SUZANNE CORY
High School

2023 Mathematical Methods Units 1 and 2
Written Examination 2

Reading time: 10 minutes
Writing time: 60 minutes

Name : ______________________ **Teacher :** ______________________

Question and Answer Booklet

Number of questions	*Number of questions to be answered*	*Number of marks*
5	5	40 **(+3)**

- Students are permitted to bring into the room: pens, pencils, highlighters, erasers, sharpeners, rulers, a CAS calculator, a scientific calculator, a bound reference.
- Students are **NOT** permitted to bring into the examination room: blank sheets of paper and/or white out liquid/tape.

Materials supplied

- Question and answer book.
- A formula sheet

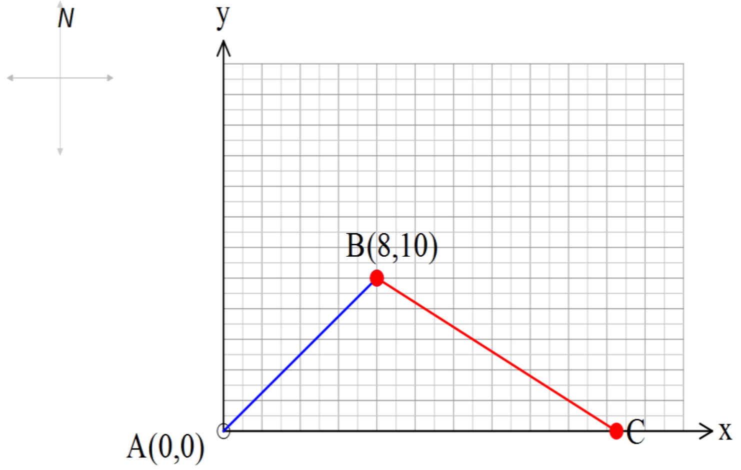
Instructions

- Write your **name** and your **teacher's** name in the space provided above on this page.
- All numerical responses must be in **exact** form unless specified otherwise.
- All written responses must be in English.

Students are NOT permitted to bring mobile phones and/or any other unauthorized electronic devices into the examination room.

Question 1 (8 marks)

A yacht sails in a straight line from $A(0,0)$ to $B(8,10)$. It then turns 90° and sails in a straight line to C which is directly east of A and finally completes the triangular course by sailing directly back to A *(all units in km).*



a Find the equation of the straight line AB. 1 mark

$y = \frac{5}{4}x$ A1

b Show that the equation of the straight line BC is $y = -\frac{4}{5}x + \frac{82}{5}$. 1 mark

$m = -\frac{4}{5}$

$y - 10 = -\frac{4}{5}(x - 8)$

$\Rightarrow y = -\frac{4}{5}x + \frac{82}{5}$ M1

Use negative reciprocal from **a.** and use either $y - y_1 = m(x - x_1)$ or $y = mx + c$ correctly.

c Find the coordinates of point C. 1 mark

$\left(\frac{41}{2}, 0\right)$

d1 Find the total distance of the triangular course to the nearest km. (That is, from A, to B, then to C) 2 marks

$d = \sqrt{8^2 + 10^2} + \sqrt{10^2 + 12.5^2} + 20.5$ M1 Sub into distance formula (one correct is enough due to change in the test)

$d \cong 49\ km$ A1 No units required

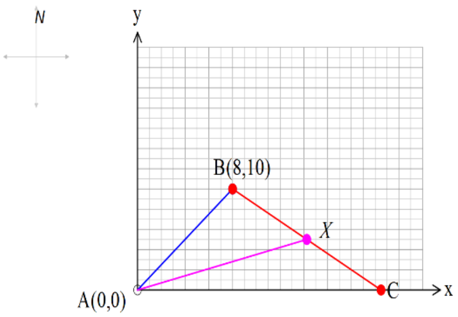
CM here as per c.

The yacht breaks down at X, the halfway point of the second leg of the triangular course. A rescue boat located at A sails directly to the yacht.

d2 Find the exact coordinates of point X. 1 mark

$\left(\frac{57}{4}, 5\right)$ A1

CM here as per c.

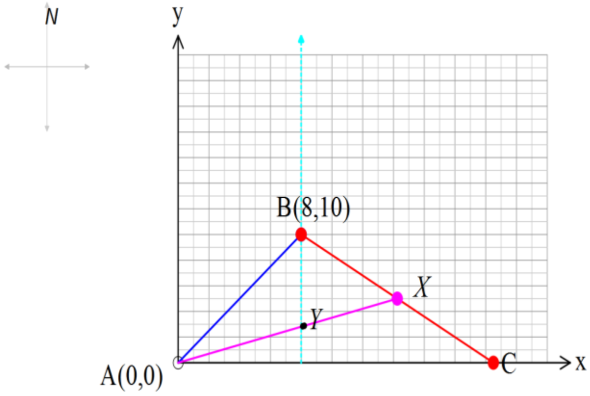


e Hence or otherwise, find the equation of the line AX. 1 mark

$y = \frac{20}{57}x$ A1

CM here as per c and d.

