VICTORIAN CURRICULUM AND ASSESSMENT AUTHORITY

Victorian Certificate of Education
2023

Print exam correction:
Section B, Question 4j., the letter 'm' after number 8 replaced with 'metres'

STUDENT NUMBER | | | | | | | | | Letter

MATHEMATICAL METHODS

Written examination 2

Thursday 2 November 2023

Reading time: 11.45 am to 12.00 noon (15 minutes)
Writing time: 12.00 noon to 2.00 pm (2 hours)

QUESTION AND ANSWER BOOK

Structure of book

Section	*Number of questions*	*Number of questions to be answered*	*Number of marks*
A	20	20	20
B	5	5	60
			Total 80

- Students are permitted to bring into the examination room: pens, pencils, highlighters, erasers, sharpeners, rulers, a protractor, set squares, aids for curve sketching, one bound reference, one approved technology (calculator or software) and, if desired, one scientific calculator. Calculator memory DOES NOT need to be cleared. For approved computer-based CAS, full functionality may be used.
- Students are NOT permitted to bring into the examination room: blank sheets of paper and/or correction fluid/tape.

Materials supplied
- Question and answer book of 23 pages
- Formula sheet
- Answer sheet for multiple-choice questions

Instructions
- Write your **student number** in the space provided above on this page.
- Check that your **name** and **student number** as printed on your answer sheet for multiple-choice questions are correct, **and** sign your name in the space provided to verify this.
- Unless otherwise indicated, the diagrams in this book are **not** drawn to scale.
- All written responses must be in English.

At the end of the examination
- Place the answer sheet for multiple-choice questions inside the front cover of this book.
- You may keep the formula sheet.

Students are NOT permitted to bring mobile phones and/or any other unauthorised electronic devices into the examination room.

SECTION A – Multiple-choice questions

Instructions for Section A

Answer **all** questions in pencil on the answer sheet provided for multiple-choice questions.
Choose the response that is **correct** for the question.
A correct answer scores 1; an incorrect answer scores 0.
Marks will **not** be deducted for incorrect answers.
No marks will be given if more than one answer is completed for any question.
Unless otherwise indicated, the diagrams in this book are **not** drawn to scale.

Question 1

The amplitude, A, and the period, P, of the function $f(x) = -\frac{1}{2}\sin(3x + 2\pi)$ are

A. $A = -\frac{1}{2},\ P = \frac{\pi}{3}$

B. $A = -\frac{1}{2},\ P = \frac{2\pi}{3}$

C. $A = -\frac{1}{2},\ P = \frac{3\pi}{2}$

D. $A = \frac{1}{2},\ P = \frac{\pi}{3}$

E. $A = \frac{1}{2},\ P = \frac{2\pi}{3}$

Question 2

For the parabola with equation $y = ax^2 + 2bx + c$, where $a, b, c \in R$, the equation of the axis of symmetry is

A. $x = -\frac{b}{a}$

B. $x = -\frac{b}{2a}$

C. $y = c$

D. $x = \frac{b}{a}$

E. $x = \frac{b}{2a}$

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DO NOT WRITE IN THIS AREA

Question 3

Two functions, p and q, are continuous over their domains, which are $[-2, 3)$ and $(-1, 5]$, respectively.

The domain of the sum function $p + q$ is

A. $[-2, 5]$

B. $[-2, -1) \cup (3, 5]$

C. $[-2, -1) \cup (-1, 3) \cup (3, 5]$

D. $[-1, 3]$

E. $(-1, 3)$

Question 4

Consider the system of simultaneous linear equations below containing the parameter k.

$$kx + 5y = k + 5$$
$$4x + (k+1)y = 0$$

The value(s) of k for which the system of equations has infinite solutions are

A. $k \in \{-5, 4\}$

B. $k \in \{-5\}$

C. $k \in \{4\}$

D. $k \in R \setminus \{-5, 4\}$

E. $k \in R \setminus \{-5\}$

Question 5

Which one of the following functions has a horizontal tangent at $(0, 0)$?

A. $y = x^{-\frac{1}{3}}$

B. $y = x^{\frac{1}{3}}$

C. $y = x^{\frac{2}{3}}$

D. $y = x^{\frac{4}{3}}$

E. $y = x^{\frac{3}{4}}$

Question 6

Suppose that $\int_3^{10} f(x)dx = C$ and $\int_7^{10} f(x)dx = D$. The value of $\int_7^{3} f(x)dx$ is

A. $C + D$

B. $C + D - 3$

C. $C - D$

D. $D - C$

E. $CD - 3$

Question 7

Let $f(x) = \log_e x$, where $x > 0$ and $g(x) = \sqrt{1-x}$, where $x < 1$.

