GOLD STRIPE

VICTORIAN CURRICULUM AND ASSESSMENT AUTHORITY

Victorian Certificate of Education 2023

SUPERVISOR TO ATTACH PROCESSING LABEL HERE

STUDENT NUMBER Letter

MATHEMATICAL METHODS

Written examination 2

Wednesday 31 May 2023

Reading time: 10.30 am to 10.45 am (15 minutes)
Writing time: 10.45 am to 12.45 pm (2 hours)

QUESTION AND ANSWER BOOK

Structure of book

Section	*Number of questions*	*Number of questions to be answered*	*Number of marks*
A	20	20	20
B	5	5	60
			Total 80

- Students are permitted to bring into the examination room: pens, pencils, highlighters, erasers, sharpeners, rulers, a protractor, set squares, aids for curve sketching, one bound reference, one approved technology (calculator or software) and, if desired, one scientific calculator. Calculator memory DOES NOT need to be cleared. For approved computer-based CAS, full functionality may be used.
- Students are NOT permitted to bring into the examination room: blank sheets of paper and/or correction fluid/tape.

Materials supplied
- Question and answer book of 23 pages
- Formula sheet
- Answer sheet for multiple-choice questions

Instructions
- Write your **student number** in the space provided above on this page.
- Check that your **name** and **student number** as printed on your answer sheet for multiple-choice questions are correct, **and** sign your name in the space provided to verify this.
- Unless otherwise indicated, the diagrams in this book are **not** drawn to scale.
- All written responses must be in English.

At the end of the examination
- Place the answer sheet for multiple-choice questions inside the front cover of this book.
- You may keep the formula sheet.

Students are NOT permitted to bring mobile phones and/or any other unauthorised electronic devices into the examination room.

SECTION A – Multiple-choice questions

Instructions for Section A

Answer **all** questions in pencil on the answer sheet provided for multiple-choice questions.
Choose the response that is **correct** for the question.
A correct answer scores 1; an incorrect answer scores 0.
Marks will **not** be deducted for incorrect answers.
No marks will be given if more than one answer is completed for any question.
Unless otherwise indicated, the diagrams in this book are **not** drawn to scale.

Question 1

The period of the function $y = 3\sin\left(2x - \frac{\pi}{3}\right) + 1$ is

A. $\frac{\pi}{2}$

B. π

C. 2π

D. $\frac{3\pi}{2}$

E. 3π

Question 2

Let $f(t) = -\frac{1}{3}t^3 + \frac{1}{2}t^2 + 50t$.

The instantaneous rate of change of f when $t = 1$ is

A. 247.5
B. 50.2
C. 50.0
D. –13.8
E. –22.0

Question 3

The function $f(x) = \sqrt{3 - 2x - x^2}$ is strictly decreasing over the domain

A. (–3, 3)
B. (–1, 1)
C. (–3, 1)
D. [–1, 1]
E. [–3, 1]

Question 4

Consider the polynomial equation $f(x) = x^n - px + 1$, where $p < 0$, and n is a positive whole number larger than zero.

The highest value of n, from the options below, for which the graph of $y = f(x)$ has exactly one x-intercept is

A. 1
B. 2
C. 3
D. 5
E. 7

Question 5

The probability distribution for the discrete random variable X is shown in the table below

x	0	1	2	3
$\Pr(X = x)$	0.2	0.4	0.3	0.1

The value of E(X) is

A. 1.0
B. 1.3
C. 1.5
D. 2.2
E. 3.0

Question 6

The range of the function $f:\left[0, \frac{1}{2}\right] \to R, f(x) = 2\tan\left(\frac{\pi x}{2}\right)$ is

A. R
B. $(0, 2)$
C. $[0, 1]$
D. $[-2, 2]$
E. $[0, 2]$

Question 7

If $\int_{-2}^{5} f(x)dx = 3$ and $\int_{1}^{5} f(x)dx = -2$, then $\int_{-2}^{1} f(x)dx$ is equal to

A. 5
B. 1
C. $\frac{3}{7}$
D. -1
E. -5

Question 8

The weights of apples that have been picked from an orchard are normally distributed with a mean of 182 grams and a standard deviation of 10 grams. 25% of the apples are not heavy enough to be sold to a supermarket.

What is the minimum weight required, in grams, for an apple to be sold to a supermarket, rounded to the nearest integer?

