Neap

VCE Mathematical Methods Units 3&4

Suggested Solutions

2024 Trial Examination 2

Section A – Multiple-choice questions

1	A	B	**C**	D
2	A	B	C	**D**
3	A	B	**C**	D
4	A	**B**	C	D
5	A	**B**	C	D
6	A	B	C	**D**
7	A	B	C	**D**
8	A	**B**	C	D
9	A	B	**C**	D
10	A	**B**	C	D
11	A	B	**C**	D
12	**A**	B	C	D
13	**A**	B	C	D
14	A	B	**C**	D
15	**A**	B	C	D
16	A	**B**	C	D
17	A	B	C	**D**
18	A	B	C	**D**
19	A	B	**C**	D
20	A	B	C	**D**

Question 1 C

Using a CAS calculator gives:

$f(x):=3\cdot x-1$	*Done*
$g(x):=6\cdot x+2$	*Done*
$f(g(3))$	59

Question 2 D

Using a CAS calculator gives:

$y(x):=m\cdot x^2-2\cdot m\cdot x+4\cdot m$	*Done*
$y\left(\frac{--2\cdot m}{2\cdot m}\right)$	$3\cdot m$

Since $m < 0$, $3m$ is the maximum value.

Question 3 C

Using a CAS calculator gives:

$\text{solve}\left(\begin{cases}\frac{m}{k+1}=\frac{2}{k}\\ \frac{2}{k}=\frac{m-3}{4}\end{cases},m,k\right)$	$m=\frac{5}{3}$ and $k=-6$

Question 4 B

Using a CAS calculator gives:

$m:=0\cdot 0.1+1\cdot 0.25+2\cdot 0.35+3\cdot 0.3$	1.85
$2\cdot m-3$	0.7

Question 5 B

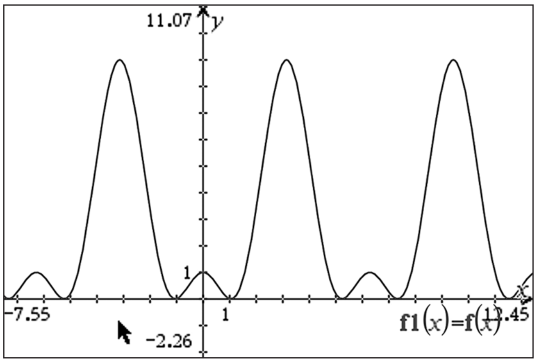
Using the chain rule gives:

$$\begin{aligned} h'(x) &= f'(4x+7)\times(4x+7)' \\ &= 4\times f'(4x+7) \end{aligned}$$

$$\begin{aligned} h'(11) &= 4\times f'(4\times 11+7) \\ &= 4\times f'(51) \end{aligned}$$

Question 6 D

Using a CAS calculator to sketch the graph of $y = f(x)$ gives:



Reading from the graph, the maximum value is 9.

This can be verified by solving $f(x)$ using a CAS calculator.

$f(x):=(2\cdot\cos(x)-1)^2$	*Done*
$\text{solve}\left(\frac{d}{dx}(f(x))=0,x\right)\|0<x<4$	$x=\frac{\pi}{3}$ or $x=\pi$
$f(\pi)$	9

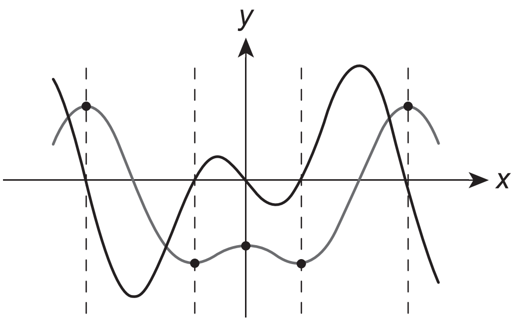
Question 7 D

Each stationary point on f will be an intercept on f'.

When f is increasing, $f' > 0$.