The domain of the derivative of $(f \circ g)(x)$ is

A. $x \in R$

B. $x \in (-\infty, 1]$

C. $x \in (-\infty, 1)$

D. $x \in (0, \infty)$

E. $x \in (0, 1)$

Question 8

A box contains n green balls and m red balls. A ball is selected at random, and its colour is noted. The ball is then replaced in the box.

In 8 such selections, where $n \neq m$, what is the probability that a green ball is selected at least once?

A. $8\left(\frac{n}{n+m}\right)\left(\frac{m}{n+m}\right)^7$

B. $1-\left(\frac{n}{n+m}\right)^8$

C. $1-\left(\frac{m}{n+m}\right)^8$

D. $1-\left(\frac{n}{n+m}\right)\left(\frac{m}{n+m}\right)^7$

E. $1-8\left(\frac{n}{n+m}\right)\left(\frac{m}{n+m}\right)^7$

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Question 9

The function f is given by

$$f(x)=\begin{cases}\tan\left(\frac{x}{2}\right) & 4\le x<2\pi\\ \sin(ax) & 2\pi\le x\le 8\end{cases}$$

The value of a for which f is continuous and smooth at $x=2\pi$ is

A. -2

B. $-\frac{\pi}{2}$

C. $-\frac{1}{2}$

D. $\frac{1}{2}$

E. 2

Question 10

A continuous random variable X has the following probability density function.

$$g(x)=\begin{cases}\frac{x-1}{20} & 1\le x<6\\ \frac{9-x}{12} & 6\le x\le 9\\ 0 & \text{elsewhere}\end{cases}$$

The value of k such that $\Pr(X<k)=0.35$ is

A. $\sqrt{14}-1$

B. $\sqrt{14}+1$

C. $\sqrt{15}-1$

D. $\sqrt{15}+1$

E. $1-\sqrt{15}$

Question 11

Two functions, f and g, are continuous and differentiable for all $x\in R$. It is given that $f(-2)=-7$, $g(-2)=8$ and $f'(-2)=3$, $g'(-2)=2$.

The gradient of the graph $y=f(x)\times g(x)$ at the point where $x=-2$ is

A. -10

B. -6

C. 0

D. 6

E. 10

Question 12

The probability mass function for the discrete random variable X is shown below.

X	-1	0	1	2
$\Pr(X = x)$	k^2	$3k$	k	$-k^2 - 4k + 1$

The maximum possible value for the mean of X is:

A. 0

B. $\frac{1}{3}$

C. $\frac{2}{3}$

D. 1

E. 2

Question 13

The following algorithm applies Newton's method using a **For** loop with 3 iterations.

```
Inputs: f(x), a function of x
        df(x), the derivative of f(x)
        x0, an initial estimate

Define newton(f(x), df(x), x0)
    For i from 1 to 3
        If df(x0) = 0 Then
            Return "Error: Division by zero"
        Else
            x0 ← x0 − f(x0) ÷ df(x0)
    EndFor
    Return x0
```

The **Return** value of the function `newton(x³ + 3x − 3, 3x² + 3, 1)` is closest to

A. 0.83333

B. 0.81785

C. 0.81773

D. 1

E. 3

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Question 14

A polynomial has the equation $y = x(3x - 1)(x + 3)(x + 1)$.

The number of tangents to this curve that pass through the positive x-intercept is

A. 0

B. 1

C. 2

D. 3

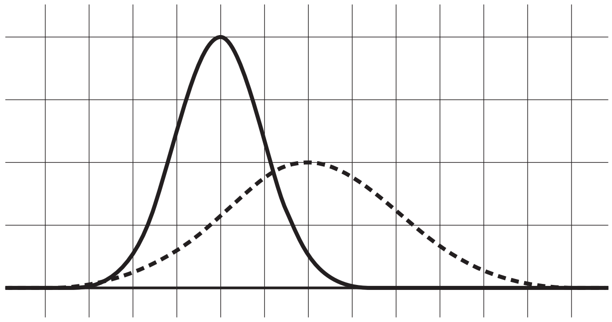
E. 4

Question 15

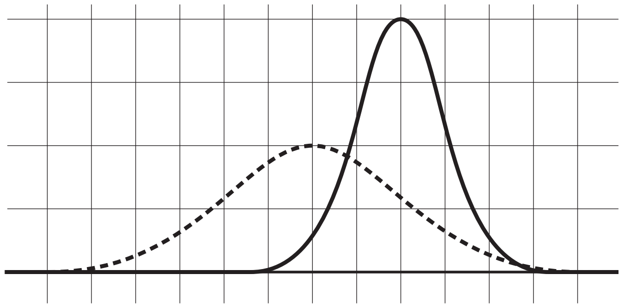
Let X be a normal random variable with mean of 100 and standard deviation of 20. Let Y be a normal random variable with mean of 80 and standard deviation of 10.

