x+1)(4x^3)-(x^4+3))/(x+1)^2

e (x^2+x)1/2(8x^3+1/x^2)(2x^4-1/x)^(-1/2)+ (2x^4-1/x)^(1/2)+(2x+1)

f. x^3 (-2)(12x^2-1-2/x^3)(4x^3-x+1/x^2)^-3+3x^2(4x^3-x+1/x^2)^-2

g. $y=\frac{(x+3)^2}{\sqrt{x^2-1}}$ **h.** $y=\frac{(x-1)^2}{(x^2-9)^3}$ **i.** $y=\left(x^{-2}-\frac{3}{x}+x^{\frac{2}{3}}\right)^{1/4}\left(x^{-\frac{1}{2}}-x^7\right)^{-1/5}$

g. (x^2-1)^(1/2).2(x+3)-(x+3)^2. (1/2.2x.(x^2-1)^(-1/2) / (x^2-1)

e (x^2-9)^3 . 2(x-1)-(x-1)^2 .(3.2x*(x^2-9)^2) / (x^2-9)^6

f. 1/4(-2x^-3+3/x^2+2/3x^-1/3)(x^-2 -3/x+x^2/3)^(-3/4). (x^-1/2 -x^7)^(-1/5) +
(-1/5) (-1/2x^-3/2-7x^6)(x^-1/2 -x^7)^(-6/5) (x^-2 -3/x+x^2/3)^(1/4)

These questions are fairly difficult; tutors should go through these questions as a class!

Extension- ***Attach separate answer sheet and show all working out! You may use a calculator***
A population of red fungi is determined by the function $P(t)=t^3(2t^2-1)^2+2,\ where\ t\ is\ days.$

a. Find the initial population and initial growth (population per day)
P=2,0 Growth= t^2 (2t-1)(14t^2-3)

b. What is the population and growth on the 6th day?
1, 088, 858 and 198396

c. When will the population reach 10,000,000?
8.22 Days
d. When will the growth be 5,000,000 population per day?
11.687 Days

e. State the times when the growth is negative and positive.
negative 0<x<0.5 positive x>0.5
f. A certain species of herbivore eats at a rate of 2,000 fungi per day, given that the function of fungi population is not affected, when is an appropriate time to expose the fungi to the herbivore and why?
At least 3.5499 days, the rate of which fungi are producing are more than the herbivore is eating

NQT EDUCATION

YEAR 12

VCE MATHS METHODS
UNIT 3 TERM 2 WORKBOOK

WEEK 14

Differentiation:

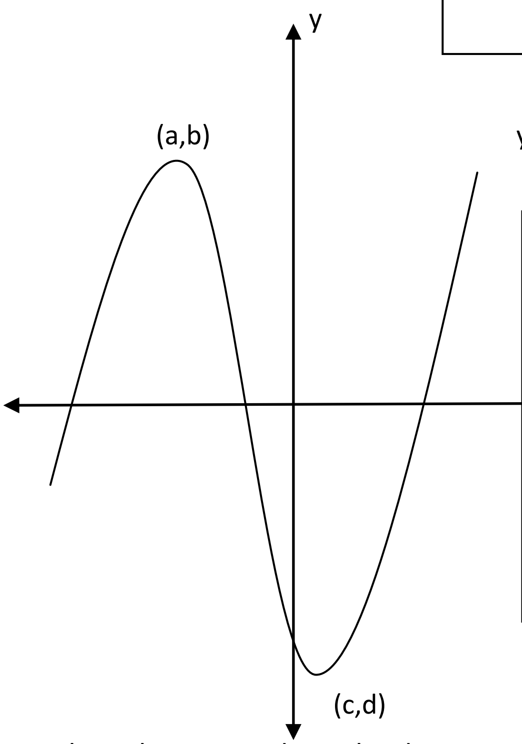
The graph of the gradient function
Differentiability
Differentiation of transcendental function

The graph of the gradient function

The graph of the gradient function can be deduced without the function. We know that curves ascending from left to right has a positive gradient and right to left has a negative gradient and lines which are parallel to the x axis has zero gradient.

Tutor should demonstrate the how to draw the dy/dx graph on the whiteboard

Here is an example



Gradients

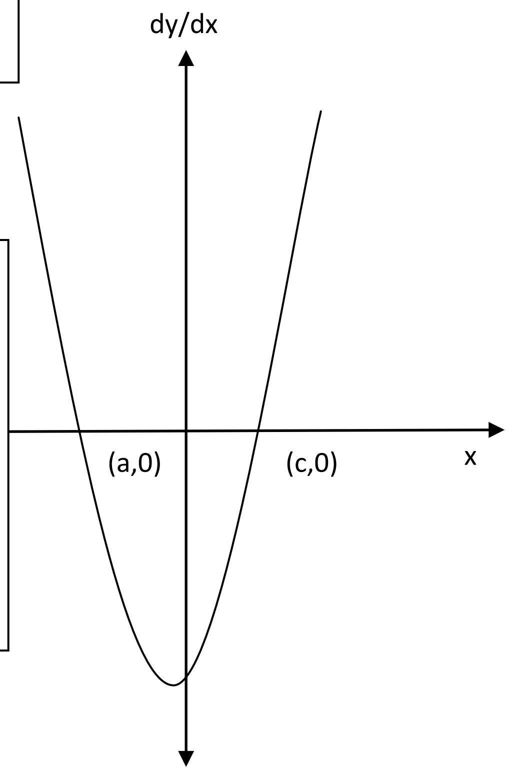
For $x < a$, $f'(x) > 0$

For $x = a$, $f'(x) = 0$

For $a < x < c$, $f'(x) < 0$

For $x = c$, $f'(x) = 0$

For $x > c$, $f'(x) > 0$



A simple technique to draw the derivative graph is:

Step 1: Locate all the zero gradients and label it on the derivative graph

In this case a and c are zero gradient, label the (a,0) and (c,0) onto the graph

Step 2: Mark the other gradients either by a negative sign or positive sign

x<a is +, a<x<c are -, x>c is +, now we know anything less than a and greater then c is on the positive y axis of the graph and between a and c is negative

Step 3: Find any change in slope, i.e. between a and c, the slope is changing!

a<x<c are changing! From a it increases steepness then decreases as it approaches c. You should set a point where the steepness is the largest then reverse the direction.

However even though curves appear continuous, it may not be so.

For example let's consider the graph

$$f: R \to R, f(x) = |x| = \begin{cases} x & if\ x \geq 0 \\ -x & if\ x < 0 \end{cases}$$

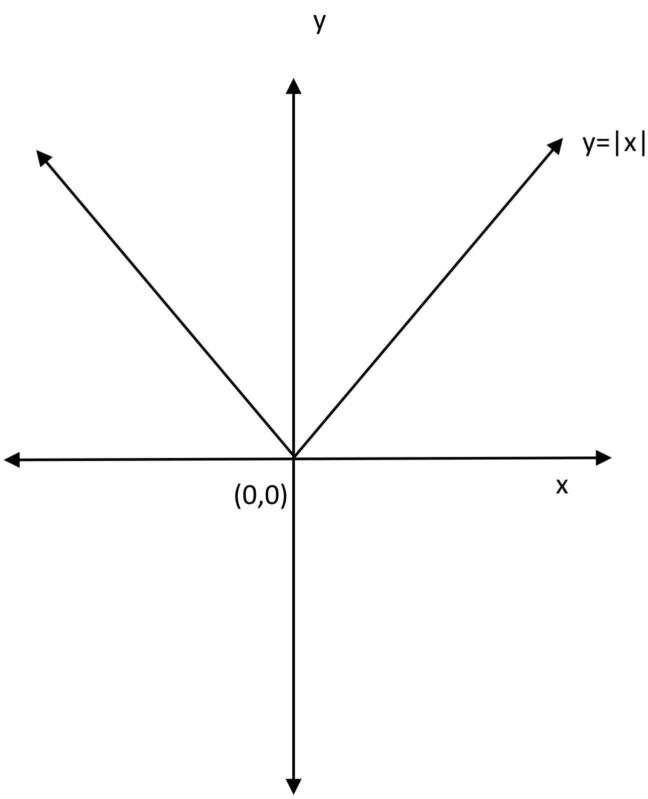
If we use first principle to find the derivative we get

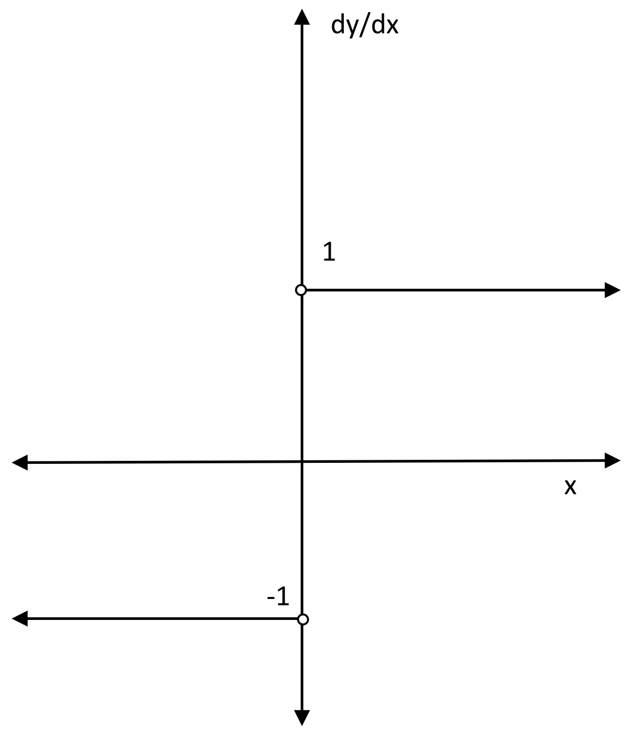
$$f'(x) = \begin{cases} 1 & if\ x > 0 \\ -1 & if\ x < 0 \end{cases}$$

At x = 0 is not unique and that the derivative at x=0 doesn't exist, in order words f is not differentiable at x = 0

Any sharp turns, fill or non fill end dots are not differentiable!