A. 175

B. 177

C. 187

D. 189

E. 190

Question 9

The area between the graph of $y = x^3 - x^2 - 2x$ and the lines $y = 0$, $x = -1$ and $x = 2$ is

A. $\frac{5}{12}$

B. $\frac{19}{12}$

C. $\frac{37}{12}$

D. $-\frac{9}{4}$

E. $-\frac{15}{4}$

Question 10

A person rolls five fair, six-sided dice. The outcome for each of the dice rolls is independent of the outcome for any of the other dice rolls.

What is the probability of an odd number landing face up on exactly three dice?

A. 0.5

B. 0.3125

C. 0.15625

D. 0.03215

E. 0.03125

Question 11

If $u = g(x)$ and $v = e^{g(2x)}$, where g is a differentiable function, then $\frac{d}{dx}(uv)$ is equal to

A. $3g(x)e^{g(2x)}$

B. $e^{g(2x)}\left(2g(x) + g'(x)\right)$

C. $e^{g(2x)}\left(g(x)g'(2x) + g'(x)\right)$

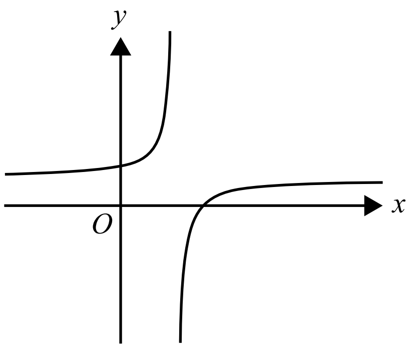
D. $e^{g(2x)}\left(2g(x)g'(2x) + g'(x)\right)$

E. $2g(x)g'(2x)e^{g'(2x)} + e^{g(2x)}g'(x)$

DO NOT WRITE IN THIS AREA

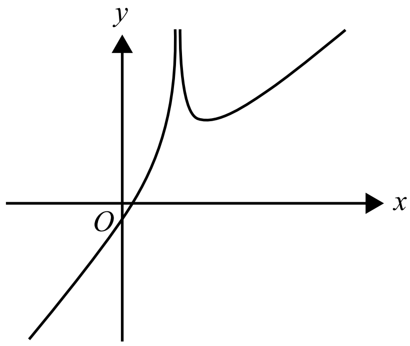
Question 12

The graph of $y = f(x)$ is shown below.

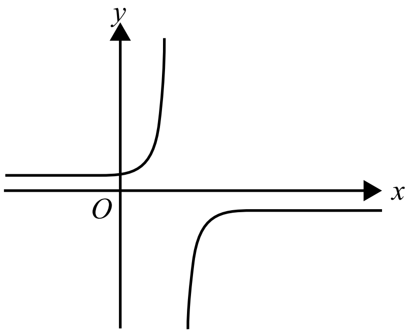


The graph of $y = f'(x)$ is best represented by

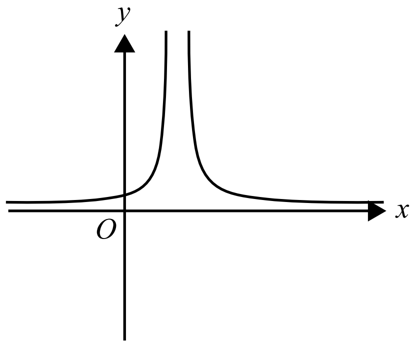
A.



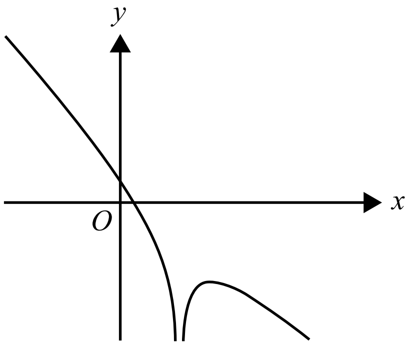
B.



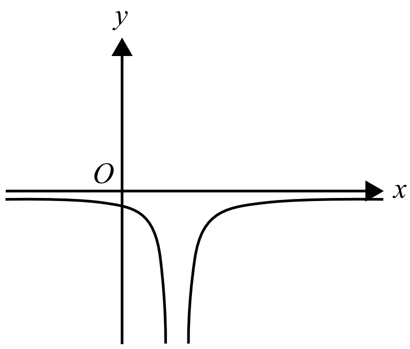
C.



D.



E.