Which of the diagrams below best represents the probability density functions for X and Y, plotted on the same set of axes?

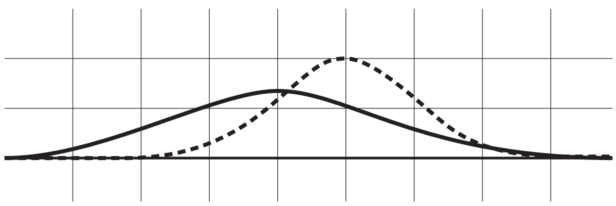
A.



B.



C.



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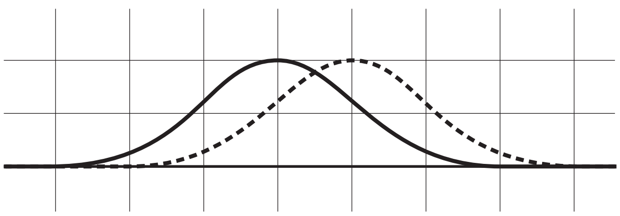
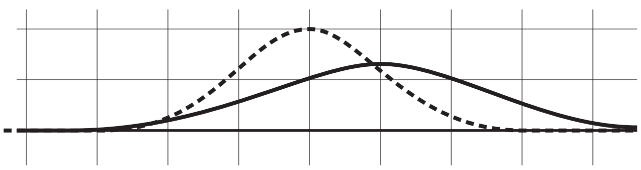
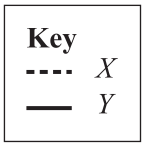


E.





Question 16

Let $f(x) = e^{x-1}$.

Given that the product function $f(x) \times g(x) = e^{(x-1)^2}$, the rule for the function g is

A. $g(x) = e^{x-1}$

B. $g(x) = e^{(x-2)(x-1)}$

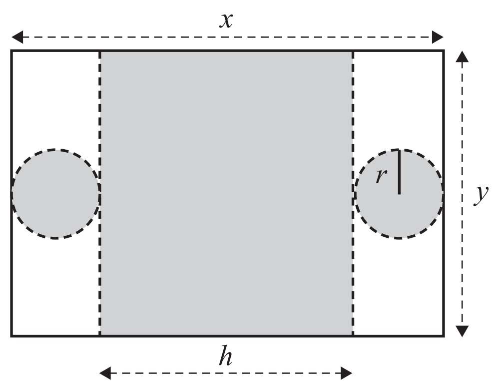
C. $g(x) = e^{(x+2)(x-1)}$

D. $g(x) = e^{x(x-2)}$

E. $g(x) = e^{x(x-3)}$

Question 17

A cylinder of height h and radius r is formed from a thin rectangular sheet of metal of length x and width y, by cutting along the dashed lines shown below.



The volume of the cylinder, in terms of x and y, is given by

A. $\pi x^2 y$

B. $\dfrac{\pi x y^2 - 2y^3}{4\pi^2}$

C. $\dfrac{2y^3 - \pi x y^2}{4\pi^2}$

D. $\dfrac{\pi x y - 2y^2}{2\pi}$

E. $\dfrac{2y^2 - \pi x y}{2\pi}$

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Question 18

Consider the function $f : [-a\pi, a\pi] \to R, f(x) = \sin(ax)$, where a is a positive integer.

The number of local minima in the graph of $y = f(x)$ is always equal to

A. 2

B. 4

C. a

D. $2a$

E. a^2

Question 19

Find all values of k, such that the equation $x^2 + (4k+3)x + 4k^2 - \frac{9}{4} = 0$ has two real solutions for x, one positive and one negative.

A. $k > -\frac{3}{4}$

B. $k \geq -\frac{3}{4}$

C. $k > \frac{3}{4}$

D. $-\frac{3}{4} < k < \frac{3}{4}$

E. $k < -\frac{3}{4}$ or $k > \frac{3}{4}$

Question 20

Let $f(x) = \log_e\left(x + \frac{1}{\sqrt{2}}\right)$.

Let $g(x) = \sin(x)$ where $x \in (-\infty, 5)$.

The largest interval of x values for which $(f \circ g)(x)$ and $(g \circ f)(x)$ both exist is

A. $\left(-\frac{1}{\sqrt{2}}, \frac{5\pi}{4}\right)$

B. $\left[-\frac{1}{\sqrt{2}}, \frac{5\pi}{4}\right)$

C. $\left(-\frac{\pi}{4}, \frac{5\pi}{4}\right)$

D. $\left[-\frac{\pi}{4}, \frac{5\pi}{4}\right]$

E. $\left[-\frac{\pi}{4}, -\frac{1}{\sqrt{2}}\right]$

END OF SECTION A
TURN OVER

SECTION B

Instructions for Section B

Answer **all** questions in the spaces provided.