So the graphs for the function and its derivative is





Differentiability

If a function is differentiable at x, then it is also continuous at x. However it is not true if it is the other way around, which was demonstrated by the example before.

For hybrid functions, we can test if it's differentiable at $\boldsymbol{x = a}$ ***1. Whether the function is continuous at a*** ***2. Whether the*** $\boldsymbol{lim_{x\to a^-} f'(x) = lim_{x\to a^+} f'(x)}$ ***exists.***

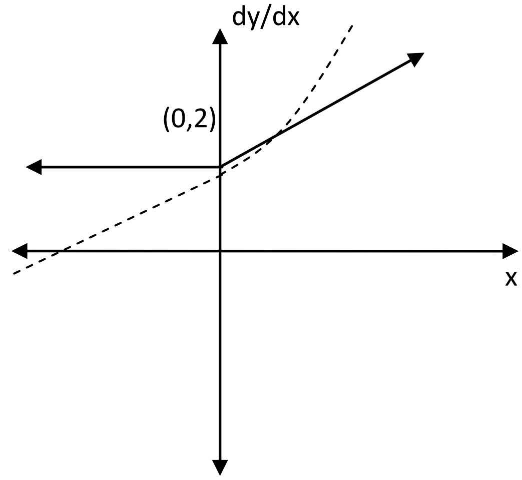
Here are examples:

1. Find the derivative and sketch the graph where $f(x) = \begin{cases} x^2 + 2x + 1 & if\ x \geq 0 \\ 2x + 1 & if\ x < 0 \end{cases}$

Solution

$$f'(x) = \begin{cases} 2x + 2 & if\ x \geq 0 \\ 2 & if\ x < 0 \end{cases}$$

In particular f'(0) is defined and is equal to 2, also f(0)=1. This means that the function is continuous and smooth as it joins at (0,1)

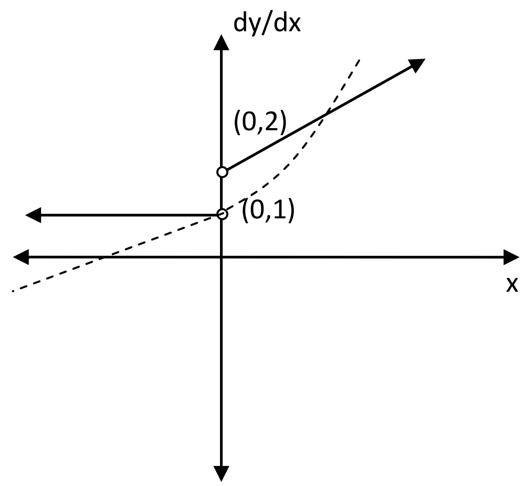


2. Find the derivative and sketch the graph where $f(x) = \begin{cases} x^2 + 2x + 1 & if\ x \geq 0 \\ x + 1 & if\ x < 0 \end{cases}$

Solution

$$f'(x) = \begin{cases} 2x + 2 & if\ x \geq 0 \\ 1 & if\ x < 0 \end{cases}$$

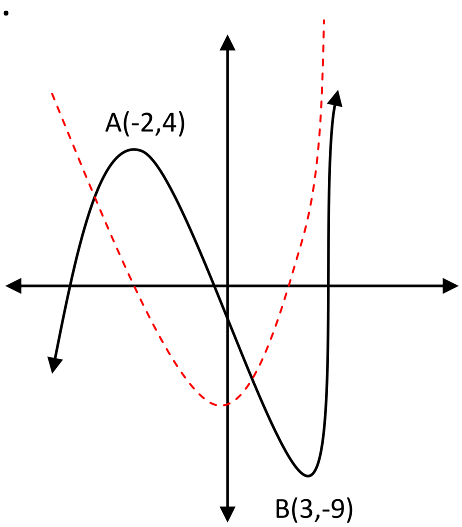
In this case f'(0) is not defined since the right and left limit are not equal. Therefore the function is differentiable everywhere except 0; $R\backslash\{0\}$.



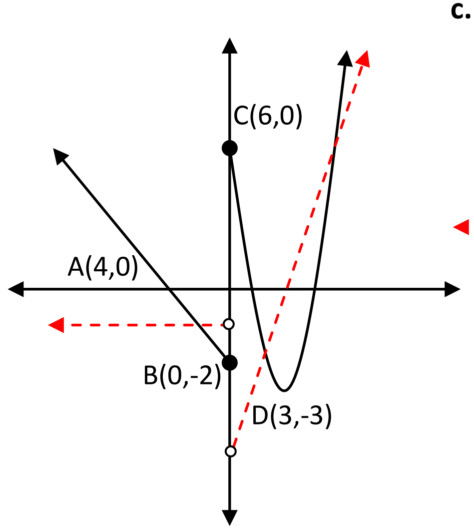
Testing Understanding

1. Sketch the derivative of the following graph on the same graph

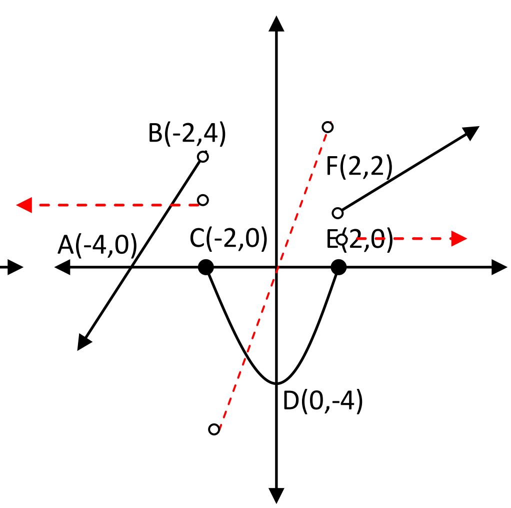
a.



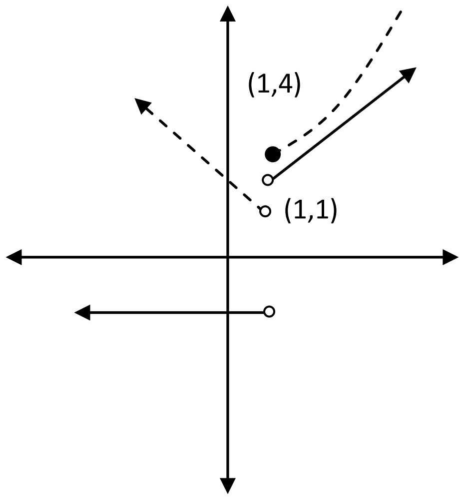
b.



c.



2. For the function with rule $f(x) = \begin{cases} x^2 + 2x + 1 & if\ x \geq 1 \\ -2x + 3 & if\ x < 1 \end{cases}$, state the set of values for which the derivative is defined, find $f'(x)$ for this set of values and sketch the graph of $f(x) and\ f'(x)$



f'(x) is defined for R\{1}

$$f(x) = \begin{cases} 2x + 2 & if\ x > 1 \\ -2x & if\ x < 1 \end{cases}$$

Differentiation of Transcendental Function

Now you know how to solve the derivatives for polynomial functions. However there are still exponential, logarithmic and even trigonometric functions to be dealt with.

Tutor should go through this section with the students!

Exponential Function

For exponential function $f(x) = e^x$ we can use the limit theorem to find the derivative

$$f'(x) = \lim_{h \to 0} \frac{f(x+h) - f(x)}{h} = \lim_{h \to 0} \frac{e^{x+h} - e^x}{h} = \lim_{h \to 0} e^x \frac{e^h - 1}{h} = e^x \lim_{h \to 0} \left(\frac{e^h - 1}{h} \right) = e^x \times 1 = e^x$$

Given $f(x) = e^x, then\ f'(x) = e^x$

In a case $y = ae^{bx+c}$ and using chain rule

We let $u = bx + c$

$$\frac{du}{dx} = b\ and\ let\ y = ae^u$$

$$\frac{dy}{dx} = \frac{dy}{du} \times \frac{du}{dx} = ae^u \times b = abe^{bx+c}$$

Therefore

$$f(x) = ae^{bx+c}\ \ and\ f'(x) = abe^{bx+c}$$

In general case with implied chain rule

$$\boldsymbol{f(x) = ae^{g(x)} \ then \ f(x) = ag'^{(x)e^{g(x)}}}$$

Example

Find the derivative of $y = \frac{2}{e^{x^2+x}}$

Solution

Change the numerator to the denominator by using exponential rules.