The rescue boat also breaks down at Y ***which is located due south of*** B.



f Find the exact coordinates of Y. 1 mark

$x = 8; \quad y = \frac{20}{57}(8)$

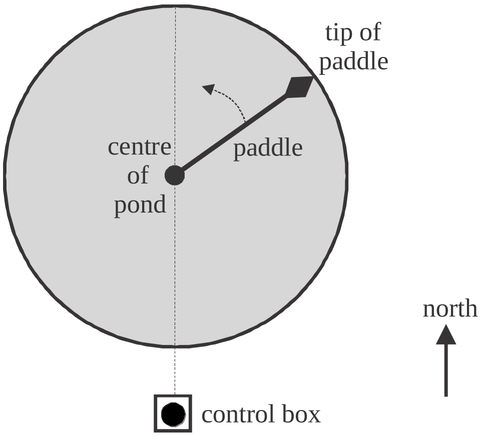
Y is at $\left(8, \frac{160}{57}\right)$ A1

Award A1 if students write $x = 8, y = \frac{160}{57}$

CM here as per c, d and e.

Question 2 (12 marks **(+1 bonus)**)

Waste water is treated in circular ponds at a treatment facility. A straight metal paddle anchored in the centre of these ponds turns anticlockwise to keep the waste water moving.
One such pond has its control box for the paddle located due south of its centre as shown below.



The position p, in metres, of the tip of the paddle north of the control box, t hours after the paddle starts moving, is given by $p(t) = 11 + 8\sin\left(\frac{\pi t}{3}\right), t \geq 0$.

a. How far north of the control box, in metres, is the tip of the paddle when the paddle starts moving? 1 mark

$p(0) = 11\ m$ A1
Unit error for not including metres

b. What is the furthest distance, north of the control box, in metres, that the tip of the paddle reaches? 1 mark

$11 + 8 = 19\ m$ A1
Unit error for not including metres

c. How far north of the control box is the tip of the paddle when $t = 7$? Give your answer correct to three decimal places. 1 mark

$p(7) = 17.928\ m$
Unit error for not including metres

d. When the distance between the tip of the paddle and the control box is a minimum, what is the smallest value of t?

2 marks

$11 - 8 = 3$ M1 Find minimum distance
$p(t) = 3$
$t = \frac{9}{2}$ A1 No unit required

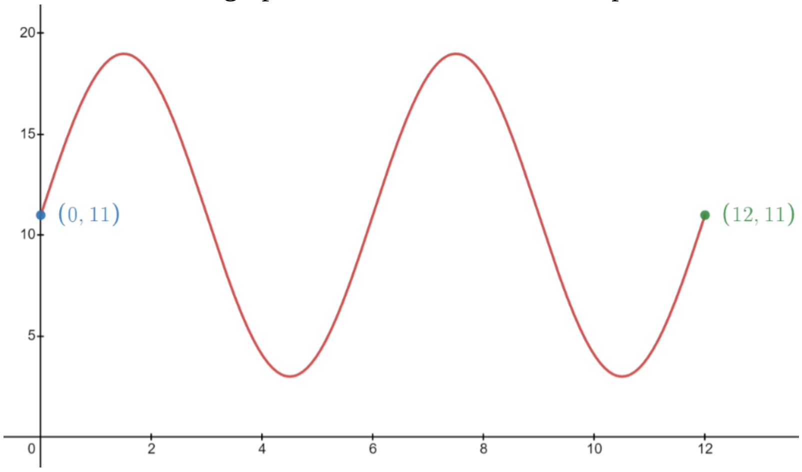
e. How many complete rotations does the paddle do in a day? 1 mark

$P = \frac{2\pi}{\left(\frac{\pi}{3}\right)} = 6\ hr\ per\ rotation$

$\frac{24\ hr}{6\ hr/rotation} = 4\ rotations$ A1 Award for just the value, 4.

f. Sketch the graph of $p(t) = 11 + 8\sin\left(\frac{\pi t}{3}\right)$ for $t \in [0,12]$ on the set of axes below. Clearly indicate on the graph the coordinates of the endpoints.

2 marks



Sine graph shape A1 ensure maximum and minimum go to 19 and 3 respectively.
Both endpoints labelled A1

g. During one complete rotation of the paddle, for how long is the tip of the paddle at least 13 metres north of the control box? Give your answer in hours, correct to three decimal places.

