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Student Name………………………………

MATHEMATICAL METHODS UNITS 3 & 4

TRIAL EXAMINATION 1

2024

Reading Time: 15 minutes
Writing time: 1 hour

Instructions to students

This exam consists of 9 questions.
All questions should be answered in the spaces provided.
There is a total of 40 marks available.
The marks allocated to each of the questions are indicated throughout.
Students may **not** bring any calculators or notes into the exam.
Where a numerical answer is required, an exact value must be given unless otherwise directed.
Where more than one mark is allocated to a question, appropriate working must be shown.
Diagrams in this trial exam are not drawn to scale.
A formula sheet can be found on the last page of this exam.

Question 1 (3 marks)

a. Let $y = \cos\left(1 - x^2\right)$.

Find $\dfrac{dy}{dx}$. 1 mark

b. If $f(x) = \dfrac{\sin(2x)}{1 + e^{2x}}$, find $f'(0)$. 2 marks

Question 2 (3 marks)

Let $f:(-1, \infty) \to R,\ f(x) = \log_e(x+1)$ and $g: R \to R,\ g(x) = x^2$.

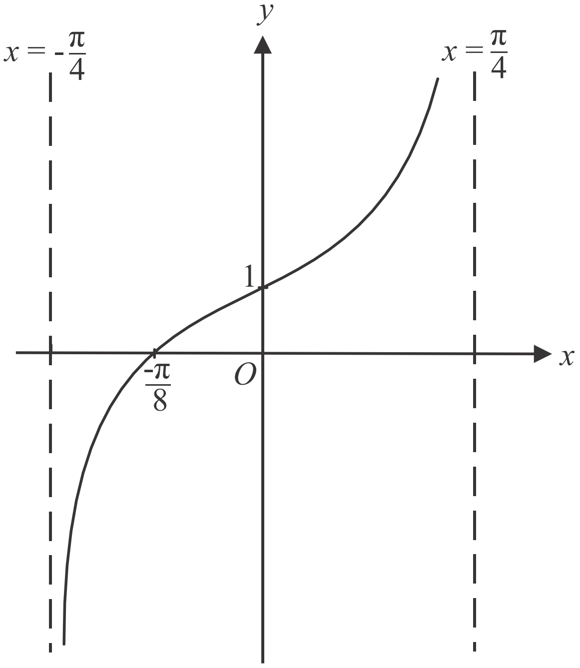
a. Find $(f \circ g)(x)$. 1 mark

b. State the domain and range of $(f \circ g)(x)$. 2 marks

Question 3 (5 marks)

Let $f:\left(-\frac{\pi}{4}, \frac{\pi}{4}\right) \to R,\ f(x) = \tan(2x) + 1.$

Part of the graph of *f* is shown below.



a. Find the average rate of change of *f* between $x = 0$ and $x = \frac{\pi}{8}$. 1 mark

b. Solve $f(x) = 1 + \frac{1}{\sqrt{3}}$ for x. 2 marks

Let $g:\left(-\frac{\pi}{4}, \frac{\pi}{4}\right) \to R, \quad g(x) = f(-x) - 2.$

c. Sketch the graph of *g* on the axes shown on page 4. Label any axis intercepts with their coordinates. 2 marks

Question 4 (4 marks)

a. Evaluate $\int_0^{e-1} \frac{3}{x+1}\,dx$. 2 marks

b. Find $f(x)$ given that $f\left(\frac{1}{3}\right)=0$ and $f'(x)=2\sin(\pi x)$. 2 marks

Question 5 (4 marks)

Let $y = x^2\log_e(2x)$.

a. Find $\frac{dy}{dx}$. 1 mark

b. Hence find the average value of the function $f(x) = x\log_e(2x)$ over the interval $x \in \left[\frac{1}{2}, 1\right]$.