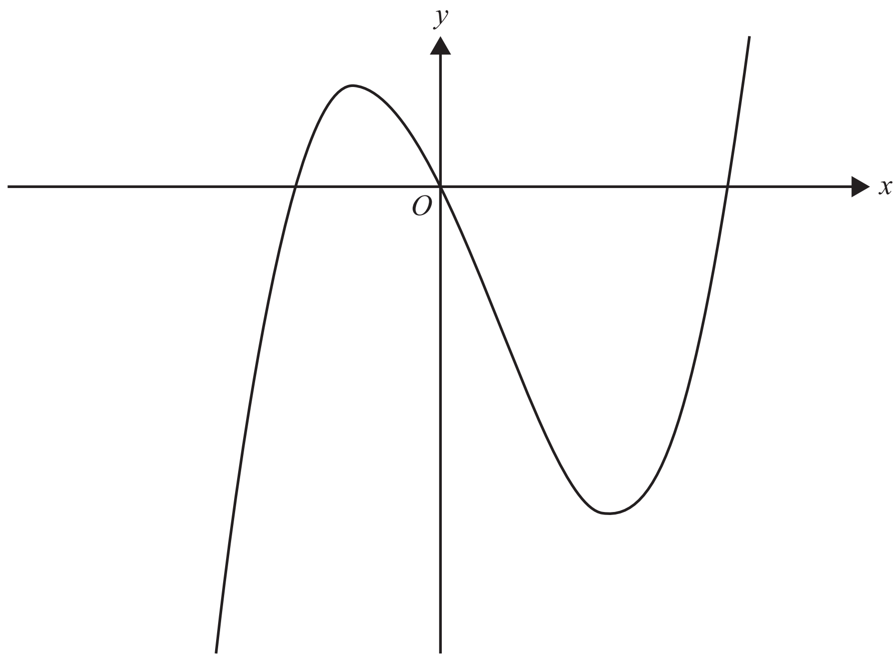
In all questions where a numerical answer is required, an exact value must be given unless otherwise specified.

In questions where more than one mark is available, appropriate working **must** be shown.

Unless otherwise indicated, the diagrams in this book are **not** drawn to scale.

Question 1 (11 marks)

Let $f : R \to R$, $f(x) = x(x - 2)(x + 1)$. Part of the graph of f is shown below.



a. State the coordinates of all axial intercepts of f. 1 mark

b. Find the coordinates of the stationary points of f. 2 marks

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c. **i.** Let $g : R \to R$, $g(x) = x - 2$.

Find the values of x for which $f(x) = g(x)$. 1 mark

ii. Write down an expression using definite integrals that gives the area of the regions bound by f and g. 2 marks

iii. Hence, find the total area of the regions bound by f and g, correct to two decimal places. 1 mark

d. Let $h : R \to R$, $h(x) = (x - a)(x - b)^2$, where $h(x) = f(x) + k$ and $a, b, k \in R$.

Find the possible values of a and b. 4 marks

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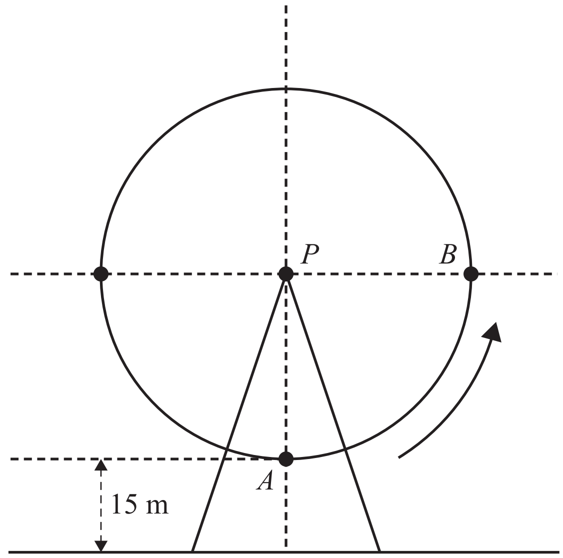
Question 2 (11 marks)

The following diagram represents an observation wheel, with its centre at point P. Passengers are seated in pods, which are carried around as the wheel turns. The wheel moves anticlockwise with constant speed and completes one full rotation every 30 minutes. When a pod is at the lowest point of the wheel (point A), it is 15 metres above the ground. The wheel has a radius of 60 metres.