Question 13

A tangent line to the graph of $y = \log_e(2x) + \log_e(x - 2)$ passes through the origin. The x-coordinate, correct to two decimal places, where the tangent line touches the graph is closest to

A. -0.66

B. 1.25

C. 2.91

D. 4.19

E. 6.33

Question 14

There are two boxes of sweets.

Box 1 contains five strawberry-flavoured sweets and seven lemon-flavoured sweets.

Box 2 contains six strawberry-flavoured sweets and 12 lemon-flavoured sweets.

You choose a box randomly and randomly select a single sweet. Given that you pick a strawberry-flavoured sweet, what is the probability that it came from Box 2?

A. $\frac{4}{9}$

B. $\frac{6}{11}$

C. $\frac{6}{30}$

D. $\frac{6}{18}$

E. $\frac{3}{18}$

Question 15

For the continuous random variable X with probability density function

$$f(x) = \begin{cases} a(6-x)(x-2)(x+2) & 2 \le x \le 6 \\ 0 & \text{elsewhere} \end{cases}$$

where a is a positive real constant, the variance of X is closest to

A. 13.7

B. 13.1

C. 4.13

D. 0.88

E. 0.78

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Question 16

Let $f:[-1, 3] \rightarrow R$, $f(x) = 3x - 1$ and $g:\left[-\sqrt{3}, \sqrt{3}\right] \rightarrow R$, $g(x) = x^2$

The largest domain for which $g(f(x))$ is defined is

A. $\left[-1, \sqrt{3}\right]$

B. $\left[\frac{1-\sqrt{3}}{3}, \frac{1+\sqrt{3}}{3}\right]$

C. $\left[-\sqrt{3}, \sqrt{3}\right]$

D. $\left[\frac{-1-\sqrt{3}}{3}, \frac{1-\sqrt{3}}{3}\right]$

E. $[-1, 3]$

Question 17

The function $f(x) = x^2 + kx + 2$ has more than one solution when

A. $k \in \left[2\sqrt{2}, \infty\right)$

B. $k \in \left(-2\sqrt{2}, 2\sqrt{2}\right)$

C. $k \in \left(-\infty, -2\sqrt{2}\right] \cup \left(2\sqrt{2}, \infty\right)$

D. $k \in \left(-\infty, -2\sqrt{2}\right] \cup \left[2\sqrt{2}, \infty\right)$

E. $k \in \left(-\infty, -2\sqrt{2}\right) \cup \left(2\sqrt{2}, \infty\right)$

Question 18

An investment firm models a particular financial investment using the equation $V = b(1 + a)^t$, where V represents the value of the investment after t years and a and b are positive real constants.

The investment firm deducts a fee from any initial investment amount before interest is calculated.

After 10 years the value of the investment was \$12 000 and after 20 years its value was \$15 000.

If the initial investment amount given to the firm was \$10 000, the approximate values of a and b are

A. $a = 0.0226$, $b = 9600$

B. $a = 0.0226$, $b = 10\,000$

C. $a = 0.0225$, $b = 9600$

D. $a = 0.0225$, $b = 10\,000$

E. $a = 0.0224$, $b = 9800$

Question 19

The graph of $y = \log_e(x - k)$, for $k \in R$, has a tangent with a maximum horizontal axis intercept of

A. $x = 1$

B. $x = k$

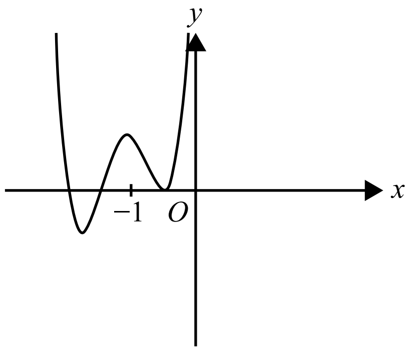
C. $x = e$

D. $x = 1 + k$

E. $x = e + k$

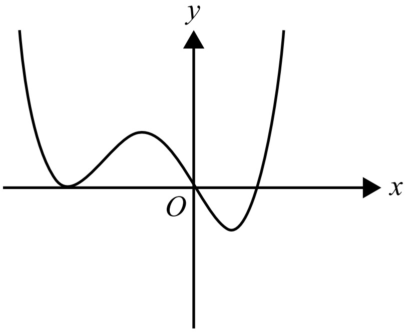
Question 20

The graph of $y = f\left(-2(x+1)\right)$ is shown below.