When f is decreasing, $f' < 0$.

This is shown in the graph below.



Question 8 B

$\Pr(A) = x \Rightarrow \Pr(B) = \frac{2}{3}x$

$\Pr(A \cup B) = \Pr(A) + \Pr(B) - \Pr(A \cap B)$

Since A and B are independent:

$\Pr(A \cup B) = \Pr(A) + \Pr(B) - \Pr(A) \times \Pr(B)$

$$0.34 = x + \frac{2}{3}x - \frac{2}{3}x^2$$

Using a CAS calculator gives:

$$\text{solve}\left(0.34 = x + \frac{2}{3} \cdot x - \frac{2}{3} \cdot x^2, x\right)$$

$$x = 0.224086 \text{ or } x = 2.27591$$

Question 9 C

Using a CAS calculator gives:

$$\text{solve}\left(p^4 = \frac{16}{2401}, p\right) \quad p = \frac{-2}{7} \text{ or } p = \frac{2}{7}$$

$$\text{binomCdf}\left(4, \frac{2}{7}, 0, 1\right) \quad 0.676801$$

Question 10 B

B is correct. Reading from the graph, the range is [–3, 9] and the period is 8. The graph crosses the y-axis at 3. Equation **B** is the only option that satisfies all three of these conditions.

A and **C** are incorrect. The period of the graphs represented by these equations is 4.

D is incorrect. The y-intercept of the graph represented by this equation is 9.

Question 11 C

$$\begin{aligned} y &= 3 \\ y_{\text{new}} &= 2 \times 3 - 1 \\ &= 5 \end{aligned}$$

$$\begin{aligned} x &= -2 \\ \frac{x_{\text{new}}}{2} + 1 &= -2 \\ x_{\text{new}} &= -6 \end{aligned}$$

Question 12 A

Using a CAS calculator gives:

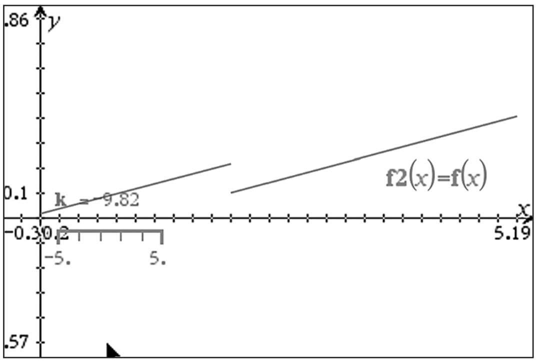
$$f(x):=\begin{cases}\frac{x+k}{10}+1, & 0\le x\le 2\\ \frac{1-x}{k}, & 2\le x\le 5\end{cases} \qquad \textit{Done}$$

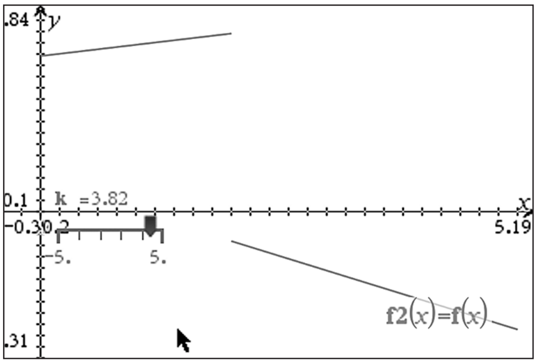
$$\text{solve}\left(\int_0^5 f(x)\,dx=1,k\right)$$

$$k=-9.81909 \text{ or } k=3.81909$$

The valid probability density function should satisfy $f(x) \ge 0$ for $0 \le x \le 5$.

Using a CAS calculator to check the graphs for each value of k gives:





Therefore, $k = -9.82$.