Consider the function $h(t) = -60\cos(bt) + c$ for some $b, c \in R$, which models the height above the ground of a pod originally situated at point A, after time t minutes.

a. Show that $b = \dfrac{\pi}{15}$ and $c = 75$. 2 marks

b. Find the average height of a pod on the wheel as it travels from point A to point B.
Give your answer in metres, correct to two decimal places. 2 marks

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c. Find the average rate of change, in metres per minute, of the height of a pod on the wheel as it travels from point *A* to point *B*. 1 mark

After 15 minutes, the wheel stops moving and remains stationary for 5 minutes. After this, it continues moving at double its previous speed for another 7.5 minutes.

The height above the ground of a pod that was initially at point A, after t minutes, can be modelled by the piecewise function w:

$$w(t) = \begin{cases} h(t) & 0 \leq t < 15 \\ k & 15 \leq t < 20 \\ h(mt+n) & 20 \leq t \leq 27.5 \end{cases}$$

where $k \geq 0$, $m \geq 0$ and $n \in R$.

d. **i.** State the values of k and m. 1 mark

ii. Find **all** possible values of n. 2 marks

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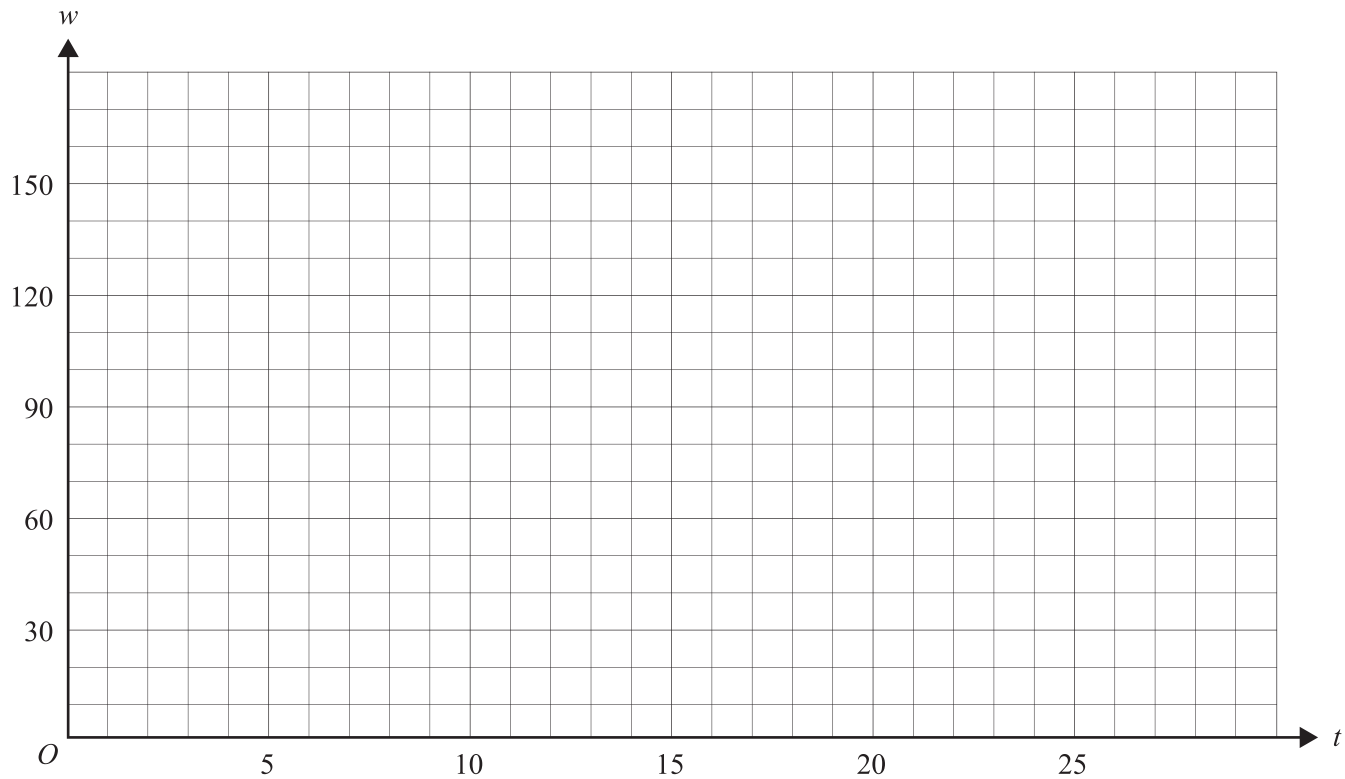
iii. Sketch the graph of the piecewise function w on the axes below, showing the coordinates of the endpoints. 3 marks



Question 3 (12 marks)

Consider the function $g : R \to R$, $g(x) = 2^x + 5$.

a. State the value of $\lim\limits_{x \to -\infty} g(x)$. 1 mark

b. The derivative, $g'(x)$, can be expressed in the form $g'(x) = k \times 2^x$.

