NAME: ____________________________

MATHEMATICAL METHODS

Unit 3 & 4 Practice Written Examination 1

Reading time: 15 minutes **Writing time: 1 hour**

QUESTION AND ANSWER BOOK

Structure of Book

Number of questions	Number of questions to be answered	Number of marks
9	9	40

- Students are permitted to bring into the examination room: pens, pencils, highlighters, erasers, sharpeners, and rulers.
- Students are NOT permitted to bring into the examination room: any technology (calculators or software), notes of any kind, blank sheets of paper and/or correction fluid/tape.

Materials supplied

- Question and Answer Book of 14 pages.
- Formula Sheet.
- Working space is provided throughout the book.

Instructions

- Write your student number in the space provided above on this page.
- Unless otherwise indicated, the diagrams in this book are not drawn to scale.
- All written responses must be in English.

At the end of the Examination

- You may keep the Formula Sheet.

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Question 1 (3 marks)

Let $f: R \to R, f(x) = \dfrac{\cos(x)}{\sin(2x)}$

a. Find $f'(x)$ 1 mark

b. Let $g: R \to R$¿$0\}, g(x) = 3\log_e(x^2)$. For what value of x is $g'(x) = 1$?

2 marks

Question 2 (3 marks)

Given $x \in [0, 8\pi]$, solve the equation:

$$\left(\sin\left(\frac{x}{2}\right)\right)^2 = 2\sin\left(\frac{x}{2}\right) - 1$$

Question 3 (7 marks)

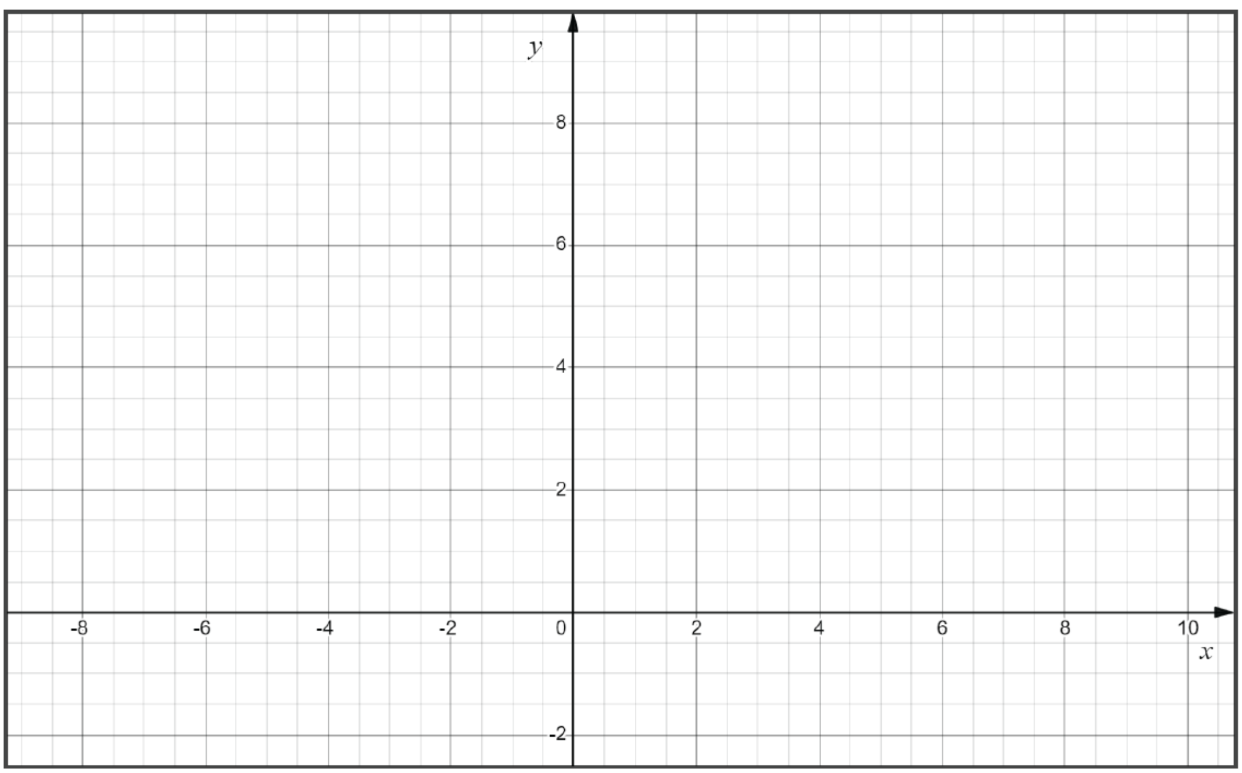
Consider $f(x) = \frac{a}{\sqrt{b-x}}, a, b \in R^{+}$¿¿