Question 13 A

Using a CAS calculator gives:

$$a(u):=\frac{1}{2}\cdot u\cdot u\cdot (u-2)^2 \qquad \textit{Done}$$

$$\text{solve}\left(\frac{d}{du}(a(u))=0,u\right) \qquad u=0 \text{ or } u=1 \text{ or } u=2$$

$$a(\{0,1,2\}) \qquad \left\{0,\frac{1}{2},0\right\}$$

Question 14 C

Pr(at most two different colours) = 1 – Pr(three different colours)

Using a CAS calculator gives:

$1-\frac{6\cdot 4\cdot 5\cdot 2}{11\cdot 10\cdot 9}$	$\frac{25}{33}$

Note that the three different colours can be drawn in six different ways.

Question 15 A

Using a CAS calculator gives:

z:=invNorm(1−0.12,0,1)	1.17499
solve$\left(\frac{22-16}{s}=z,s\right)$	s=5.10644
s:=5.10644	5.10644
normCdf(-∞,15,16,s)	0.422371

Question 16 B

$$x_1 = x_0 - \frac{f(x_0)}{f'(x_0)}$$

Let $f(x) = x^2 - n$.

Thus, $f'(x) = 2x$.

$$x_1 = x_0 - \frac{x_0^2 - n}{2x_0}$$

In pseudocode, this is equivalent to `x ← x – (x * x – num) / (2 * x)`.

Question 17 D

$$\int_1^4 (f(x) + 2x)\,dx = 15$$

$$\int_1^4 f(x)dx + \left[x^2\right]_1^4 = 15$$

$$\int_1^4 f(x)dx = 0$$

This means that the sum of the signed areas is zero. Unless *f* is a constant function that is equal to zero (which is not an option), this is only possible if *f* has at least one *x*-intercept.

Therefore, while all options could possibly be correct, only option **D** must be correct.

Question 18 D

The derivative function, which is quadratic, should not have any roots. This means that the discriminant must be negative.

Using a CAS calculator gives:

$f(x):=x^3+p\cdot x^2-p\cdot x$	*Done*
$\frac{d}{dx}(f(x))$	$3\cdot x^2+2\cdot p\cdot x-p$
$\text{solve}((2\cdot p)^2-4\cdot 3\cdot -p<0,p)$	$-3<p<0$

Question 19 C

Using a CAS calculator gives:

$f(x):=\ln(x-2)$	*Done*
$y(x):=\text{tangentLine}(f(x),x=a)$	*Done*
⚠ $y(0)$	$\ln(a-2)-\frac{2}{a-2}-1$
⚠ $\text{solve}(y(0)=0,a)$	$a=6.31914$

Question 20 D

$$\log_{a\sqrt{b}}(c)=x$$

$$\frac{\log_c(c)}{\log_c\left(a\sqrt{b}\right)}=x$$

$$\frac{1}{\log_c(a)+\frac{1}{2}\log_c(b)}=x$$

$$\frac{1}{2}\log_c(b)=\frac{1}{x}-\log_c(a)$$

$$\frac{1}{2}\log_c(b)=\frac{1-x\log_c(a)}{x}$$

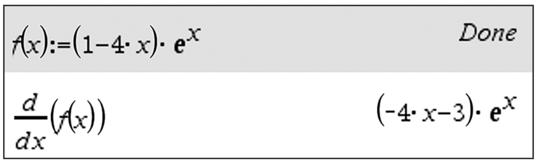
$$x\log_c(b)=2-2x\log_c(a)$$

$$\log_c\left(b^x\right)=2-\log_c\left(a^{2x}\right)$$

Section B

Question 1 (11 marks)

a. Using a CAS calculator gives:

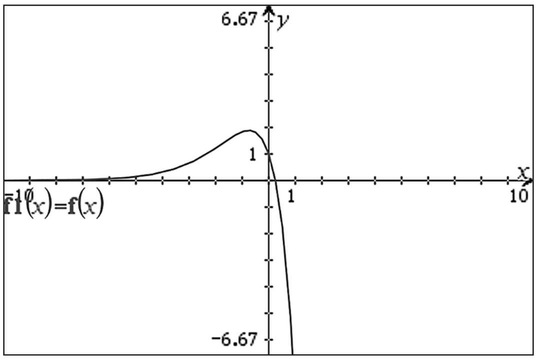


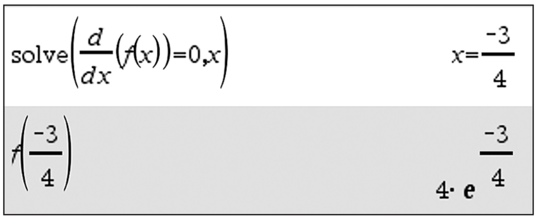
$f'(x) = (-4x - 3)e^{x}$ A1

b. Using a CAS calculator gives:

$y = -3x + 1$ A1

c. Using a CAS calculator gives:





Reading from the graph, the minimum value does not exist.

The maximum value occurs at $x = -\frac{3}{4}$. A1

$\therefore \text{range} = \left(-\infty, 4e^{-\frac{3}{4}}\right]$ A1

d. **i.** Using a CAS calculator gives:

solve(f(x)=1,x)	x=-2.33666 or x=0.

$$\int_{-2.34}^{0} (f(x)-1)\,dx$$

correct boundaries A1
correct integrand A1

ii. Using a CAS calculator gives:

$$\int_{-2.33666}^{0} (f(x)-1)\,dx \quad 1.27674$$

1.28 A1

e. Using a CAS calculator gives:

y(x):=tangentLine(f(x),x=a)	*Done*
y(x)	$(4\cdot a^2-a+1)\cdot \mathbf{e}^a-(4\cdot a+3)\cdot \mathbf{e}^a\cdot x$
solve(y(0)=2.5,a)	a=-2.19936 or a=-1.35165 or a=0.50360

The equation of the tangent at $x = a$:

$y = (4a^2 - a + 1)e^a - (4a + 3)e^a x$ M1

$y(0) = 2.5$

$a = -2.20, -1.35, 0.50$ A1

f. $f(x) = (1 - 4x)e^x \to (ax + b)e^{-2x+4}$

From the index of e:

$x \to -2x + 4$ M1

$\therefore (1 - 4x) \to 1 - 4(-2x + 4)$

$1 - 4(-2x + 4) = ax + b$

$8x - 15 = ab + b$

$\therefore a = 8, b = -15$ A1

Question 2 (12 marks)

a. Using a CAS calculator gives:

$$f(w):=\frac{1}{15}\cdot e^{\frac{-w}{15}} \qquad \textit{Done}$$

$$\int_0^{\infty}\left(w\cdot f(w)\right)dw \qquad 15$$

15 minutes A1

b. Using a CAS calculator gives:

$$\int_{18}^{\infty} f(w)\,dw \qquad 0.301194$$

$$0.30119421186582\cdot 120 \qquad 36.1433$$

$\Pr(W > 18) = 0.3012$ A1

36 passengers A1

c. Using a CAS calculator gives:

$$\frac{\int_{18}^{20} f(w)\,dw}{\int_{18}^{\infty} f(w)\,dw} \qquad 0.124827$$

$$\Pr(W < 20 \mid W > 18) = \frac{\Pr(18 < W < 20)}{\Pr(W > 18)}$$ M1

$$= 0.1248$$ A1

d. Using a CAS calculator gives:

$$\text{normCdf}(10,15,28.3,5.6) \qquad 0.008233$$

0.0082 A1

e. Using a CAS calculator gives:

invNorm(0.2,28.3,5.6)	23.5869

$T = 23.6$ A1

f. Using a CAS calculator gives:

p:=normCdf(-∞,25,28.3,5.6)	0.277835
binomCdf(7,p,3,7)	0.303292

$\Pr(T < 25) = 0.2778 = p$ A1

$X \sim \text{Bi}(7, p)$ M1

$\Pr(X \geq 3) = 0.3033$ A1

g. $\hat{p} = \dfrac{0.72 + 0.84}{2}$

$= 0.78$ A1

$0.06 = 1.64\sqrt{\dfrac{0.78 \times 0.22}{n}}$

$n = 129$ A1

Note: Accept answers that are rounded to 128.