$y = 2e^{-x^2-x}$

Use Chain Rule or the implicit rule from above

$$g(x) = -x^2 - x\,, so\ g'(x) = -2x - 1$$

Therefore

$$f'(x) = -2(2x+1)e^{-x^2-x}$$

Logarithmic Function

Given that $f(x) = \log_e x$, we can investigate the gradient function by using our graphics calculator. Soon we can deduce that $\frac{d(\log_e x)}{dx} = f'(x) = \frac{1}{x}$

Let's try through some examples

Find the derivative of the following function

a. $y = \log_e(2-5x)$

Solution

Using Chain Rule we substitute

$u = 2 - 5x$ so that $\frac{du}{dx} = -5$ and that

$y = \log_e u$ and that $\frac{dy}{du} = \frac{1}{u}$

$$\frac{dy}{dx} = \frac{dy}{du} \times \frac{du}{dx} = \left(\frac{1}{u} \times (-5)\right)$$

$$= -\frac{5}{2-5x} = \frac{5}{5x-2}$$

b. $y = 2\log_e(x^2 - x)$

Solution

Using Chain Rule we substitute

$u = x^2 - x$ so that $\frac{du}{dx} = 2x - 1$ and that

$y = 2\log_e u$ and that $\frac{dy}{du} = \frac{2}{u}$

$$\frac{dy}{dx} = \frac{dy}{du} \times \frac{du}{dx} = \frac{2(2x-)1}{u}$$

$$= \frac{2(2x-1)}{x^2-x} = \frac{4x-2}{x^2-x}$$

So from the example we can deduce

Note that the derivative of $f(x) = \log_e kx$ is $f'(x) = \frac{1}{x}$.

Do you know why?

Given that $y = alog_e(bx + c)$, then $\frac{dy}{dx} = \frac{ab}{bx+c}$

Where a is a constant and b and c are coefficients

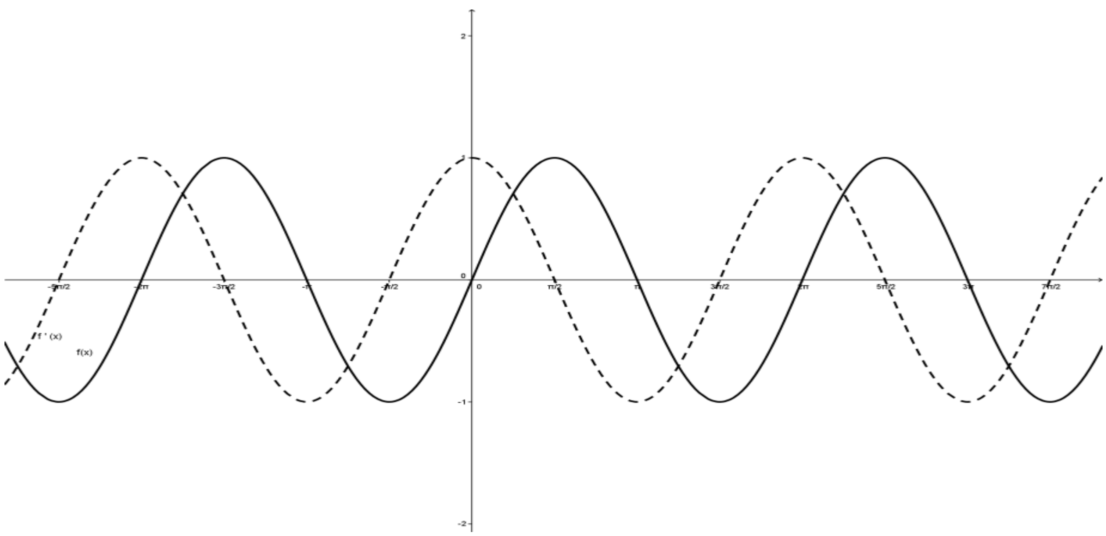
In general using the implied chain rule:

$f(x) = alog_e\big(g(x)\big)$, then $f'(x) = ag'(x)\log_e(g(x))$ Where a is a constant

Trigonometric Function

The derivative of sine and cosine can be observed on the graphics calculator.

You will see that the derivative of the sine is simply the resemblance of cosine!



Let's investigate by doing some examples

Differentiate $y = -sin(2x - 5)$

Solution

Using Chain Rule we substitute

$u = 2x - 5$ so that $\frac{du}{dx} = 2$ and that $y = -\sin u$ and that $\frac{dy}{du} = -\cos u$

$$\frac{dy}{dx} = \frac{dy}{du} \times \frac{du}{dx} = -\cos u \times 2$$

$$= -\cos(2x - 5) \times 2 = -2\cos(2x - 5)$$

Therefore

$$f(x) = a\sin(bx + c), then \;\; f'(x) = abcos(bx + c)$$

where a is a constant.

In general

$$f(x) = a\sin(g(x)), then \;\; f'(x) = ag'(x)cos(g(x))$$

where a is a constant.

For the other trigonometric functions like cosine, by using the same method and chain rule we can deduce that

$$y = \cos x \;\; , then \; \frac{dy}{dx} = -\sin x$$

$$y = \mathrm{a}\cos(bx + c) \;\; , then \; \frac{dy}{dx} = -\,\mathrm{a}\sin(bx + c)$$

In general

$$y = \mathrm{a}g'(x)\cos(g(x)) \;\; , then \; \frac{dy}{dx} = -\,\mathrm{a}g'(x)\sin(g(x))$$

Example

Find the derivative, given that $f(x) = 4\cos(x^2 + x - 1)$

Solution

Substitution by letting $u = x^2 + x - 1$ so that $\frac{du}{dx} = 2x + 1$

Such that $y = 4\cos(u)$, then $\frac{dy}{du} = -4\sin u$

$$\frac{dy}{dx} = \frac{dy}{du} \times \frac{du}{dx} = -4\sin u \times (2x + 1)$$

$$= -4\sin(x^2 + x - 1) \times (2x + 1) = -4(2x + 1)\sin(x^2 + x - 1)$$

Find the derivative, given that $f(x) = \sin^3 x$

Solution

Substitution by letting $u = \sin x$ so that $\frac{du}{dx} = \cos x$

Such that $y = u^3$, then $\frac{dy}{du} = 3u^2$

$$\frac{dy}{dx} = \frac{dy}{du} \times \frac{du}{dx} = 3u^2 \times \cos x$$

$$= 3(sin\ x)^2 \times \cos x = 3\cos x \sin^2 x$$

The derivative of tangent, tan x

The differentiating of tangent requires chain rule and quotient rule that you have learnt previously.

If $y = \tan x$, $then\ \frac{dy}{dx} = \frac{1}{\cos^2 x}$ since $\cos x = \frac{1}{\sec x}$ then

Given $y = \tan x$, $then\ \frac{dy}{dx} = \frac{1}{\cos^2 x} = \sec^2 x$

In general

$$f(x) = \tan(g(x)),\ \ then\ f'(x) = \frac{g'(x)}{\cos^2 g(x)} = g'(x)\sec^2 g(x)$$

Example

Find the derivative of function $f(x) = \sqrt{\tan x}$

Solution

Substitution by letting $u = \tan x$ so that $\frac{du}{dx} = \sec^2 x$

Such that $y = \sqrt{u}$, then $\frac{dy}{du} = \frac{1}{2\sqrt{u}}$

$$\frac{dy}{dx} = \frac{dy}{du} \times \frac{du}{dx} = \frac{1}{2\sqrt{u}} \times \sec^2 x$$

$$= \frac{1}{2\sqrt{\tan x}} \times \sec^2 x = \frac{\sec^2 x}{2\sqrt{\tan x}}$$

Note: The reciprocal of the trigonometric and its inverse functions are

$cosec\ x = \frac{1}{\sin x}$ $\quad$ $cosec^{-1}\ x = \frac{1}{\sin^{-1} x}$

$sec\ x = \frac{1}{\cos x}$ $\quad$ $sec^{-1}\ x = \frac{1}{\cos^{-1} x}$

$cot\ x = \frac{1}{\tan x}$ $\quad$ $\cot^{-1} x = \frac{1}{\tan^{-1} x}$

When solving the inverse for the reciprocal trigonometric function you should convert it to sine, cosine and tan for simplicity!

Testing Understanding

3. Solve for the derivative for the following equation (Hint you can solve some question by using some logarithm rules)

a. $y = 3e^{2x}$

$6e^{2x}$

b. $y = -e^{-6x+1}$

$6e^{-6x+1}$

c. $y = \frac{1}{3}e^{x^3+1}$

$\frac{1}{3}(3x^2)e^{x^3+1}$

d. $y = \frac{1}{e^{x^2-x^{-1}}}$

$-(2x + x^{-2})e^{-x^2+x^{-1}}$

e. $y = \log_e 2x$

$\frac{1}{x}$

f. $y = \log_e x^2$

$\frac{2x}{x^2} = \frac{2}{x}$

g. $y = -\log_e(3x^5 + x)$

$-\frac{15x^4+1}{3x^5+x}$

h. $y = -\log_e(\frac{1}{x+1})$

$\frac{1}{(x+1)}$

i. $y = \log_e(x^{-2} + \frac{1}{x})$

$(-2x^{-3} - \frac{1}{x^2})/(x^{-2} + \frac{1}{x})$

j. $y = \sin(4x + 1)$

$4\cos(4x + 1)$

k. $y = \cos(x^2 - 1)$

$-(2x)\sin(x^2 - 1)$

l. $y = \tan(4x^2 + 2)$

$(8x)\sec^2(4x^2 + 2)$

The following questions are quite difficult. These questions will test your knowledge of using the differentiation rules as well as rule from previous topics.

Tutors should go through the solution and working out in class.