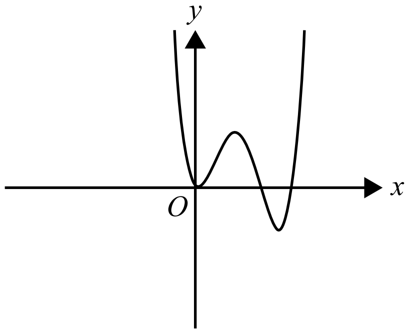


If all axes are to the same scale as the graph above, the graph of $y = f(x)$ is best represented by

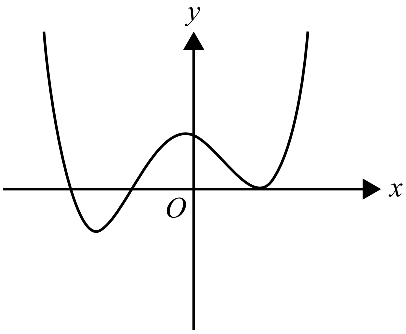
A.



B.



C.



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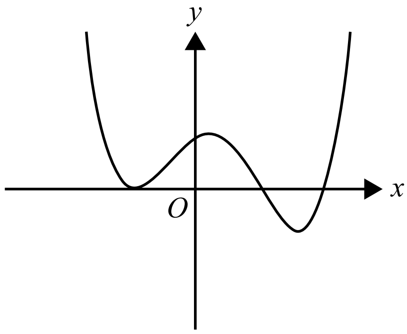

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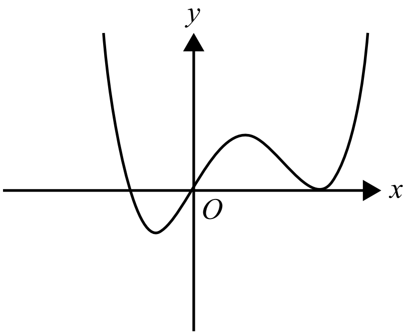

END OF SECTION A

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SECTION B

Instructions for Section B

Answer **all** questions in the spaces provided.

In all questions where a numerical answer is required, an exact value must be given unless otherwise specified.

In questions where more than one mark is available, appropriate working **must** be shown.

Unless otherwise indicated, the diagrams in this book are **not** drawn to scale.

Question 1 (16 marks)

Let $g : R \to R, g(x) = (x + 2)^2 - 1$.

a. Express the rule for g in the form $g(x) = ax^2 + bx + c$, where $a, b, c \in R$. 1 mark

b. The function g can also be written in the form $g(x) = (x - p)(x - q)$, where $p, q \in Z$. Give the values of p and q. 1 mark

c. Find the value of k for which the graph of $y = g(x) + k$ passes through the origin. 2 marks

d. Using algebra, find the value(s) of d such that the graph of $y = g(x - d)$ will pass through the origin. 2 marks

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e. Describe the transformation from the graph of $y = g(x)$ to the graph of $y = g(3x)$. 1 mark

Let $h : R \to R$, $h(x) = mx + n$, where m and n are real numbers.

f. Find the value of m, such that the graph of the sum function $y = g(x) + h(x)$ has a turning point on the y-axis. 2 marks

g. Find n in terms of m, such that the graph of the sum function $y = g(x) + h(x)$ has a turning point on the x-axis. 2 marks

h. Find **two** pairs of values for m and n, such that the graph of the product function $y = g(x)h(x)$ has exactly two x-intercepts. 3 marks

i. Find the coordinates of the turning point of the graph of $y = g\left(h(x)\right)$, giving your answer in terms of m and n. 2 marks

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Question 2 (13 marks)

The amount of caffeine present in Kim's body after they drink espresso coffee can be modelled mathematically.

Students suggest that the amount of caffeine, C in milligrams, in Kim's body t hours after consuming an espresso coffee can be modelled by the function $C(t) = 65e^{-\frac{t}{8}}$.

a. How much caffeine will be present in Kim's body 2 hours after they consume an espresso? Give your answer in milligrams, correct to one decimal place. 1 mark

b. How long will it take for the amount of caffeine in Kim's body to reach 10 milligrams after drinking an espresso? Give your answer in hours and minutes, correct to the nearest minute. 2 marks

c. At what rate is the amount of caffeine in Kim's body decaying 4 hours after they drink an espresso? Give your answer in milligrams per hour in the form $\frac{a}{b\sqrt{e}}$, where a and b are positive integers. 2 marks

DO NOT WRITE IN THIS AREA

Kim consumes another espresso coffee 4 hours after consuming the first.