Question 3 (14 marks)

a. maximum = 16 and minimum = –4 A1

b. 15 A1

c. $\text{period}_g = \dfrac{2\pi}{\frac{\pi}{60}}$

$= 120$

$\text{period}_f = \dfrac{2\pi}{\frac{\pi}{80}}$

$= 160$ A1

Therefore, the period of *h* is:

$LCM(120, 160) = 10 \times 2^4 \times 3$

$= 480$ A1

d. Finding the *t*-values for the stationary points gives:

$h(t):=g(t)-f(t)$ Done

$\text{solve}\left(\frac{d}{dt}(h(t))=0,t\right)|0\le t\le 400$

$t=24.013$ or $t=79.4467$ or $t=142.123$ or $t=$ ▸

$$\frac{d}{dt}h(t)=0 \Rightarrow t=24.013, 79.4467, 142.123, \ldots$$ M1

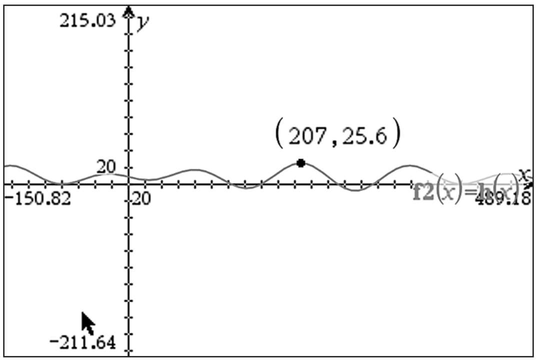
Finding the respective values for *h*(*t*) gives:

$h(\{24.013, 79.4467, 142.123\})$ $\{5.15255, 17.6639, -4.68183\}$

$h(\{207.195, 272.805, 337.877\})$ $\{25.6148, -7.61478, 22.6818\}$

The maximum value of *h*(*t*) occurs at $t=207.195$.

This can be justified by analysing the graph of *h*(*x*).



Thus, the maximum difference occurs at $t=207.195$. M1

The maximum difference is 25.6°C. A1

e. Using a CAS calculator gives:

$\frac{1}{60}\cdot\int_0^{60} h(t)\, dt$ 7.70543

$$\frac{1}{60}\int_0^{60} h(t)dt$$ M1

$=7.7°\text{C}$ A1

f. Using a CAS calculator gives:

$$\text{solve}\left(\frac{d}{dt}(f(t))=\frac{d}{dt}(g(t)),t\right)|0\le t\le 100$$
$$t=24.013 \text{ or } t=79.4467$$

$f'(t)=g'(t)$

$t=24.013, 79.4467, ...$ M1

The temperatures will decrease at the same rate for the first time at $t = 79.4$ minutes. A1

g. $$10\sin\left(\frac{\pi t}{60}\right)+6 \xrightarrow[y_{\text{new}}=\frac{7}{10}y]{} 7\sin\left(\frac{\pi t}{60}\right)+\frac{21}{5}$$

$$\xrightarrow[y_{\text{new}}=y+\frac{54}{5}]{} 7\sin\left(\frac{\pi t}{60}\right)+15$$

$$\xrightarrow[t_{\text{new}}=\frac{3}{4}t]{} 7\sin\left(\frac{\pi t}{80}\right)+15$$

dilation by a factor of $\frac{7}{10}$ *from the t-axis* A1

translation of $\frac{54}{5}$ *units upwards* A1

dilation by a factor of $\frac{4}{3}$ *from the y-axis* A1

Question 4 (11 marks)

a. $f(-1)=-\frac{11}{12}<0$

$f(0) = 1 > 0$

f is continuous.