Express your answer in the form $\log_e(a) - \frac{b}{a^3}$, where $a, b \in Z^+$. 3 marks

Question 6 (2 marks)

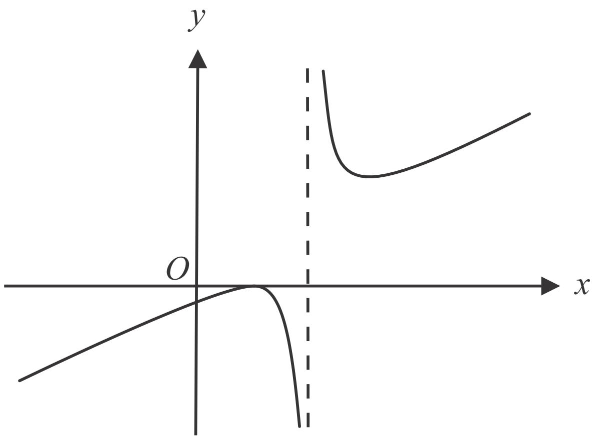
For random samples of four check-outs at a certain brand of supermarkets, $\hat{P}$ is the random variable that represents the proportion of check-outs that are occupied by customers.

It is known that $\Pr(\hat{P} = 0) = 0.25$.

Find the expected value of the proportion $\mathrm{E}(\hat{P})$.

Question 7 (4 marks)

Consider the function f with rule $f(x) = x + \frac{1}{x-2}$. Part of the graph of f is shown below.



a. Find the coordinates of the stationary points of f. 2 marks

b. **i.** Find the values of c, where $c \in R$, for which $f(x) + c = 0$ has no solutions. 1 mark

ii. Find the value of a, where $a \in R$, for which the graph of $y = 1 + f(a - x)$ has no y-intercepts. 1 mark

Question 8 (7 marks)

A random variable X has the probability density function f given by

$$f(x)=\begin{cases} e^{x}-1 & 0\le x\le a \\ 6e^{-x} & a<x\le b \\ 0 & \text{elsewhere} \end{cases}$$

where a and b are real constants. The function is continuous at $x=a$.

a. Show that $a=\log_e(3)$. 2 marks

b. **i.** Evaluate $\Pr(a < X < b)$. 2 marks

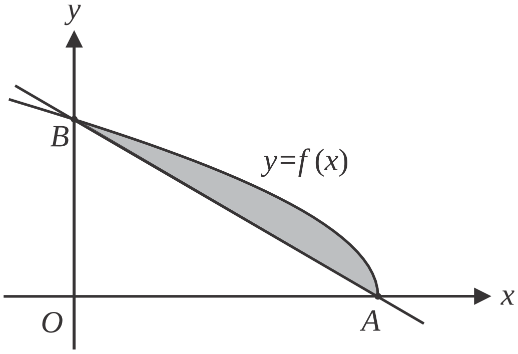
ii. Hence find the value of b. Express your answer in the form

$$b = \log_e\left(\frac{m}{n - \log_e(n)}\right) \quad \text{where } m,\ n \in N$$

3 marks

Question 9 (8 marks)

Consider the function $f:(-\infty, 3] \to R,\ f(x) = \sqrt{3-x}$. Part of the graph of f is shown below.



The points A and B represent the x and y intercepts of f respectively.
The shaded region between $y = f(x)$ and the straight line that passes through points A and B is also shown.

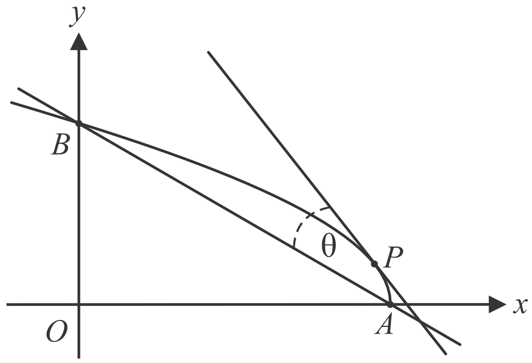
a. Show that the equation of the line through A and B is given by $y = -\frac{\sqrt{3}}{3}x + \sqrt{3}$. 2 marks

b. Find the area of the shaded region. 2 marks

c. Find the rule for the derivative of *f*. 1 mark

The point $P(x, f(x))$ lies on the graph of *f*.