Differentiate:

a. $e^{\sin x}$

$\boldsymbol{\cos x\, e^{\sin x}}$

b. $\frac{e^{2\sin(x^3)}}{e^{\cos(x^4)}}$

$\boldsymbol{(3x^2\cos(x^3) + 4x^3\sin x^4)e^{2\sin(x^3)+\cos(x^4)}}$

c. $\log_e(\tan(x)^2)$

$\boldsymbol{\frac{2\sec^2 x}{\tan x}}$

d. $y = \cos^3(x^2 + e^{\frac{1}{x}})$

$3\left(2x - \frac{1}{x^2}e^{\frac{1}{x}}\right)\sin\left(x^2 + e^{\frac{1}{x}}\right)\cos^2\left(x^2 + e^{\frac{1}{x}}\right)$

e. $y = \log_e(x^3\sin(x))$

$(3x^2\sin x + x^3\cos x)/x^3\sin x$

c. $y = \sin(x^2)\cos\left(\frac{1}{x}\right)$

$x^2\cos(x^2)\cos\left(\frac{1}{x}\right) - \frac{1}{x^2}\cos\left(\frac{1}{x}\right)\sin(x^2)$

NQT EDUCATION

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VCE MATHS METHODS
UNIT 3 TERM 2 WORKBOOK

WEEK 15

Differentiation:

Application of differentiation
Stationary points
Linear approximation
Maxima and minima problems

Application of differentiation

The derivative of a function gives you a new function that measure the gradient of the curve. With the gradient function, we can solve for the tangent for any given point. Thus we can make a tangent and normal linear equation once solved for derivative.

Example

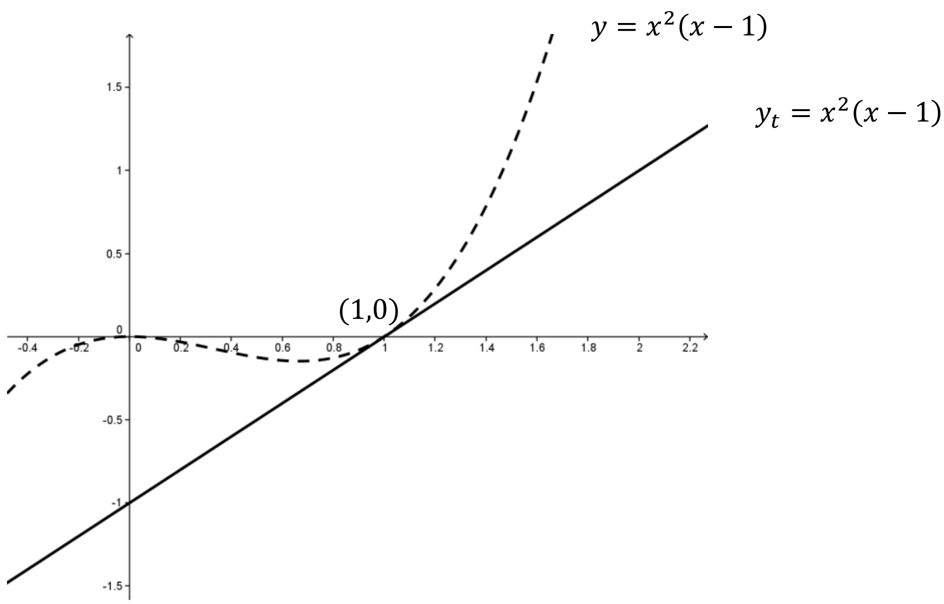
Consider $y = x^2(x-1)$ at point $x = 1$, find the equation of the tangent at that point.

Solution:

Differentiate the function$y = x^3 - x^2$, $\frac{dy}{dx} = 3x^2 - 2x = x(3x-2)$

When $x = 1, y = 0$ and $\frac{dy}{dx} = 1$ for a tangent linear equation consider $y - y_1 = m(x - x_1)$

Thus, tangent equation is $y = x - 1$



For a normal curve we use the rule $m_1 m_2 = -1$

In this case the normal gradient is $m_2 = -1$ and the equation of the line will be $y = -x + 1$

You can also solve the tangent line by using a graphic calculator:

- Graph function and ensure the point of interest is visible in the screen
- In the draw menu select the tangent option.

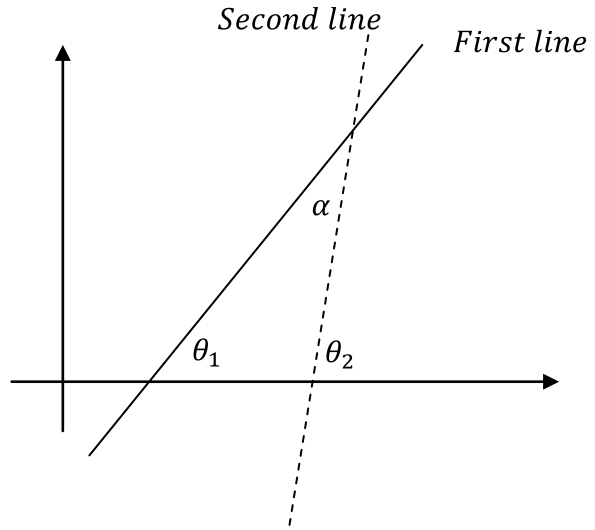
Angles between two straight lines

When two straight lines makes angles θ_1 *and* θ_2 with the positive direction of the axis. Then

gradient of first line $= m_1 = \tan\ \theta_1$

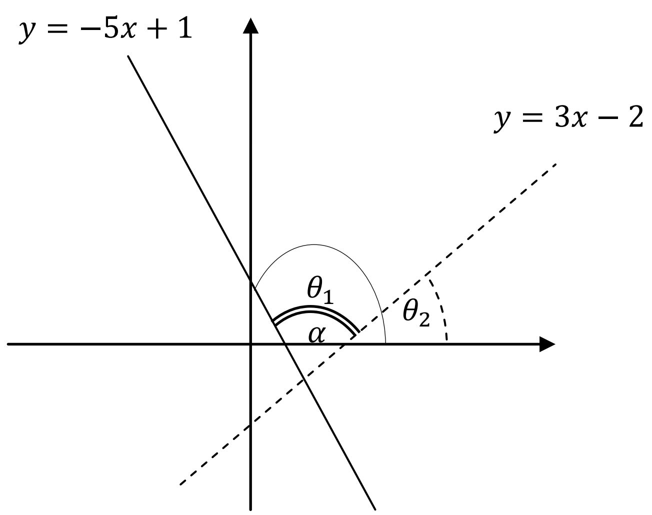
gradient of first line $= m_2 = \tan\ \theta_2$

Note that $\tan\alpha = \tan(\theta_1 - \theta_2) = \frac{\tan\ \theta_1 - \tan\ \theta_2}{1+\tan\ \theta_1 \tan\ \theta_2}$



Example

Two straight lines have equation $y = -5x + 1$ and $y = 3x - 2$ find the angle between both lines.



Solution 1

$\tan\ \theta_1 = -5$ $\qquad$ $\tan\ \theta_2 = 3$

$\theta_1 = 180 - 78.69 = 101.31$ $\qquad$ $\theta_1 = 71.57$

Angle between two lines is

$101.31 - 71.57 = 29.74°$

Solution 2

$m_1 = -5$ $\qquad$ $m_2 = 3$

If α is the acute angle between the two lines

$$\tan\alpha = \frac{m_1 - m_2}{1 + m_1 m_2} = \frac{-5-3}{1-15} = -\frac{-8}{-14} = \frac{4}{7}$$

$$\beta = \tan^{-1}\frac{4}{7} = 29.74$$

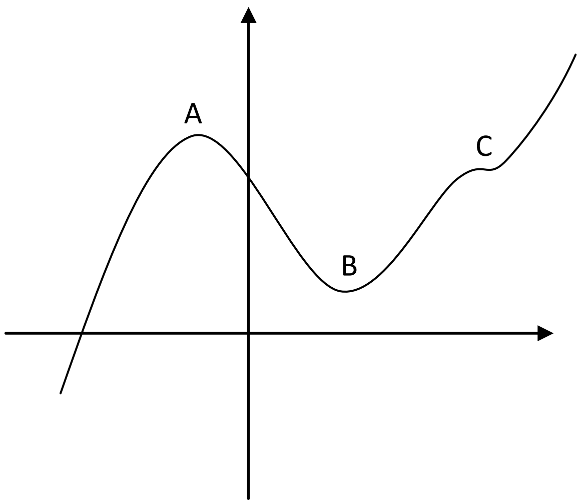
Be careful! If you have a negative gradient change it to positive and subtract it from 180° to obtain the degree from the right side of the x axis.

It is always good to draw the lines on the graph!

Stationary Points

We know that a stationary point the gradient is zero but there are different types of stationary points

Consider this graph below



A - local maximum point

- immediate left is positive gradient
- immediate right is negative gradient

B - local minimum point

- immediate left is negative gradient
- immediate right is positive gradient

C - inflexion point

- Either both direction is negative or positive gradient

Note that A and B are turning point because the gradient changes direction.

Example

Consider the function $f: R \rightarrow R, f(x) = 3x^3 - 4x + 1$, state the stationary points and their nature.

Solution

Stationary point where $f'(x) = 0 \; such \; that \; f'(x) = 9x^2 - 4 = 0, thus \; when \; x = \pm\frac{2}{3}$

The stationary points are $\left(-\frac{2}{3}, \frac{25}{9}\right) and \left(\frac{2}{3}, -\frac{7}{9}\right)$

Now we need to figure the gradient of each

$$\left\{x: x < \frac{2}{3}\right\}, \left\{x: -\frac{2}{3} < x < \frac{2}{3}\right\} and \left\{x: x > \frac{2}{3}\right\}$$

By substituting any values in each set.