Find the real number k. 1 mark

c. **i.** Let a be a real number. Find, in terms of a, the equation of the tangent to g at the point $(a, g(a))$. 1 mark

ii. Hence, or otherwise, find the equation of the tangent to g that passes through the origin, correct to three decimal places. 2 marks

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Let $h : R \to R$, $h(x) = 2^x - x^2$.

d. Find the coordinates of the point of inflection for h, correct to two decimal places. 1 mark

e. Find the largest interval of x values for which h is strictly decreasing.
Give your answer correct to two decimal places. 1 mark

f. Apply Newton's method, with an initial estimate of $x_0 = 0$, to find an approximate x-intercept of h.
Write the estimates x_1, x_2 and x_3 in the table below, correct to three decimal places. 2 marks

x_0	0
x_1	
x_2	
x_3	

g. For the function h, explain why a solution to the equation $\log_e(2) \times (2^x) - 2x = 0$ should not be used as an initial estimate x_0 in Newton's method. 1 mark

h. There is a positive real number n for which the function $f(x) = n^x - x^n$ has a local minimum on the x-axis.

Find this value of n. 2 marks

Question 4 (15 marks)

A manufacturer produces tennis balls.

The diameter of the tennis balls is a normally distributed random variable D, which has a mean of 6.7 cm and a standard deviation of 0.1 cm.

a. Find $\Pr(D > 6.8)$, correct to four decimal places. 1 mark

b. Find the minimum diameter of a tennis ball that is larger than 90% of all tennis balls produced.

Give your answer in centimetres, correct to two decimal places. 1 mark

Tennis balls are packed and sold in cylindrical containers. A tennis ball can fit through the opening at the top of the container if its diameter is smaller than 6.95 cm.

c. Find the probability that a randomly selected tennis ball can fit through the opening at the top of the container.

Give your answer correct to four decimal places. 1 mark

d. In a random selection of 4 tennis balls, find the probability that at least 3 balls can fit through the opening at the top of the container.

Give your answer correct to four decimal places. 2 marks

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A tennis ball is classed as grade A if its diameter is between 6.54 cm and 6.86 cm, otherwise it is classed as grade B.

e. Given that a tennis ball can fit through the opening at the top of the container, find the probability that it is classed as grade A.

Give your answer correct to four decimal places. 2 marks

f. The manufacturer would like to improve processes to ensure that more than 99% of all tennis balls produced are classed as grade A.

Assuming that the mean diameter of the tennis balls remains the same, find the required standard deviation of the diameter, in centimetres, correct to two decimal places. 2 marks

g. An inspector takes a random sample of 32 tennis balls from the manufacturer and determines a confidence interval for the population proportion of grade A balls produced.

The confidence interval is (0.7382, 0.9493), correct to 4 decimal places.

Find the level of confidence that the population proportion of grade A balls is within the interval, as a percentage correct to the nearest integer. 2 marks

A tennis coach uses both grade A and grade B balls. The serving speed, in metres per second, of a grade A ball is a continuous random variable, V, with the probability density function

$$f(v) = \begin{cases} \dfrac{1}{6\pi}\sin\left(\sqrt{\dfrac{v-30}{3}}\right) & 30 \le v \le 3\pi^2 + 30 \\ 0 & \text{elsewhere} \end{cases}$$

h. Find the probability that the serving speed of a grade A ball exceeds 50 metres per second. Give your answer correct to four decimal places. 1 mark

i. Find the **exact** mean serving speed for grade A balls, in metres per second. 1 mark

The serving speed of a grade B ball is given by a continuous random variable, W, with the probability density function $g(w)$.

A transformation maps the graph of f to the graph of g, where $g(w) = af\left(\dfrac{w}{b}\right)$.

j. If the mean serving speed for a grade B ball is $2\pi^2 + 8$ metres per second, find the values of a and b. 2 marks

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Question 5 (11 marks)

Let $f : R \to R, f(x) = e^x + e^{-x}$ and $g : R \to R, g(x) = \frac{1}{2} f(2 - x)$.