Therefore, $f(x)$ crosses the x-axis between –1 and 0. M1

Note: All three conditions are required to obtain full marks.

b. Using a CAS calculator gives:

$$fd(x):=\frac{d}{dx}(f(x)) \quad \textit{Done}$$
$$-0.5-\frac{f(-0.5)}{fd(-0.5)} \quad -0.579235$$
$$-0.5792349726776-\frac{f(-0.5792349726776)}{fd(-0.5792349726776)} \quad -0.577835$$

$x_1 = x_0 - \frac{f(x_0)}{f'(x_0)}$ M1

$x_1 = -0.5792$ A1

$x_2 = -0.5778$ A1

c. $f'(x) = \frac{1}{8}x^2 - \frac{3}{4}x + \frac{3}{2}$

$= 0$

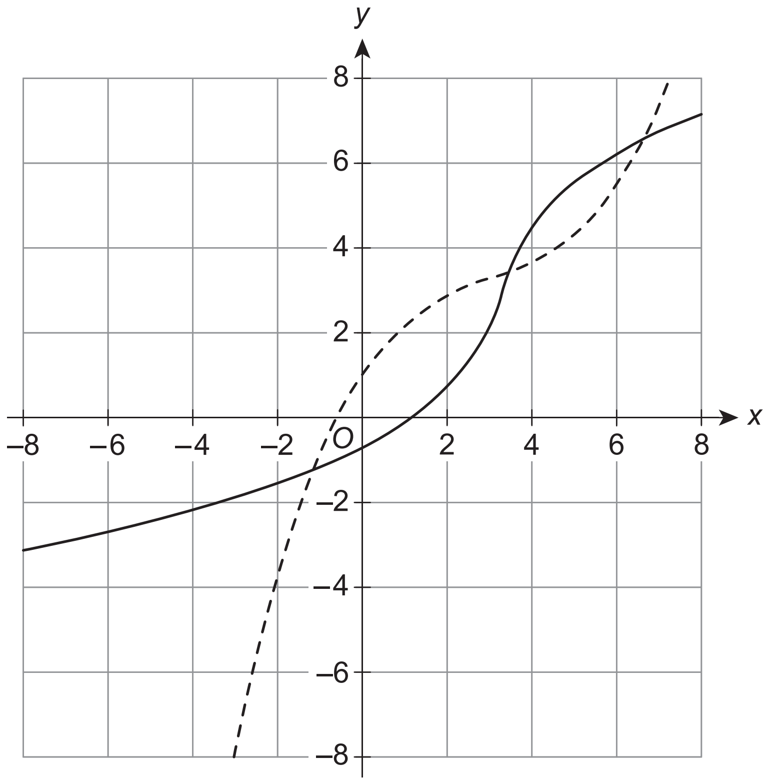
$\Delta = \left(-\frac{3}{4}\right)^2 - 4 \times \frac{1}{8} \times \frac{3}{2}$

$= -\frac{3}{16} < 0$

$f'(x)$ has no solutions and $f > 0$ for all values of x. M1

d. Since f is increasing for all values of x and its graph crosses the x-axis between –1 and 0, it will have only one root. M1

e.



an increasing function with correct intercepts A1

an increasing function with correct points of intersection A1

Note: The original graph is shown by the dashed line. The correct shape is required in order to obtain full marks.

f. Using a CAS calculator gives:

$$\text{solve}(f(x)=x,x)$$
$$x=-1.05932 \text{ or } x=3.40441 \text{ or } x=6.65491$$
$$2\cdot\left(\int_{-1.05932}^{3.40441}(f(x)-x)\,dx+\int_{3.40441}^{6.65491}(x-f(x))\,dx\right)$$
$$9.67664$$

$f(x) = x$

$x = -1.05932, 3.40441, 6.65491$ M1

$2\left(\int_{-1.06}^{3.40}(f(x)-x)\,dx+\int_{3.40}^{6.65}(x-f(x))\,dx\right)$ A1