Lets use

$f'(-1) = 5 > 0$

$f'(0) = -4 < 0$

$f'(1) = 5 > 0$

x		$-\frac{2}{3}$		$\frac{2}{3}$	
$f'(x)$	+	0	-	0	+
Shape of f	/	___	\	___	/

Local maximum at $\left(-\frac{2}{3}, \frac{25}{9}\right)$ and local minimum at $\left(\frac{2}{3}, -\frac{7}{9}\right)$

Linear approximation

Sometimes we are unable to solve for the exact gradient but we can use the linear approximation technique determine the approximate increase.

Consider

$f'(x) = \lim_{h\to 0} \frac{f(x+h)-f(x)}{h}$, when $h\ is\ very\ small\ then\ \ f'(x) \approx \frac{f(x+h)-f(x)}{h}$

So once rearranged

$f(x+h) \approx hf'(x) + f(x)$ to solve for the next point

And $f(x+h) - f(x) = hf'(x)$ is the approximate increase in $f(x)$ which you will denote as $\delta y \approx \frac{dy}{dx}\delta x$

δ in this situation means small change.

In practice, we often required to solve for the percentage change in quantity

$100\left(\frac{f(a+h)-f(a)}{f(a)}\right)$ given that $f(a) \neq 0$

Using the results of $f(a+h) \approx hf'(a) + f(a)$

he percentage change is approximately equal to $\frac{100hf'(a)}{f(a)}$

Example

Given that $y = 10 - \frac{5}{x}$ and that the value of y increase from 5 by a small amount $\frac{p}{10}$, find in terms of *p:*

a. *the approximate change in x.*

b. *the corresponding percentage change in x*

Solution

a. the derivative is $\frac{dy}{dx} = \frac{5}{x^2}$ and when

$y = 5, x = 1$ therefore

$\delta y \approx \frac{5}{1}\delta x$ Given that $\delta y = \frac{p}{10}$ then

$\delta x = \frac{1}{5}\delta y = \frac{p}{50}$

Solution

b. percentage change in x

$$\approx \frac{\delta x}{x} \times 100\%$$
$$= \left(\frac{p}{50} \times 100\right)\%$$
$$= 2p\%$$

Testing Understanding

1. Find the equation of the tangent and normal to the curve $y = x^2 + x - 1$ at the point (1,1)

$$\mathbf{y_t = 3x - 2}$$
$$\mathbf{y_n = -\frac{1}{3}x + \frac{4}{3}}$$

2. Find the equation of the tangent and normal to the curve $y = x^3 - 2x - 4$ at the point where it cuts the y axis.

$$\mathbf{y_t = -2x - 4}$$
$$\mathbf{y_n = \frac{1}{2}x - 4}$$

3. The graphs $y = x^2 - 2x$ and $y = x$ intersect at the point (3,3). Find the acute angle lying between them at this point.

30.96

4. Differentiate $\frac{1}{\sqrt{x}}$ with respect to x and use the result to find an approximate value for $\frac{1}{\sqrt{100.2}}$

0.0999

Maxima and minima problems

You will experience many practical problem requires minimisation or maximisation. We can use differential calculus to solve these type of problem. Most questions you will find very difficult, the only way to prepare for these types of questions is by practicing and attempting it.

Example

A farmer has sufficient fencing to make a rectangular pen of perimeter 300 metres. What dimension will give an enclosure of maximum area?

Solution

The perimeter $300 = 2(L + W)$ transpose that $W = 150 - L$

The area of a rectangular shape $Area = L \times W$

$$Area = L \times (150 - L) = 150L - L^2$$

Maximum value is when derivative $\frac{dA}{dx} = 0$

$$\frac{dA}{dx} = 150 - 2L = 0, \quad so\ L = 75\ thus\ W = 75$$

When the width and length is 75 metres (square) the maximum area is 5625 metres square

Example

A phone company has 500 subscribers who are paying $20 per month. They can get 100 more subscribers for each $0.50 decrease in the monthly fee. What rate will yield maximum revenue and what will this revenue be?

Solution

Number of subscribers $= 500 + 100\frac{(20-x)}{0.5}$

The revenue $R = x\big(500 + 200(20-x)\big) = 500x + 4000x - 200x^2 = 4500x - 200x^2$

So the derivative of the revenue $\frac{dR}{dx} = 4500 - 400x$*, minima when derivative is 0*

Therefore $x = 11.25$ *and the total revenue is $25 312.5*

Testing Understanding

5. Find the maximum area of a rectangular piece of ground that can be enclosed by 50 m of fencing.

625

6. Find the point on the parabola $y = x^2$that is closest to point $(3,0)$

(1,0) (use distance formula)

7. The number flies, N, in a certain area depends on the rainfall, x, measured in mm is approximately given by

$$N(x) = 20(1 - 16x + 8x^2 - x^3)$$

Solve for the rainfall that will produce the maximum and minimum number of flies.

(1.333,-169.63)(4,20)

8. A cuboid has a total surface area of 150 cm^2 with a square base of side $x\ cm$

a. Show that the height, h cm, of the cuboid is given by $h = \frac{75-x^2}{2x}$

b. Express the volume of the cuboid in terms of x

c. Hence determine its maximum volume as x varies

b.$\frac{x^2(75-x^2)}{2x} = \frac{x}{2}(75 - x^2)$

c.125

NQT EDUCATION

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VCE MATHS METHODS
UNIT 3 TERM 2 WORKBOOK

WEEK 16

Differentiation:

Rate of change
Related rates of change
Extension Questions

Rate of change

The rate of change or derivative of a function has been used to solve for the gradient of the corresponding curve. Thus, the process of differentiation is used to solve problems involving rate of changes, such as speed, leakage and growth.

The $\frac{dy}{dx}$, derivative of y with respect to x, gives the rate of change of x, $y$with respect to x.

If $\frac{dy}{dx} > 0$ the change is an increase in the value of y corresponding to an increase in x.

If $\frac{dy}{dx} < 0$ the change is a decrease in the value of y corresponding to an increase in x.

Kinematics

Displacement, velocity and acceleration are measurement of a body moving in a straight line. Displacement was specific with respect to a reference point, O, on that line.

For velocity $(v\ m/s)$ $v = \frac{ds}{dt}$

For acceleration $(a\ m/s^2)$ $a = \frac{d^2s}{dt^2} = \frac{dv}{dt}$

Note that $\frac{d^2s}{dt^2}$ means the second derivative of the displacement. It simply means differentiate the 's' function twice.

Be careful! Velocity is the measurement of displacement over time while speed is the measurement of distance over time. You must know how to distinguish the difference between distance, displacement, velocity and speed!

Example

An atomic particle is moving along straight line so that its distance x cm from a point at time t seconds is given by $x = -t^3 - 5t^2 + 8t$.

a. Find at what times and in what positions the point will have zero velocity

b. its acceleration at those instants

c. its velocity when acceleration is zero

Solution

Firstly solve for the velocity and acceleration with the given distance equation.

displacement $x = -t^3 - 5t^2 + 8t.$
velocity $v = \frac{dx}{dt} = -3t^2 - 10t + 8.$

acceleration $a = \frac{dv}{dt} = \frac{d^2x}{dt^2} = -6t - 10.$

a. Solve for the velocity equation when $v = 0$

$$0 = -3t^2 - 8t + 8 = -(3t^2 + 8t - 8) = -(3t - 2)(t + 4)$$

Using null theorem $t = \frac{2}{3}$ *and* $t = -4$ (note that this means 4 seconds before the reference time)

Substitute this value into the displacement

$x = -\left(\frac{2}{3}\right)^3 - 5\left(\frac{2}{3}\right)^2 + 8\left(\frac{2}{3}\right) = \frac{76}{27}\ cm$ $\qquad$ $x = -(-4)^3 - 5(-4)^2 + 8(-4) = -48\ cm$

b. The solution and information obtain from previous questions we can find the acceleration for the instant.

$a = -6\left(\frac{2}{3}\right) - 10 = -14\ cm/s^2$ $\qquad$ $a = -6(-4) - 10 = 14\ cm/s^2$

c. Similar to question **a** but instead we will have to solve for the acceleration using null theorem when acceleration is zero.

$$0 = -6t - 10$$
$$t = -\frac{5}{3}$$

Substitute the value into the velocity function $v = \frac{dx}{dt} = -3\left(-\frac{5}{3}\right)^2 - 10\left(-\frac{5}{3}\right) + 8 = -17$

Technically, acceleration is zero when $-\frac{5}{3}$ seconds prior to the time frame reference. Be careful with the wording of the questions, you may find that the question requires only one solution even though there are two or more solution

Example

The throat of the balloon is been released. Its volume $V\ cm^3$ at time t seconds $V = 500 - 10t - \frac{2}{3}t^2$, when $t > 0$. Find the rate of change of volume after 10 and 30 seconds and how long the model could be valid.

Solution

Firstly, solve for the derivative of the function.

$\frac{dV}{dt} = -10 - \frac{4}{3}t$

Then substitute the time values $t = 10\ and\ 30$

$\frac{dV(10)}{dt} = -10 - \frac{40}{3} = -23\frac{1}{3}$

$\frac{dV(30)}{dt} = -10 - \frac{120}{3} = -50$

At time 10 seconds the volume is decreasing at a rate of $23\frac{1}{3}cm^3 per\ second$ and at 30 seconds the volume is decreasing at a rate of $50cm^3 per\ second$.