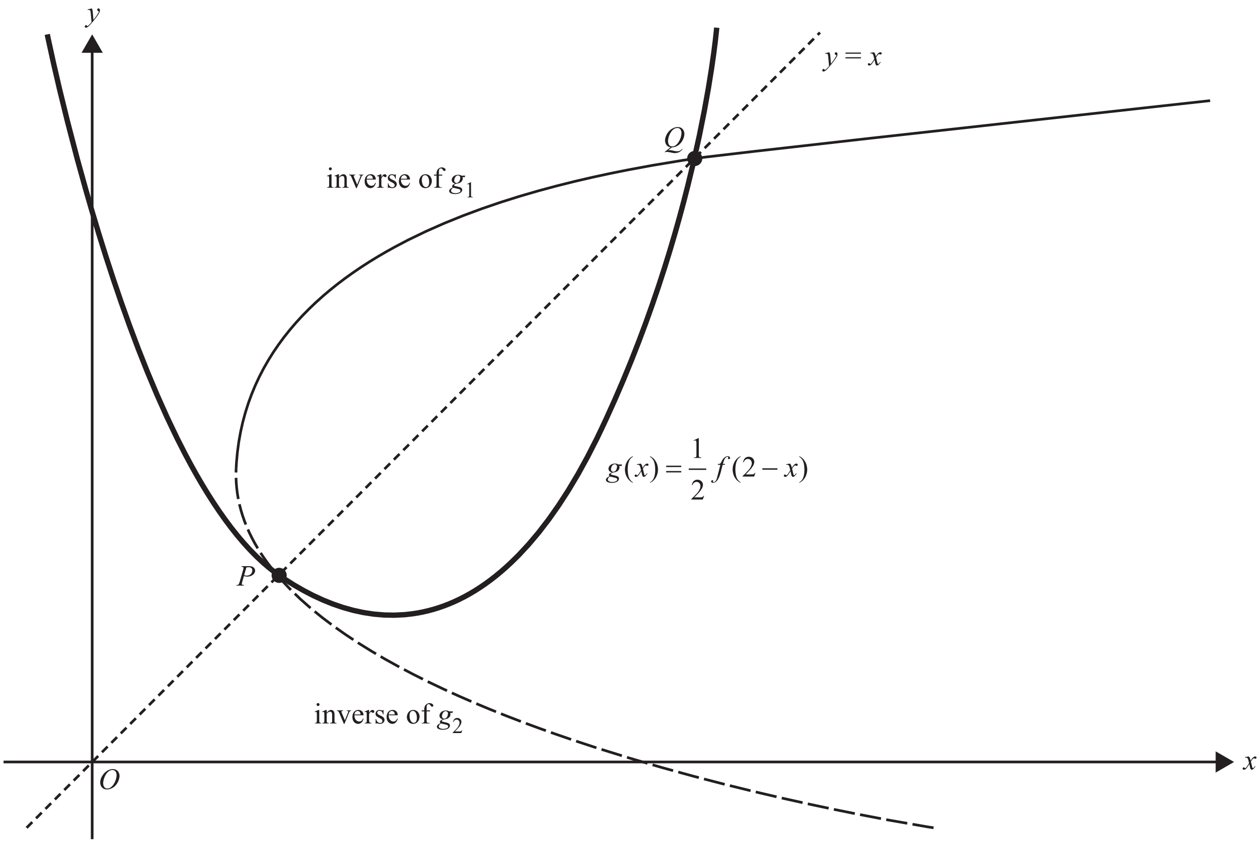
a. Complete a possible sequence of transformations to map f to g. 2 marks

- Dilation of factor $\frac{1}{2}$ from the x axis.
-
-

Two functions g_1 and g_2 are created, both with the same rule as g but with distinct domains, such that g_1 is strictly increasing and g_2 is strictly decreasing.

b. Give the domain and range for the inverse of g_1. 2 marks

Shown below is the graph of g, the inverses of g_1 and g_2, and the line $y = x$.



The intersection points between the graphs of $y = x$, $y = g(x)$ and the inverses of g_1 and g_2, are labelled P and Q.

c. **i.** Find the coordinates of P and Q, correct to two decimal places. 1 mark

ii. Find the area of the region bound by the graphs of g, the inverse of g_1 and the inverse of g_2. Give your answer correct to two decimal places. 2 marks

Let $h: R \to R,\ h(x) = \frac{1}{k} f(k - x)$, where $k \in (0, \infty)$.

d. The turning point of h always lies on the graph of the function $y = 2x^n$, where n is an integer.
Find the value of n. 1 mark

Let $h_1 : [k, \infty) \to R,\ h_1(x) = h(x)$.

The rule for the **inverse** of h_1 is $y = \log_e\left(\frac{k}{2}x + \frac{1}{2}\sqrt{k^2x^2 - 4}\right) + k$

e. What is the smallest value of k such that h will intersect with the inverse of h_1?
Give your answer correct to two decimal places. 1 mark

It is possible for the graphs of h and the inverse of h_1 to intersect twice. This occurs when $k = 5$.

f. Find the area of the region bound by the graphs of h and the inverse of h_1, when $k = 5$.
Give your answer correct to two decimal places. 2 marks

END OF QUESTION AND ANSWER BOOK

VICTORIAN CURRICULUM
AND ASSESSMENT AUTHORITY

Victorian Certificate of Education
2023

MATHEMATICAL METHODS

Written examination 2

FORMULA SHEET

Instructions

This formula sheet is provided for your reference. A question and answer book is provided with this formula sheet.