The students then suggest that a more appropriate model for the absorption of caffeine, in milligrams, in Kim's body t hours after they consume the first espresso is

$$C_2(t) = \begin{cases} 65e^{-\frac{t}{8}} & 0 \le t < 4 \\ 65\left(\dfrac{1-e}{1-\sqrt{e}}\right)e^{-\frac{t}{8}} & t \ge 4 \end{cases}$$

d. When $t \ge 4$, is the function C_2 strictly increasing, strictly decreasing or neither? 1 mark

e. Show that the function C_2 is not continuous for $t > 0$. 1 mark

f. Using C_2, find the maximum amount of active caffeine in Kim's body and the time at which this level was reached. Give the maximum amount of caffeine, in milligrams, correct to one decimal place. 2 marks

g. Find the derivative $C_2'(t)$, giving your answer as a hybrid function that includes the relevant domains. 2 marks

h. Use the derivative $C_2'(t)$ to find the times during which the amount of active caffeine is decreasing by at least 8 milligrams per hour. Express your answer in interval notation, correct to one decimal place. 2 marks

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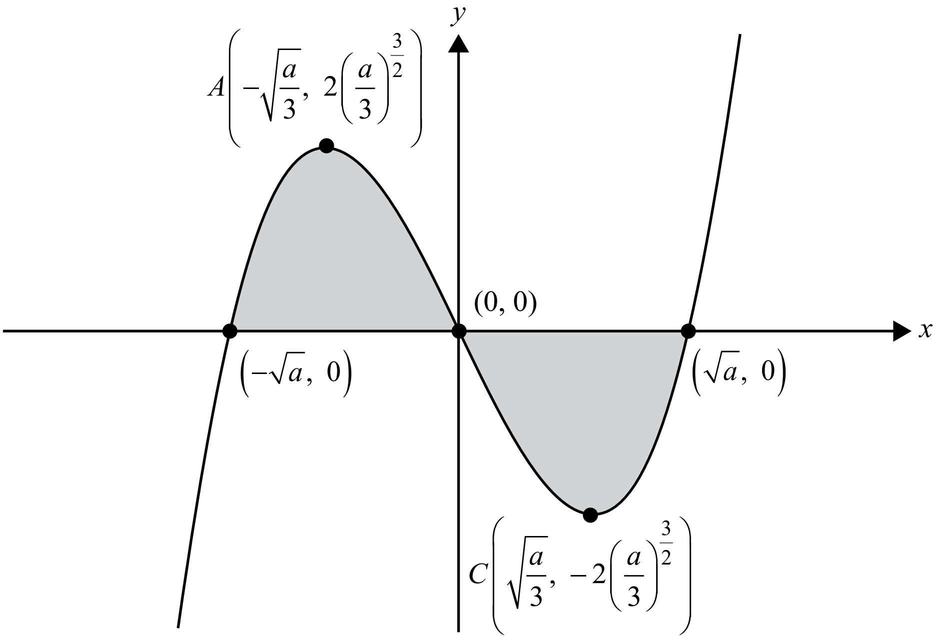
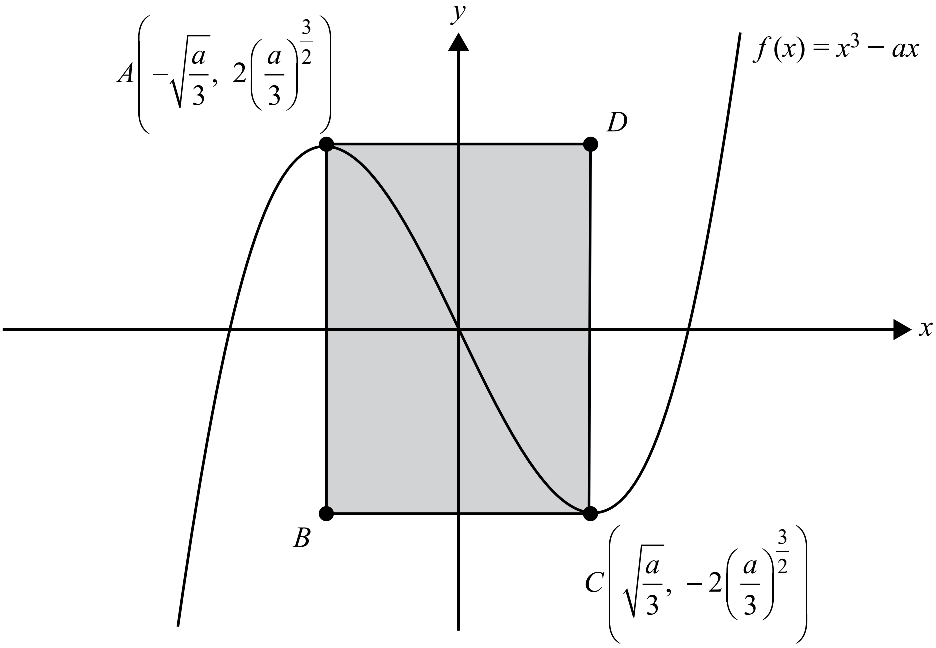
Question 3 (11 marks)