The model is only valid if volume is positive or zero and cannot be negative.

Use any quadratic method to solve the solution for $time, t$. Consider the quadratic formula

$$t = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}$$

$The\ quadratic\ formula\ for\ V = 500 - 10t - \frac{2}{3}t^2$

$$t = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a} = \frac{\left((10) \pm \sqrt{100 - 4 \times \frac{-2}{3} \times 500}\right)}{-\frac{4}{3}} = \frac{\left(10 \pm \sqrt{\frac{4300}{3}}\right)(3)}{4} = \frac{30 + 30\sqrt{\frac{43}{3}}}{4}$$

Checking Understanding

1. Express the following in symbols

a. the rate of change of volume (V) with respect to time (t) dv/dt

b. the rate of change of area (A) of a circle with respect to radius (r) da/dr

c. the rate of change of volume (V) of water, with respect of depth of water (h) dv/dh

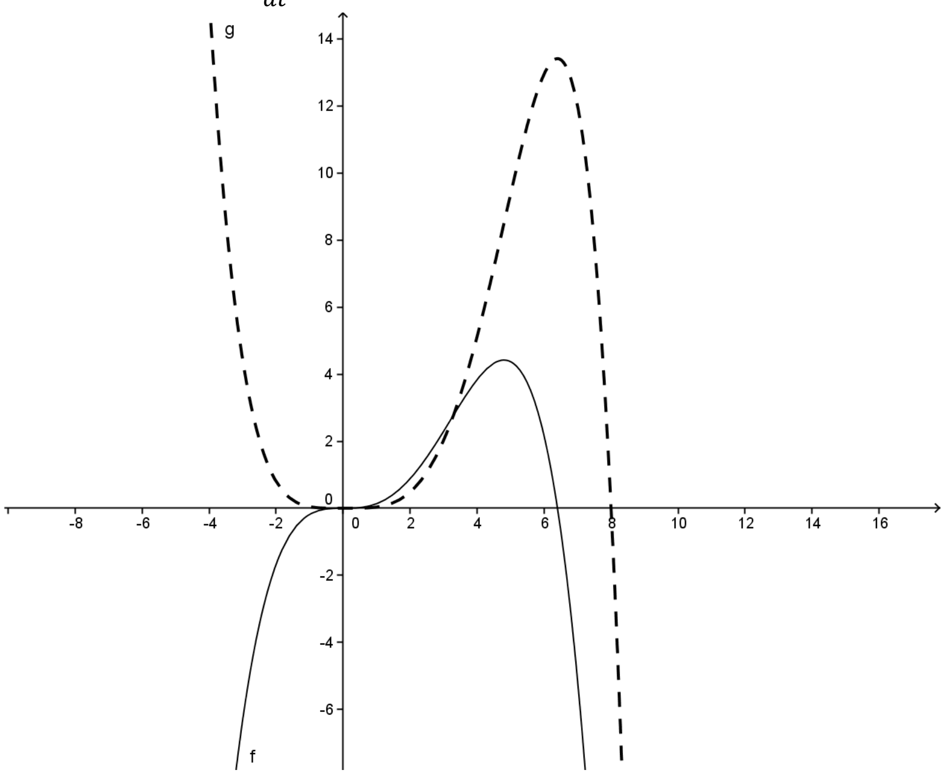
2. A filter allows 500mL of water to flow into a bucket in 20 minutes. The volume which has flowed into the bucket t minutes from the start is given by:

$$V(t) = \frac{1}{100}\left(4t^4 - \frac{t^5}{2}\right) \qquad for\ \ 0 \le t \le 8$$

a. At what rate is the water flowing into the bucket at t minutes?

1/100(16t^3-5/2*t^4)

b. Sketch the graph $\frac{dV}{dt}$ against $t\ for\ \ 0 \le t \le 8$.



c. When is the rate of flow greatest? 6.4

3. A particle moves in a straight line such that its position $x\ cm$ from a point O at time t seconds is given by $x = \sqrt{3t^2 + 1}$

a. Find the velocity as a function of t

v=(6t)/(3t^2+1)^(1/2)

b. Find the acceleration as a function of t

(6)/(3t^2+1)^(1/2) -(18t)/(3t^2+1)^(3/2)

c. Solve for the acceleration when $t = 2$

0.89605

4. A water reservoir is being emptied, and the quantity of water $V\ m^3$, remaining in the reservoir t days after it starts to empty is given by $V(t) = 1000(100 - 3t)^2$

a. At what rate is the water reservoir being emptied at time t and how long does it take to empty the reservoir?

-6000(100-3t) t=100/3

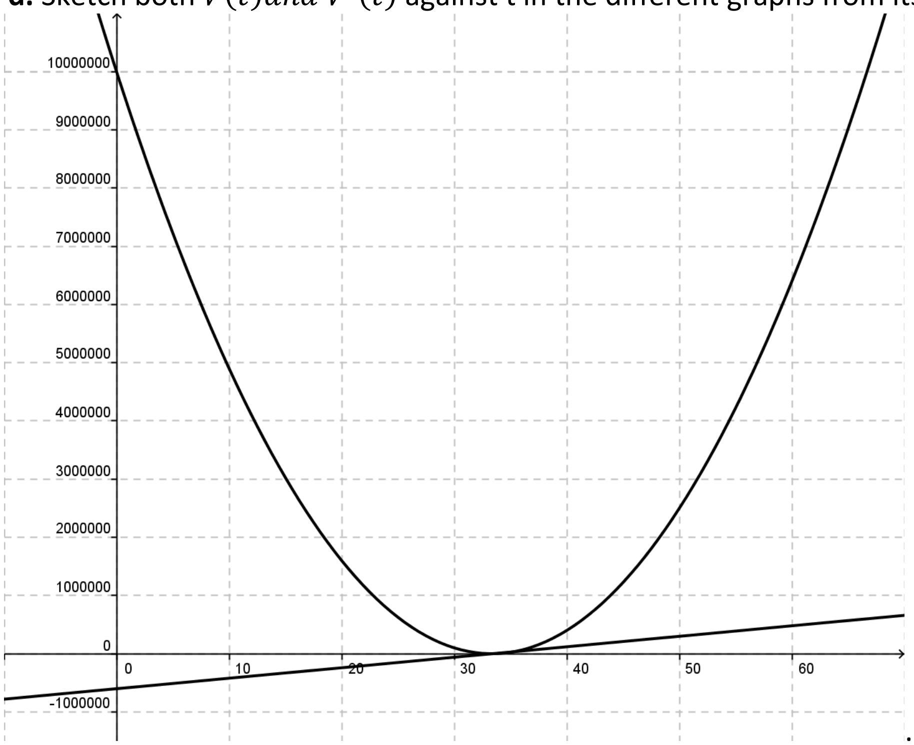
b. What is the volume of water in the reservoir when $t = 0$

10,000,000m^3

c. After what time is the reservoir being emptied at 20,000 m^3/day

34.44 day

d. Sketch both $V(t) and\ V'(t)$ against t in the different graphs from its initial time until first emptied



May need students to draw 2 separate graphs, as the scale is quite extreme.

Related rate of change

Most students will find related rate of change difficult to understand, but the technique to solve for the rate of change with respect to a variable is fairly simple. We use then chain rule to aid us. Generally the rate of change of one variable will lead to the rate of change of another related variable. Thus, this involves the notion of related rates of change.

$$For\ example$$

If we are required to solve for $\frac{dV}{dt}$ and was given $\frac{dr}{dt}$ and $V(r)$ function we need to combine this rates to find the solvable missing rate.

$$\frac{dV}{dt} = ? \times \frac{dr}{dt}$$

Knowing the chain rule expression we deduce the missing rate is $\frac{dV}{dr}$

Consider the situation of a container, which is a right circular cone, being filled from a tap.

Consider the possible variables in this situation.

- $t\ seconds$
- $V\ cm^3\ Volume$
- $height\ of\ teh\ water\ in\ the\ cone\ h\ cm$
- $the\ radius\ of\ the\ circular\ water\ surfrace\ is\ r\ \ cm$

Now consider the changes of $V, h\ and\ r$ with respect with time

$\frac{dV}{dt} = the\ rate\ of\ change\ of\ volume\ with\ respect\ to\ time$

$\frac{dh}{dt} = the\ rate\ of\ change\ of\ height\ with\ respect\ to\ time$

$\frac{dr}{dt} = the\ rate\ of\ change\ of\ radius\ with\ respect\ to\ time$

Examples

An inverted cone of base radius 5 cm and height is 15 cm is being filled such that its height is changing at 4 cm/min. Find the rate at which the volume is changing at height 1 cm.