Students are NOT permitted to bring mobile phones and/or any other unauthorised electronic devices into the examination room.

Mathematical Methods formulas

Mensuration

area of a trapezium	$\frac{1}{2}(a+b)h$	volume of a pyramid	$\frac{1}{3}Ah$
curved surface area of a cylinder	$2\pi rh$	volume of a sphere	$\frac{4}{3}\pi r^3$
volume of a cylinder	$\pi r^2 h$	area of a triangle	$\frac{1}{2}bc\sin(A)$
volume of a cone	$\frac{1}{3}\pi r^2 h$		

Calculus

$\frac{d}{dx}\left(x^n\right) = nx^{n-1}$		$\int x^n dx = \frac{1}{n+1}x^{n+1} + c,\ n \neq -1$	
$\frac{d}{dx}\left((ax+b)^n\right) = an(ax+b)^{n-1}$		$\int (ax+b)^n dx = \frac{1}{a(n+1)}(ax+b)^{n+1} + c, n \neq -1$	
$\frac{d}{dx}\left(e^{ax}\right) = ae^{ax}$		$\int e^{ax} dx = \frac{1}{a}e^{ax} + c$	
$\frac{d}{dx}\left(\log_e(x)\right) = \frac{1}{x}$		$\int \frac{1}{x} dx = \log_e(x) + c,\ x > 0$	
$\frac{d}{dx}\left(\sin(ax)\right) = a\cos(ax)$		$\int \sin(ax)dx = -\frac{1}{a}\cos(ax) + c$	
$\frac{d}{dx}\left(\cos(ax)\right) = -a\sin(ax)$		$\int \cos(ax)dx = \frac{1}{a}\sin(ax) + c$	
$\frac{d}{dx}\left(\tan(ax)\right) = \frac{a}{\cos^2(ax)} = a\sec^2(ax)$			
product rule	$\frac{d}{dx}(uv) = u\frac{dv}{dx} + v\frac{du}{dx}$	quotient rule	$\frac{d}{dx}\left(\frac{u}{v}\right) = \frac{v\frac{du}{dx} - u\frac{dv}{dx}}{v^2}$
chain rule	$\frac{dy}{dx} = \frac{dy}{du}\frac{du}{dx}$	Newton's method	$x_{n+1} = x_n - \frac{f(x_n)}{f'(x_n)}$
trapezium rule approximation	$Area \approx \frac{x_n - x_0}{2n}\left[f(x_0) + 2f(x_1) + 2f(x_2) + \ldots + 2f(x_{n-2}) + 2f(x_{n-1}) + f(x_n)\right]$		

Probability

$\Pr(A) = 1 - \Pr(A')$		$\Pr(A \cup B) = \Pr(A) + \Pr(B) - \Pr(A \cap B)$	
$\Pr(A\|B) = \dfrac{\Pr(A \cap B)}{\Pr(B)}$			
mean	$\mu = \mathrm{E}(X)$	variance	$\mathrm{var}(X) = \sigma^2 = \mathrm{E}((X-\mu)^2) = \mathrm{E}(X^2) - \mu^2$
binomial coefficient	$\binom{n}{x} = \dfrac{n!}{x!(n-x)!}$		

Probability distribution		Mean	Variance
discrete	$\Pr(X = x) = p(x)$	$\mu = \sum x\, p(x)$	$\sigma^2 = \sum (x-\mu)^2\, p(x)$
binomial	$\Pr(X = x) = \binom{n}{x} p^x (1-p)^{n-x}$	$\mu = np$	$\sigma^2 = np(1-p)$
continuous	$\Pr(a < X < b) = \int_a^b f(x)\,dx$	$\mu = \int_{-\infty}^{\infty} x\, f(x)\,dx$	$\sigma^2 = \int_{-\infty}^{\infty} (x-\mu)^2 f(x)\,dx$

Sample proportions

$\hat{P} = \dfrac{X}{n}$		mean	$\mathrm{E}(\hat{P}) = p$
standard deviation	$\mathrm{sd}(\hat{P}) = \sqrt{\dfrac{p(1-p)}{n}}$	approximate confidence interval	$\left(\hat{p} - z\sqrt{\dfrac{\hat{p}(1-\hat{p})}{n}},\ \hat{p} + z\sqrt{\dfrac{\hat{p}(1-\hat{p})}{n}}\right)$

END OF FORMULA SHEET