a. Determine the equation of $f^{-1}(x)$ and state its domain. 2 marks

b. If $f^{-1}(1) = 2$ and $f(5) = 2$, find the values of a and b. 2 marks

c. Hence sketch the graph of $f(x)$ on the axes provided. Include the coordinate of all intercepts and equations of any asymptotes. Hint: The graph of $f^{-1}(x)$ may be of assistance.

3 marks



Question 4 (3 marks)

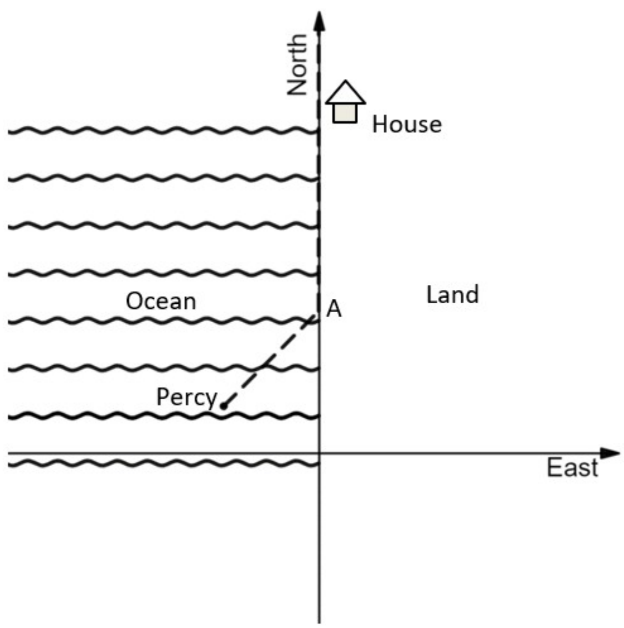
Consider $f(x)=2\cos(2x)$

Determine the area bound by $f(x)$, the x-axis, $x = 0$ and $x = 3\pi/4$.

Question 5 (6 marks)

Percy was swimming at the beach and suddenly remembered he had an appointment.

His location at that moment was 4 km West and 10 km south of his house, as shown in the diagram below.



Percy needed to get home as quickly as possible. He decided to swim in a straight line towards the beach, then run north along the beach towards home. His swimming speed is 3 km/hr and his running speed along the sandy beach is 4 km/hr.

Let A be the point where he reaches the beach and begins his run. Let x be the distance of A north of his original position.

a. If L is the total distance Percy travelled to get home, develop an equation for L in terms of x.

2 marks

b. Hence, develop an equation for T, the total time taken for Percy to reach the house.

1 mark

c. Determine the shortest time that Percy needs to get home. Express your answer in the form $\frac{a+b\sqrt{d}}{d}, a, b, c, d \in Z$

3 marks

Question 6 (5 marks)

A game of 'Doubles' involves a player rolling two regular six-sided dice. The player wins if the number rolled on one die, is double the number rolled on the other die. All other combinations result in a loss.

a. What is the probability that a player wins a particular game of Doubles? 2 marks

b. In a particular game of Doubles, one of the rolls was greater than 3. What is the probability that the player won the game? 1 mark

c. A game of Doubles costs \$1 to play. How much should the prize payment be so that the game is fair? A fair game is one where the average amount won or lost is \$0. 2 marks

Question 7 (4 marks)

X is a random variable representing the number of mosquito bites a hiker may receive while traversing the Swamp of Despair. The probability density function of X is provided below.

$$f(x) = ax(b - x), 0 < x < b$$

a. Assuming a hiker cannot receive more than 10 mosquito bites, determine the value of the parameters a and b in $f(x)$.