2 marks

$p(t) = 13 \mid 0 \le t \le 6$ M1 Set up equation, domain not needed to award the method mark

$\Rightarrow t = 0.24\ldots, 2.75\ldots$

$\therefore Duration = 2.75\ldots - 0.24\ldots = 2.517\ hours$ A1 No unit required

h. Simplify the expression as a sine function: $\cos\left(\frac{\pi}{2} - \theta\right)$

1 mark

$\sin(\theta)$ A1

The function p can be expressed in the form: $p(t) = 11 + 8\cos\left(-\frac{\pi}{6}(2t - b)\right)$ where $b \in [0,10]$.

i. Find the value of b. 1 mark

$p(t) = 11 + 8\cos\left(\frac{b\pi}{6} - \frac{\pi t}{3}\right)$

From above: $p(t) = 11 + 8\sin\left(\frac{\pi t}{3}\right)$

If $\cos\left(\frac{\pi}{2} - \theta\right) = \sin(\theta)$ such that $\theta = \frac{\pi t}{3}$, then $\frac{\pi}{2} = \frac{b\pi}{6}$

$\Rightarrow b = 3$ A1

At a second identical pond, the position, q, in metres, of the tip of the paddle north of the control box, t hours after the paddle starts moving is given by $q(t) = 11 + 8\sin(kt),\ t \geq 0$, where $k \in \mathbb{R}$.

j. **(BONUS marks)**
At this second pond, the optimal number of complete rotations of the paddle per day is between six and eight.
Find the possible values of k.

1 mark

$3 < P < 4$
$3 < \frac{2\pi}{k} < 4$

$\Rightarrow \frac{2\pi}{k} > 3\ or\ \frac{2\pi}{k} < 4$
$\Rightarrow k < \frac{2\pi}{3}\ or\ k > \frac{\pi}{2}$

$\therefore \frac{\pi}{2} < k < \frac{2\pi}{3}$ A1

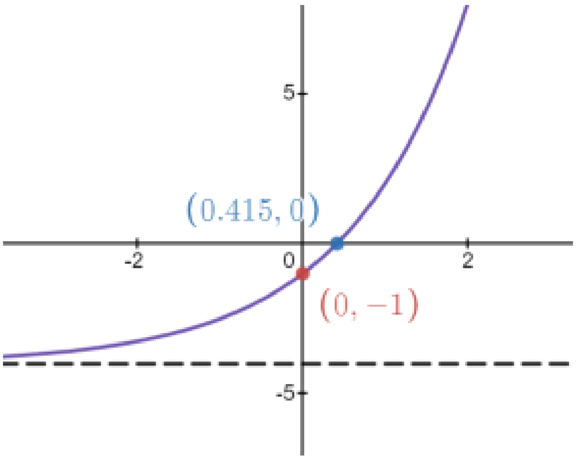
Question 3 (4 marks **(+1 bonus)**)

An exponential function g is defined as:

$g: \mathbb{R} \to \mathbb{R}, g(x) = 3 \times 2^x - 4$

a. Sketch the graph of $y = g(x)$.
Label the asymptote with its equation and all axial intercepts with their coordinates, correct to 3 decimal places.

3 marks



Exponential graph shape A1
Asymptote labelled, $y = -4$ A1
Both axial intercepts labelled A1
Take 1 mark if extra asymptotes have been given (ex $x = 5$)

b. Solve $g(x) = \frac{1}{2}$ for x in the form $\log_2\left(\frac{a}{b}\right)$ where a and b are integers.

1 mark

$\Rightarrow 3 \times 2^x = \frac{9}{2}$

$\Rightarrow x = \log_2\left(\frac{3}{2}\right)$ A1

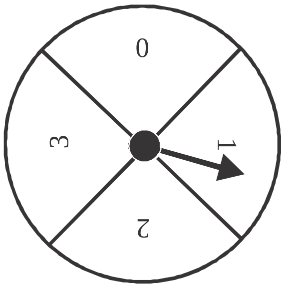
c. **(BONUS mark)**
Hence or otherwise, state all values of x such that $g(x) < \frac{1}{2}$.

1 mark

$x < \log_2\left(\frac{3}{2}\right)$ A1 Conseq to **b.**

Question 4 (9 marks)

A board game uses a spinner and a fair six-sided die. The spinner has equal sectors labelled 0, 1, 2 and 3 as shown below.



A player takes a turn by spinning the spinner and rolling the die. The sum of the resulting two numbers is the players score.

a. Complete the table below that gives the sum of the die and the spinner for each of the possible outcomes of a players turn.