$= 9.7$ A1

Question 5 (12 marks)

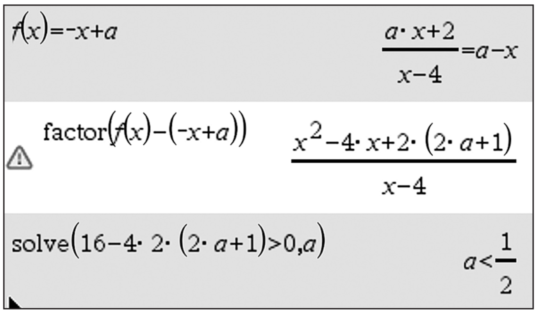
a. Using a CAS calculator gives:

$$f(x):=\frac{a\cdot x+2}{x-4} \quad \textit{Done}$$
$$\text{solve}(f(x)=y,x) \quad x=\frac{2\cdot(2\cdot y+1)}{y-a}$$

$f^{-1}(x) = \frac{4x+2}{x-a}$ A1

$x \in R \setminus \{a\}$ A1

b. Using a CAS calculator gives:



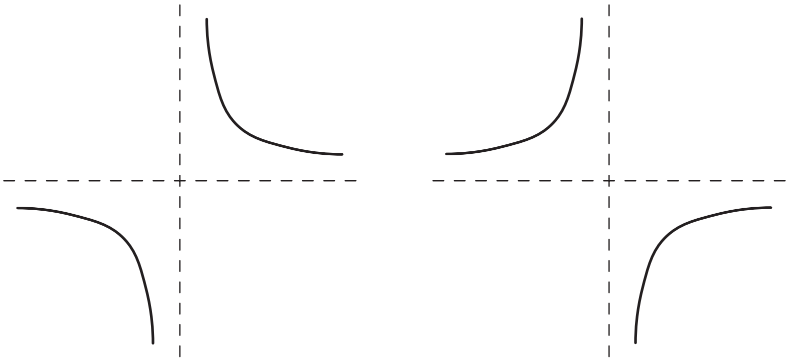
There are two conditions for $S_1 > 0$.

Condition 1: $f(x) = -x + a$ must have two solutions.

$\Delta > 0$

$a < \frac{1}{2}$ A1

Condition 2: The graph of $y = f(x)$ must have the shape of Type 1, as shown below.



Type 1 Type 2

A change between these types occurs when $y = f(x)$ is linear.

$$\frac{ax+2}{x-4} = \frac{a(x-4)}{x-4} = \frac{ax-4a}{x-4}$$

Hence:

$-4a = 2$

$a = -\frac{1}{2}$

By observation, Type 1 is satisfied when $a > -\frac{1}{2}$.

Therefore, $S_1 > 0$ when $-\frac{1}{2} < a < \frac{1}{2}$. A1

c. Using a CAS calculator gives:

$a:=0$	0
$\text{solve}(f(x)=-x,x)$	$x=-(\sqrt{2}-2)$ or $x=\sqrt{2}+2$
$\int_{2-\sqrt{2}}^{2+\sqrt{2}} (f(x)+x)\,dx$	$2\cdot\ln(-(2\cdot\sqrt{2}-3))+4\cdot\sqrt{2}$