Solution

First specify the information you are given in mathematic terms.

$$Volume\ of\ a\ cone\ ,\ V = \frac{1}{3}\pi r^2 h$$

$$r = 5\ cm\ and\ h = 15\ cm\ and\ \frac{dh}{dt} = 4$$

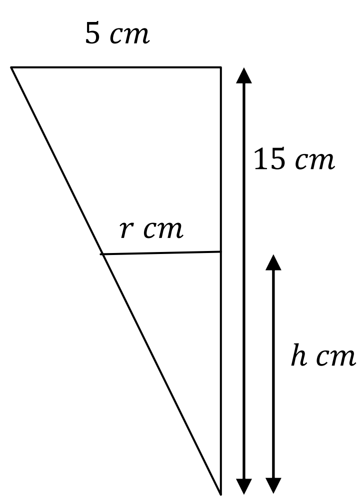
Secondly specify which rate of change is required and use chain rule to connect the terms

$\frac{\mathrm{dV}}{\mathrm{dt}}$ is needed and $\frac{dh}{dt}$ *is given*

So $\frac{dV}{dt} = ?\ \times \frac{dh}{dt} = \frac{dV}{dh} \times \frac{dh}{dt}$

We know $\frac{dh}{dt}$ but not $\frac{dV}{dh}$, however we can express the function V as a derivative

$V = \frac{1}{3}\pi r^2 h$ since the function consist two variables we need to express the other variable in terms of the other by ratios.



$$\frac{5}{15} = \frac{r}{h}$$
$$h = 3r$$

$$Thus, r = \frac{h}{3}$$

Substitute the terms into the V

$$V = \frac{1}{3}\pi\left(\frac{h}{3}\right)^2 h = \frac{1}{27}\pi h^3$$

So the derivative of V in terms of h is

$$\frac{dV}{dh} = \frac{1}{9}\pi h^2$$

$$Thus, \quad \frac{dV}{dt} = \frac{dh}{dt} \times \frac{dV}{dh} = 4\ \times \frac{1}{9}\pi h^2 = \frac{4\pi h^2}{9}$$

At height 1cm the volume is increasing at a rate of $\frac{4\pi}{9}\ cm^3/min$

Testing Understanding

1. A circular patch of oil spreads out from a point on a lake so that the area of the patch grows at a rate of $4cm^2/h$. At what rate is the radius increasing when the radius oil patch is $5\ cm$

=1/(2pi*r) * (4) = 4/2pi(5) =2/5pi

2. When the depth of liquid in a container is $x\ cm$ the volume is $x(x^2+20)cm^3$. Liquid is added to the container at a rate of $2cm^3/h$. Find the rate of change of the depth of liquid at the instant when $x = 5$.

1/(3x^2+20) *2= 1/(75+20) *2 =1/45

3. Variables $x\ and\ y$ are related by the equation $y = (x^2 - 5x - 1)^5$. Given that $x\ and\ y$are functions of t and that $\frac{dx}{dt} = 2$

a. Find $\frac{dy}{dt}\ in\ terms\ of\ x$

dy/dx=5(2x-5)(x^2-5x-1)^4
dy/dt=10(2x-5)(x^2-5x-1)^4

b. Find $\frac{dy}{dt}\ when\ x = 2$

=10(-1)(4-20-1)^4=-10(-17)^4

4. Consider an inverted cone container with diameter $24\ cm$ and height $36\ cm$ is filled with water. Water is leaked from the container at the rate of $15\ cm^3/sec$

a. Find the volume, $V\ cm$, of water in the container when the depth is $x\ cm$ 4/27 pi h^3

b. Find the rate at which the depth is decreasing when $x = 3$ =15/(4pi)

5. Sam and Emma decided to make a cone made from ice, with radius $6\ cm$ and height $8\ cm$. They decided to go and have lunch an finish the ice cone later. However it melt as the water is sunny and the radius is decreasing at a constant rate of $0.5cm/min$.

a. At what rate of the cone decreasing with respect to time when the radius is $2cm$

dV/dt= 4/3pi (2)^2 *0.5 = 2/(3 pi 4)=1/6(pi)

b. Assuming the rate of cone decreasing when radius is 2cm is considered, when will there be no cone left.

V=1/3 pi r^2 h =1/3 (pi 36*8)

V/ answer (a) =576 mins

NQT EDUCATION

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VCE MATHS METHODS
UNIT 3 TERM 2 WORKBOOK

WEEK 17

TEST ONE:

Reading Time: 10 Minutes
Writing Time: 45 Minutes
Short Answers **SCORE: /45**

You may use calculators.

Question 1

$If\ f(x) = x^2, x > 0, and\ g(x) = \sqrt{1-x},\ x \leq 1\ and\ if\ h(x) = (fg)(x) and\ k(x) = g\big(f(x)\big),$
$write\ h(x) and\ k(x) as\ general\ function.$

- **By Product and By Substitution**

$h: (0,1] \to R, h(x) = x^2\sqrt{1-x}$ **1 MARK**

$k: (0,1] \to R, k(x) = \sqrt{1-x^2}$ **1 MARK**

Note:
$g\ o\ f\ (x) \to ran\ f\ is\ subset\ of\ dom\ g$

/2

Question 2

$State\ the\ range\ of\ f: [-3, \infty) \to R, f(x) = x^2 - 4.$

$Can\ an\ inverse\ function\ exist\ within\ the\ given\ domain?$
$If\ not, restrict\ the\ domain\ such\ that\ inverse\ can\ exist.\ Hence\ sketch\ the\ function\ and\ its\ inverse.$

- **Sketch the graph in the calculator to observe the range and domain of function**
- **Inverse exist if it is a one to one function, usually restrict domain from turning point**

Range $[-4, \infty)$ **1 MARK**

$f^{-1}(x) = \sqrt{x+4}\ \ for\ x \geq 0$ **1 MARK**

SKETCH **4 MARK**

/6

Question 3

$If P(x) = x^3 - 2x^2 + x + 3\ is\ divded\ by\ x + 2, what\ is\ the\ remainder\ ?$

- **Either long division or substitution**

$Remainder = -15$ **1 MARK**
Working out 1 MARK

Note:
$P(-2) = Remainder$
$(x+2)\ factor\ form$
$x = -2\ \ root\ form$

/2

Question 4

$Given\ the\ function\ f(x) = 2x^2 + 5x - 5$

$a)\ Show\ f(x)\ in\ turning\ point\ form$

- **Complete the square**

$$f(x) = 2\left(x + \frac{5}{2}\right)^2 - \frac{35}{2}$$ **1 MARK**

WORKING OUT 1 MARK

Note:

$Take\ 2\ as\ a\ common\ factor\ out\ first!\ this\ is\ a\ common\ mistake$

$i.e\ 2\left(x^2 + \frac{5}{2}x - \frac{5}{2}\right)$

$b)\ Find\ the\ derivative\ of\ the\ function$

$$f'(x) = 4x + 5$$ **1 MARK**

$c)\ State\ the\ stationary\ point\ and\ determine\ why\ it\ is\ a\ maximum\ or\ minimum\ point.$

- **Sign nature rule to check if the gradient between the stationary point represents the maxima or minima**

Minimum

1 MARK

Note:

Construct a table

$\boldsymbol{x}$	-2	-5/2	-3
$\boldsymbol{f'(x)}$	-	0	+
SIGN	/	_	\

/5

Question 5

$Consider\ this\ function\ f(x) = \frac{2x-3}{x+1}.$

$a)\ State\ the\ range\ and\ domain\ of\ the\ function\ f(x)$

$$Domain = R\backslash\{-1\}$$ **1 MARK**

$$Range = R\backslash\{2\}$$ **1 MARK**

$b)\ Show\ that\ the\ function\ can\ be\ expressed\ as\ f(x) = a + \frac{b}{x+1}$

- **Either long division or by parts**

1 MARK for working out

$$2 + \frac{-5}{x+1}$$ **1 MARK for results**

$c)\ Find\ the\ derivative\ of\ f(x)\ and\ state\ the\ range\ and\ domain\ of\ f'(x)$

$f'(x) = \frac{5}{(x+1)^2}$ **1 MARK**

$Domain = R\backslash\{-1\}\ Range\ = R\backslash\{0\}$ **2 MARKS**

$d)\ Sketch\ the\ function\ f(x)\ and\ f'(x).$

4 MARKS

/11

Question 6

$If\ g(x) = x - 4 + \frac{x+1}{x-1}$

$a)\ Solve\ for\ g(x^2)$

$x^2 - 4 + \frac{x^2+1}{x^2-1}$

1 MARK

$a)\ Solve\ for\ g(1/x)$

$1/x - 4 + \frac{1+x}{1-x}$

1 MARK

$a)\ Solve\ for\ g(g(x))$

$\frac{x+1}{x-1} - 4 + x$

2 MARKS

/4

Question 7

$Find\ the\ value\ of\ x\ that\ satisfy\ the\ equation\ \sin(2x) = -\frac{1}{2}\ for\ 0 \leq 2x \leq 2\pi$

- **Trigonometric , careful! The domain is for 2x not x!**

$$\frac{7\pi}{6}\ and\ \frac{11\pi}{6}$$

2 MARKS for answers and 1 MARK for working out

/3

Question 8

$a)\ Given\ the\ coordinates\ of\ A(0,1), B(1,2) and\ C(3,0), solve\ for\ the\ parabolic\ equation\ y = ax^2 + bx + c$

- **Substitution or Elimination**

$y = -x^2 + 2x + 1$ **1 MARK for answers and 2 MARKS for Working out**

Note:
Write down 3 equations first then solve

$1 = c$
$2 = a + b + c$
$0 = 9a + 3b + c$

/3

Question 9

$Solve\ for\ x\ when\ log_{10}5 + log_{10}x - log_{10}2 = 2$

$x = 40$

1 MARK for answers and 1 MARK for working out

/2

Question 10

$Tim\ walked\ along\ the\ beach\ according\ to\ this\ function$

$x(t) = x^{\frac{1}{2}} + 2^{x/30} + 1, where\ x\ is\ displacement\ in\ metre\ and\ t\ is\ minutes\ after\ 7am$