Consider the graph of $f(x) = x^3 - ax$, where a is a real number, with its turning points, A and C, as shown below.



a. Find, using calculus, an expression for the shaded area, in terms of a, bounded by the function f and the horizontal axis on the graph above. 2 marks

The turning points of the graph of f are also the vertices of a rectangle $ABCD$, as shown below. The line segment AC is a diagonal of the rectangle.



b. Calculate the area of the rectangle $ABCD$ when $a = 4$. 2 marks

c. Explain why the value of a must be greater than zero for the area of the rectangle $ABCD$ to be positive. 1 mark

d. Show that the area of rectangle $ABCD$ can be written in the form $8\left(\frac{a}{3}\right)^2$. 1 mark

e. Find, in terms of a, an expression for the length of AC. 2 marks

f. Find the x-values of the points on the graph of f in terms of a, where the tangent to the graph of f is perpendicular to AC. 3 marks

Question 4 (10 marks)

A bakery makes thousands of cupcakes during their morning shift. On average, 70% of the cupcakes are vanilla flavoured. A large random sample of 100 cupcakes was inspected by the manager to calculate the proportion, $\hat{p}$, that were vanilla flavoured during the morning shift.

Let $\hat{P}$ be the random variable representing the sample proportion of vanilla-flavoured cupcakes.

a. State the mean of $\hat{P}$. 1 mark

b. Calculate the standard deviation of $\hat{P}$. 1 mark

c. If the sample size were 600 instead of 100, what changes, if any, would occur to the mean and standard deviation of the sampling distribution? 2 marks

d. What is the probability, in a sample of 100 cupcakes, that more than 66 and fewer than 72 of the cupcakes are vanilla flavoured? Give your answer correct to three decimal places. 1 mark

DO NOT WRITE IN THIS AREA

During the afternoon shift, the bakery makes a batch of muffins. The number of muffins made per batch may be modelled by a normal distribution, X, with a mean of 69 and a standard deviation of 7.

There is a real number, a, such that $\Pr(\mu - a \leq X \leq \mu + a) = 0.64$.

e. Calculate a. Give your answer correct to three decimal places. 3 marks

A local delivery company claims that there is a 90% probability that each delivery it makes from the bakery will arrive on time or earlier.

Assume that each delivery is independent of all other deliveries.

f. Find the probability that on a day in which the delivery company makes 20 deliveries from the bakery, more than 15 deliveries are on time or earlier. Give your answer correct to three decimal places. 2 marks

Question 5 (10 marks)

Let $f(x) = 2\sin(3x) + 1$.

a. State the amplitude of f. 1 mark

b. Find the range of f. 1 mark

c. **i.** Give the general solution for $f(x) = 0$. 1 mark

ii. Calculate the sum of solutions for $f(x) = 0$, where $x \in \left[-\frac{10\pi}{3}, -\frac{5\pi}{3}\right]$. 1 mark

DO NOT WRITE IN THIS AREA

Let the definite integral $g(a)=\int_0^a (f(x)-1)\,dx$ where $a \in \left[0, \frac{\pi}{2}\right]$ be given by

$g(a)=\frac{-2(\cos(3a)-1)}{3}$.

d. **i.** Find the maximum value of g and the value of a for which this occurs. 2 marks

ii. Find the minimum value of g and the value of a for which this occurs. 2 marks

iii. Find the value of a, such that $\frac{1}{a}\times g(a)=\frac{1}{\frac{\pi}{2}-a}\times\int_a^{\frac{\pi}{2}}(f(x)-1)\,dx$. Give your answer correct to three decimal places. 1 mark