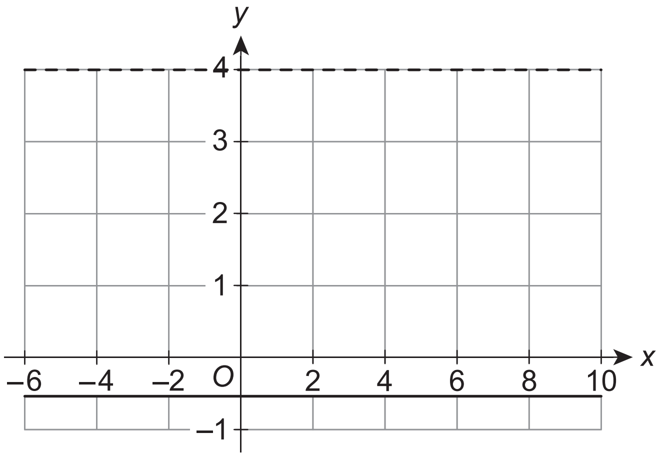
$f(x) = -x$

$x = 2 \mp \sqrt{2}$ M1

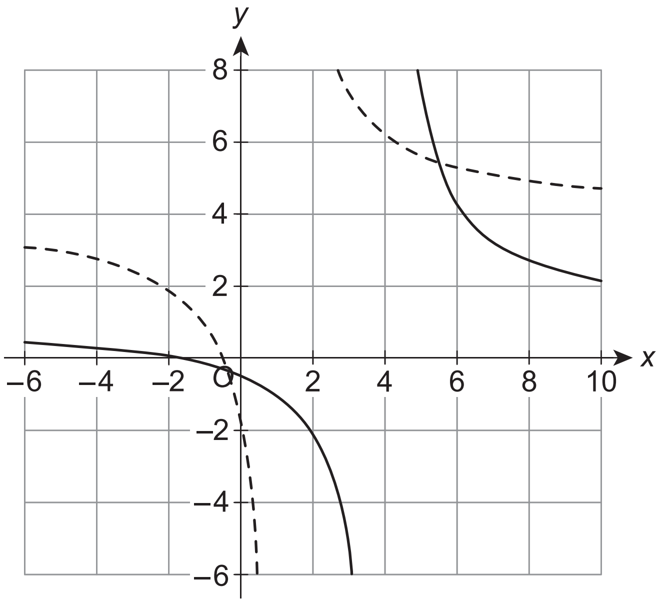
$\int_{2-\sqrt{2}}^{2+\sqrt{2}} (f(x)+x)\,dx$ A1

$= 2\log_e\left(3-2\sqrt{2}\right)+4\sqrt{2}$ A1

d. When $a = -0.5$, both graphs are parallel lines. M1



When $a > -0.5$, there is no region enclosed between the graphs. M1



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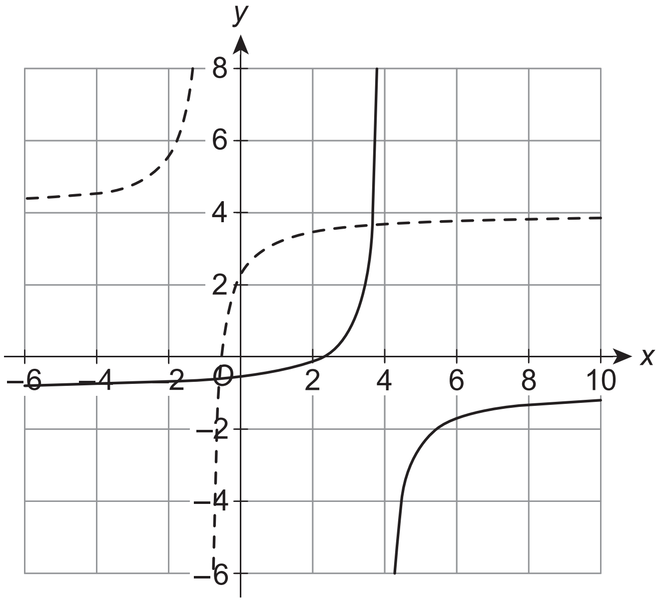
When $a < -0.5$, there is an enclosed area between the graphs. A1



Therefore, $S_2 > 0$ when $a < -0.5$.

e. S_2 is between the asymptotes $x = 4$, $x = a$, $y = 4$ and $y = a$. M1

The region between these four asymptotes forms a square and contains S_2.

Since the area of this square is $(4 - a)^2$, $S_2 < (4 - a)^2$. M1