	Possible outcomes					
Spinner	Die					
	1	2	3	4	5	6
0	1	2	3	4	5	6
1	2	3	4	5	6	7
2	3	4	5	6	7	8
3	4	5	6	7	8	9

All table values correct A1 1 mark

Mohan and his friend Nithin play the game together.

b. Find the probability that on his next turn, Mohan

i. scores five. 1 mark

$\frac{4}{24} = \frac{1}{6}$ A1

ii. scores five given that the spinner landed on two. 1 mark

$\frac{1}{6}$ A1

The probability that either player scores two on their next turn is $\frac{1}{12}$.

c. Find the probability that on Nithin's next three turns he scores

i. a two on each turn. 1 mark

$\left(\frac{1}{12}\right)^3 = \frac{1}{1728}$ A1

ii. a two at least once. 1 mark

$1 - \left(\frac{11}{12}\right)^3 = \frac{397}{1728}$ A1 Conseq to **ci.** if appropriate working is shown

Mohan has six different board games. These board games are contained in boxes and are stacked on top of one another.

d. In how many different ways can these board games be arranged in a stack of 6.

iii. if there are no restrictions? 1 mark

$6! = 720$ A1

iv. if Mohan's favourite and least favourite board games are together in the stack? 1 mark

${}^5P_2 \times 2! = 240$ A1

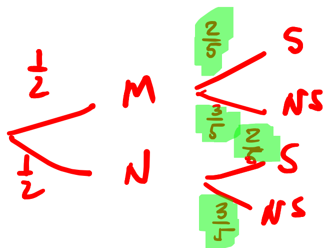
The two boys, Mohan and Nithin, each have two sisters.
All six children are at Mohan's house and his mother randomly selects two of them to help her.

e. Find the probability that the two children chosen are siblings. 2 marks

$\frac{{}^3C_2 \times {}^3C_0 \times 2}{{}^6C_2}$ M1

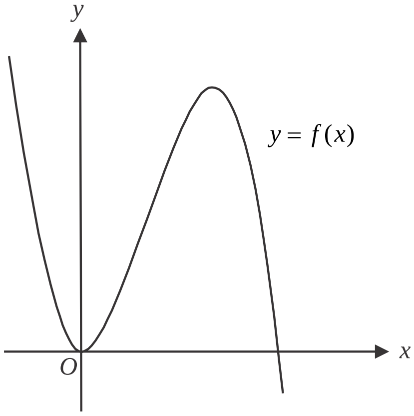
$= \frac{2}{5}$ A1

Method could include probability tree or listing. If PT used at least one highlighted probabilities given correctly.



Question 5 (7 marks **(+1 bonus)**)

Part of the graph of $f(x)=6x^2-2x^3$ is shown on the axes below.



a. Find the x-intercepts of the graph of f. 1 mark

$(0,0)$ and $(3,0)$ A1 Allow $x=0, x=3$.

b. Find the coordinates of the turning points of the graph of f. 1 mark

$(2,8)$ and $(0,0)$ A1

c. Consider the function g which has the same rule as function f.
The function g has an inverse function g^{-1}.
If the domain of g is $(-\infty, a]$, find the maximum value of a. 1 mark

0 A1

d. Find the values of h such that $f(x-h)=0$ has no negative solutions for x. 1 mark

$h \geq 0$ A1

e. Find the values of k such that $f(x)+k=0$ has three solutions for x. 1 mark

$-8 < k < 0$ A1

Recall that $f(x) = 6x^2 - 2x^3$

f. The graph of $y = f(x)$ undergoes a series of transformations to become the graph of $y = t(x)$. This series of transformations involves

- a dilation by a factor of 3 from the *y*-axis
- a reflection in the *x*-axis.

Find the rule for the new function, *t*.

2 marks

$t(x) = \frac{2x^3}{27} - \frac{2x^2}{3}$

2 out of 2 marks for correct rule (or equivalent) with relevant working.

If the final answer is incorrect, 1 out of 2 marks can be awarded for any one of:

- $t(x) = -f\left(\frac{x}{3}\right)$
- Correct final expression except for one error.
  E.g. $t(x) = \frac{2x^3}{27} + \frac{2x^2}{3}$; sign of one term is incorrect

g. (BONUS marks)

Let $u(x) = -5 + nx - 2x^3$, where $n \in R$.
Find all values of n where the graphs of $y = f(x)$ and $y = u(x)$ do not intersect.

1 mark

$f(x) = u(x)$

$\Rightarrow 6x^2 - nx + 5 = 0$

$\Delta = n^2 - 120$; Require $\Delta < 0$ for no solution to $f(x) = u(x)$ for x

$\Rightarrow n \in \left(-2\sqrt{30}, 2\sqrt{30}\right)$ A1

END OF EXAM