$a)\ How\ far\ did\ Tim\ walked\ from\ 7am\ to\ 8\ am$

$10.75\ metres$ **1 MARK**

$b)\ What\ is\ Tim's\ average\ speed\ from\ 7am\ to\ 8\ am$

- **Gradient formula**

$0.17\ m/min$ **2 MARKS**

$Note$:
$$m = \frac{y_2 - y_1}{x_2 - x_1}$$

$c)\ What\ is\ Tim's\ instantaneously\ speed\ at\ 7am\ and\ 8\ am$

- **Differentiation**
  1 MARK for answer and 1 MARK for working out

$$v = \frac{1}{2x^{\frac{1}{2}}} + \frac{1}{30}\log_e(2) \times 2^{\frac{x}{30}}$$

$7am - undefined\ or\ infinity$
$8am - 0.15m/mins$

$Note$:

$Differenation\ of\ 2^{\frac{x}{30}}$

$$2^{\frac{x}{30}} = e^{\log_e 2^{\frac{x}{30}}}$$

$= e^{\frac{x}{30}\log_e 2} = e^{kx}$ where

$k = \frac{1}{30}\log_e 2$

$\frac{dy}{dx} = ke^{kx} = \frac{1}{30}\log_e 2\ \times e^{\frac{x}{30}\log_e 2}$

$\frac{1}{30}\log_e(2)\ \times 2^{\frac{x}{30}}$

$d)\ Comment\ about\ the\ instantaneously\ speed\ at\ 7am\ and\ 8am$

$The\ speed\ at\ 7am\ is\ unrealistic\ or\ unknown, however\ the\ speed\ at\ 8am\ is\ similar\ to\ the\ average\ speed$

1 MARK for 7am and 1 MARK for 8am

/7

NQT EDUCATION

YEAR 12

VCE MATHS METHODS
UNIT 3 TERM 2 WORKBOOK

WEEK 18

TEST TWO:

Reading Time: 10 Minutes
Writing Time: 45 Minutes
Short Answers **SCORE: /45**

You may use calculators.

Extended Response

$A\ researcher\ observed\ that\ the\ population\ of\ insects\ increased\ dramatically. The\ population\ was\ modelled\ by$

$$P(t) = \frac{a}{1 + be^{-rt}}\ , \quad t \geq 0$$

$where\ P(t)\ is\ population\ after\ t\ days. Let\ a = 10\ 000, b = 5\ and\ r = 0.1$

$a)\ Find\ the\ inital\ insect\ population.$

$P(0) = 1667$ **1mark**

$b)\ What\ is\ the\ expected\ number\ of\ insects\ as\ time\ increases?$

$P(\infty) = 10{,}000$ **1mark**

$c)\ Sketch\ the\ graph\ of\ P(t)$

4 marks

$However\ the\ insect\ population\ can\ be\ modelled\ by\ P_2(t) = ktlog_e(t+1) + m\ between\ the\ inital\ day$ *and when the population reaches to 6000, where k and m are constants.*

$a)\ Calulate\ the\ values\ of\ k\ and\ m$

$$(20.149{,}6000)\ and\ (0{,}1667)$$

$$P_2(t) = 70.47tlog_e(t+1) + 1667$$

3 mark

$b)\ Using\ the\ second\ model, find\ the\ growth\ rate\ of\ the\ population\ per\ days$

- **Product Rule**

$$P_2'^{(t)} = 70.47log_e(t+1) + \frac{70.47t}{t+1}$$

4 mark

$c)\ Using\ your\ calculator\ find\ the\ maximum\ growth\ rate\ of\ the\ insect\ population.$

$$\left(t, \frac{dp}{dt}\right) \rightarrow (20.149{,}282)$$

1mark

A cone made out of ice is left out on a sunny day. The cone melts at the rate of $1.5m^3/min$. Originally, the radius is 2.5 metres and the height is 5 metres.

$a)$ $Express\ the\ radius\ in\ terms\ of\ height$

$$\frac{r}{h} = \frac{2.5}{5} \rightarrow r = \frac{1}{2}h$$ **1mark**

$b)$ $Express\ the\ volume\ of\ the\ ice, in\ terms\ of\ height\ of\ the\ cone.$

$$v = \frac{1}{3}\pi r^2 h = \frac{1}{3}\pi\left(\frac{1}{4}h^2\right)h = \frac{1}{12}\pi h^3$$ **1mark**

$c)$ $Find\ the\ expression\ for\ the\ rate\ of\ change\ of\ height\ of\ the\ cone.$

$$\frac{dh}{dt} = \frac{dh}{dv} \times \frac{dv}{dt} = \frac{1}{\frac{1}{4\pi}h^2} \times -1.5 = -\frac{6\pi}{h^2}$$ **2marks**

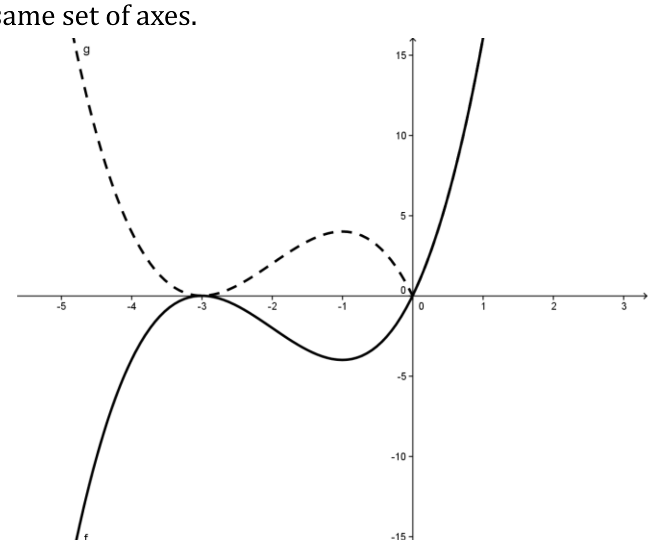
$d)$ $Find\ the\ rate\ at\ which\ the\ radius\ is\ changing\ when\ the\ height\ of\ the\ cone\ is\ three\ metres.$

$$\frac{dr}{dt} = \frac{dr}{dv} \times \frac{dv}{t} = \frac{1}{2\pi r} \times -1.5 = -\frac{3}{4\pi(\frac{3}{4})} = -\frac{1}{\pi}$$ **2marks**

Question 3

$Consider\ the\ graph\ f: [-5,1] \rightarrow R\ where\ F(x) = x^3 + 6x^2 + 9x$

$a)$ Sketch f(x)and |f(x)|on the same set of axes.



/4

$b)$ State the range of the function with rule y = |f(x)|and domain $[-5,1]$

[0,20] **1mark**

$c)$ A normal graph of the f(x) has equation $\mathrm{y} = \frac{1}{3}\mathrm{x} + \mathrm{c}$, where c is a real constant. Find the value for c.

$$note\ \frac{dy}{dx} = -3\ as\ m_1 m_2 = -1\ since\ normal$$

$$\frac{dy}{dx} = 3(x^2 + 4x + 3) = -3$$
$$0 = 3(x^2 + 4x + 4) \rightarrow x = -2$$

$y = -2\ therefore\ using\ subsitution\ c = -4/3$ **3mark**

$d)$ A normal graph of the |f(x)|has equation $\mathrm{y} = \frac{1}{24}\ \mathrm{x} + \mathrm{b}$, where b is a real constant. Find the values for b.

$$\frac{dy}{dx} = 3(x^2 + 4x + 3) = 24$$
$$0 = 3(x^2 + 4x - 5) \rightarrow x = -5\ ,1$$

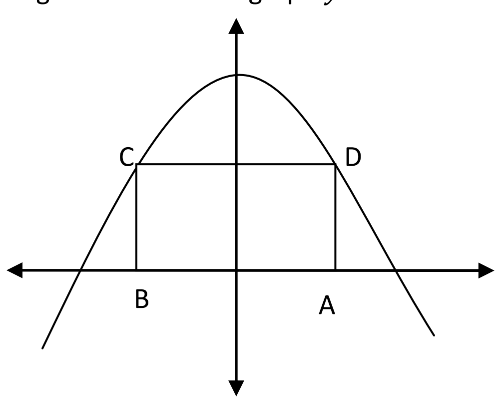
$$|y| = |-20|, |16|$$

$c = \frac{485}{24}, \frac{143}{24}$ **3mark**

Question 4 /7

A rectangle ABCD is inside the enclosed area of the x-axis and graph of $y = 9 - 3x^2$. It has two vertices ,B and A, on the x-axis and the other two vertices, C and D, on the graph. The coordinates of D are (a, b) and C are $(-a, b)$, where $a\ and\ b$ are positive real numbers

$a)$ Illustrate the rectangle ABCD and the graph $y = 9 - 3x^2$ by sketching the function first.



4mark

$b)$ Find the area of rectangle ABCD in terms of a

$$length \times width = 2a \times b = 2a \times (9 - 3a^2) = 2a(9 - 3a^2)$$

1mark

c) Find the maximum area and the value of a for which this occurs.

$$When\frac{dArea}{da} = 0 = 18 - 12a^2 \quad a = +\sqrt{\frac{3}{2}}$$

$$Area_{max} = 2\sqrt{\frac{3}{2}}\left(9 - \frac{3}{2}\right) = \frac{15}{2}\sqrt{6}\,Unit^2$$

4mark