Let θ be the angle between the line *AB* and the tangent to *f* at *P* such that $0° < \theta < 60°$ as shown in the diagram below.



d. Find the coordinates of *P* when $\theta = 30°$. 3 marks

Mathematical Methods formula sheet

Mensuration

area of a trapezium	$\frac{1}{2}(a+b)h$	volume of a pyramid	$\frac{1}{3}Ah$
curved surface area of a cylinder	$2\pi rh$	volume of a sphere	$\frac{4}{3}\pi r^3$
volume of a cylinder	$\pi r^2 h$	area of a triangle	$\frac{1}{2}bc\sin(A)$
volume of a cone	$\frac{1}{3}\pi r^2 h$		

Calculus

$\frac{d}{dx}\left(x^n\right)=nx^{n-1}$		$\int x^n dx = \frac{1}{n+1}x^{n+1}+c,\ n\neq -1$	
$\frac{d}{dx}\left((ax+b)^n\right)=an(ax+b)^{n-1}$		$\int (ax+b)^n dx = \frac{1}{a(n+1)}(ax+b)^{n+1}+c,\ n\neq -1$	
$\frac{d}{dx}\left(e^{ax}\right)=ae^{ax}$		$\int e^{ax}dx=\frac{1}{a}e^{ax}+c$	
$\frac{d}{dx}\left(\log_e(x)\right)=\frac{1}{x}$		$\int \frac{1}{x}dx=\log_e(x)+c,\ \ x>0$	
$\frac{d}{dx}\left(\sin(ax)\right)=a\cos(ax)$		$\int \sin(ax)dx=-\frac{1}{a}\cos(ax)+c$	
$\frac{d}{dx}\left(\cos(ax)\right)=-a\sin(ax)$		$\int \cos\ (ax)\,dx=\frac{1}{a}\sin(ax)+c$	
$\frac{d}{dx}\left(\tan(ax)\right)=\frac{a}{\cos^2(ax)}=a\sec^2(ax)$			
product rule	$\frac{d}{dx}(uv)=u\frac{dv}{dx}+v\frac{du}{dx}$	quotient rule	$\frac{d}{dx}\left(\frac{u}{v}\right)=\frac{v\frac{du}{dx}-u\frac{dv}{dx}}{v^2}$
chain rule	$\frac{dy}{dx}=\frac{dy}{du}\frac{du}{dx}$	Newton's method	$x_{n+1}=x_n-\frac{f(x_n)}{f'(x_n)}$
trapezium rule approximation	$Area\approx\frac{x_n-x_0}{2n}\left[f(x_0)+2f(x_1)+2f(x_2)+...+2f(x_{n-2})+2f(x_{n-1})+f(x_n)\right]$		

Probability

$\Pr(A)=1-\Pr(A')$		$\Pr(A\cup B)=\Pr(A)+\Pr(B)-\Pr(A\cap B)$	
$\Pr(A\mid B)=\frac{\Pr(A\cap B)}{\Pr(B)}$			
mean	$\mu=\mathrm{E}(X)$	variance	$\mathrm{var}(X)=\sigma^2=\mathrm{E}((X-\mu)^2)=\mathrm{E}(X^2)-\mu^2$
binomial coefficient	$\binom{n}{x}=\frac{n!}{x!(n-x)!}$		

Probability distribution		Mean	Variance
discrete	$Pr(X=x)=p(x)$	$\mu=\Sigma x\,p(x)$	$\sigma^2=\Sigma(x-\mu)^2\,p(x)$
binomial	$\Pr(X=x)=\binom{n}{x}p^x(1-p)^{n-x}$	$\mu=np$	$\sigma^2=np(1-p)$
continuous	$\Pr(a<X<b)=\int_a^b f(x)dx$	$\mu=\int_{-\infty}^{\infty}x\,f(x)\,dx$	$\sigma^2=\int_{-\infty}^{\infty}(x-\mu)^2\,f(x)\,dx$

Sample proportions

$\hat{P}=\frac{X}{n}$		mean	$\mathrm{E}(\hat{P})=p$
standard deviation	$\mathrm{sd}(\hat{P})=\sqrt{\frac{p(1-p)}{n}}$	approximate confidence interval	$\left(\hat{p}-z\sqrt{\frac{\hat{p}(1-\hat{p})}{n}},\ \hat{p}+z\sqrt{\frac{\hat{p}(1-\hat{p})}{n}}\right)$

End of formula sheet