Let $h(a)=\int_0^a \left(f\left(x-\frac{k\pi}{3}\right)-1\right)dx$, where $k>0$ and $a \in \left[0, \frac{\pi}{2}\right]$.

e. Give the values of $k>0$, given that $h(a)=\frac{4}{3}$ for some value of a. 1 mark

END OF QUESTION AND ANSWER BOOK

VICTORIAN CURRICULUM AND ASSESSMENT AUTHORITY

Victorian Certificate of Education
2023

MATHEMATICAL METHODS

Written examination 2

FORMULA SHEET

Instructions

This formula sheet is provided for your reference.
A question and answer book is provided with this formula sheet.

Students are NOT permitted to bring mobile phones and/or any other unauthorised electronic devices into the examination room.

Mathematical Methods formulas

Mensuration

area of a trapezium	$\frac{1}{2}(a+b)h$	volume of a pyramid	$\frac{1}{3}Ah$
curved surface area of a cylinder	$2\pi rh$	volume of a sphere	$\frac{4}{3}\pi r^3$
volume of a cylinder	$\pi r^2 h$	area of a triangle	$\frac{1}{2}bc\sin(A)$
volume of a cone	$\frac{1}{3}\pi r^2 h$		

Calculus

$\frac{d}{dx}\left(x^n\right) = nx^{n-1}$		$\int x^n dx = \frac{1}{n+1}x^{n+1} + c,\ n \neq -1$	
$\frac{d}{dx}\left((ax+b)^n\right) = an(ax+b)^{n-1}$		$\int (ax+b)^n dx = \frac{1}{a(n+1)}(ax+b)^{n+1} + c, n \neq -1$	
$\frac{d}{dx}\left(e^{ax}\right) = ae^{ax}$		$\int e^{ax} dx = \frac{1}{a}e^{ax} + c$	
$\frac{d}{dx}\left(\log_e(x)\right) = \frac{1}{x}$		$\int \frac{1}{x} dx = \log_e(x) + c,\ x > 0$	
$\frac{d}{dx}\left(\sin(ax)\right) = a\ \cos(ax)$		$\int \sin(ax)dx = -\frac{1}{a}\cos(ax) + c$	
$\frac{d}{dx}\left(\cos(ax)\right) = -a\ \sin(ax)$		$\int \cos(ax)dx = \frac{1}{a}\sin(ax) + c$	
$\frac{d}{dx}\left(\tan(ax)\right) = \frac{a}{\cos^2(ax)} = a\ \sec^2(ax)$			
product rule	$\frac{d}{dx}(uv) = u\frac{dv}{dx} + v\frac{du}{dx}$	quotient rule	$\frac{d}{dx}\left(\frac{u}{v}\right) = \frac{v\frac{du}{dx} - u\frac{dv}{dx}}{v^2}$
chain rule	$\frac{dy}{dx} = \frac{dy}{du}\frac{du}{dx}$		

Probability

$\Pr(A) = 1 - \Pr(A')$		$\Pr(A \cup B) = \Pr(A) + \Pr(B) - \Pr(A \cap B)$	
$\Pr(A\|B) = \dfrac{\Pr(A \cap B)}{\Pr(B)}$			
mean	$\mu = \mathrm{E}(X)$	variance	$\mathrm{var}(X) = \sigma^2 = \mathrm{E}((X-\mu)^2) = \mathrm{E}(X^2) - \mu^2$

Probability distribution		Mean	Variance
discrete	$\Pr(X = x) = p(x)$	$\mu = \sum x\, p(x)$	$\sigma^2 = \sum (x-\mu)^2\, p(x)$
continuous	$\Pr(a < X < b) = \int_a^b f(x)dx$	$\mu = \int_{-\infty}^{\infty} x\, f(x)dx$	$\sigma^2 = \int_{-\infty}^{\infty} (x-\mu)^2 f(x)dx$

Sample proportions

$\hat{P} = \dfrac{X}{n}$		mean	$\mathrm{E}(\hat{P}) = p$
standard deviation	$\mathrm{sd}(\hat{P}) = \sqrt{\dfrac{p(1-p)}{n}}$	approximate confidence interval	$\left(\hat{p} - z\sqrt{\dfrac{\hat{p}(1-\hat{p})}{n}},\ \hat{p} + z\sqrt{\dfrac{\hat{p}(1-\hat{p})}{n}}\right)$

END OF FORMULA SHEET