2 marks

b. Hence what is the probability that a hiker will receive not more than two mosquito bites when crossing the swamp? 2 marks

Question 8 (3 marks)

50 people from the city of Wombat were surveyed regarding their favourite breed of dog. A certain proportion, less than half, of the respondents indicated that they preferred Cocker Spaniels. Let $\hat{p}$ be the sample proportion of people who prefer Cocker Spaniels.

a. If the standard deviation of $\hat{p}$ is $\sqrt{42}/100$, how many of the 50 people surveyed preferred cocker spaniels? 2 marks

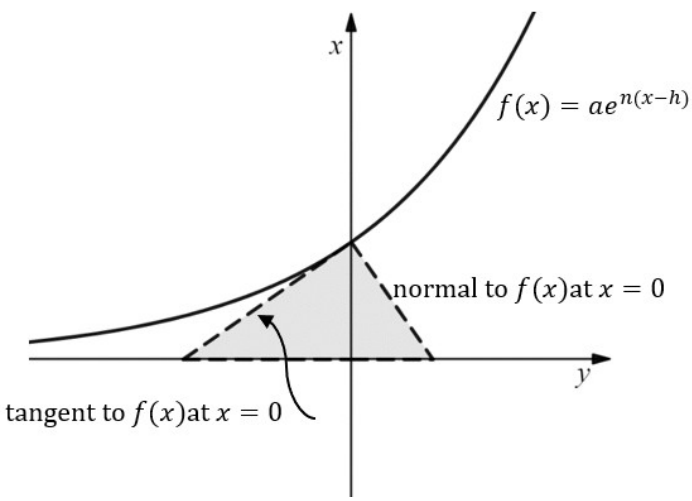
b. If the standard deviation of $\hat{p}$ was to be halved, how many people would need to be surveyed?

1 mark

Question 9 (6 marks)

Consider $f(x)=ae^{n(x-h)}, a,h,n\in R^{+¿¿}$

A triangle can be formed whose vertices correspond to the point of intersection of the tangent and normal lines at $x = 0$, the point of intersection of the tangent line with the x-axis and the point of intersection of the normal line with the x-axis. This triangle is shown in the graph below.



a. Show that the equation for the area of the triangle formed by the x-axis and tangent and normal lines of $f(x)$ at $x = 0$ is:

$$Area=\frac{a^3n^3+ae^{2nh}}{2ne^{3nh}}$$

3 marks

b. If the area of the triangle is k units squared, and $h = 0$ in $f(x)$, develop an equation that will provide the value of n for any given value of a and k. Include any restrictions on a.

3 marks

END OF QUESTION AND ANSWER BOOK

MATHEMATICAL METHODS

Written Examinations 1 and 2

FORMULA SHEET

Instructions
This Formula Sheet is provided for your reference.

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Mensuration

area of a trapezium	$\frac{1}{2}(x+b)h$	volume of a pyramid	$\frac{1}{3}Ah$
curved surface area of a cylinder	$2\pi rh$	volume of a sphere	$\frac{4}{3}\pi r^3$
volume of a cylinder	$\pi r^2 h$	area of a triangle	$\frac{1}{2}bc\sin(A)$
volume of a cone	$\frac{1}{3}\pi r^2 h$		

Calculus

$\frac{d}{dx}(x^n)=nx^{n-1}$	$\int x^n\,dx=\frac{1}{n+1}x^{n+1}+c, n\neq -1$
$\frac{d}{dx}((ax+b)^n)=an(ax+b)^{n-1}$	$\int$ ¿¿
$\frac{d}{dx}(e^{ax})=ax^{ax}$	$\int e^{ax}\,dx=\frac{1}{a}e^{ax}+c$
$\frac{d}{dx}(\log_e(x))=\frac{1}{x}$	$\int \frac{1}{x}\,dx=\log_e(x)+c$
$\frac{d}{dx}(\sin(ax))=a\cos(ax)$	$\int \sin(ax)\,dx=\frac{-1}{a}\cos(ax)+c$
$\frac{d}{dx}(\cos(ax))=-a\sin(ax)$	$\int \cos(ax)\,dx=\frac{1}{a}\sin(ax)+c$
$\frac{d}{dx}(\tan(ax))=\frac{a}{\cos^2(ax)}=a\,sec^2(ax)$	

product rule	$\frac{d}{dx}(uv)=u\frac{dv}{dx}+v\frac{du}{dx}$	quotient rule	$\frac{d}{dx}\left(\frac{u}{v}\right)=\frac{v\frac{du}{dx}-u\frac{dv}{dx}}{v^2}$
chain rule	$\frac{dy}{dx}=\frac{dy}{du}\frac{du}{dx}$		

Probability

$\Pr(A$ ¿ $=1-Pr\left(A'\right)$		$\Pr(A\cup B$ ¿ $=\Pr(A$ ¿ $+\Pr(B$ ¿ $-\Pr(A\cap B$ ¿	
$\Pr(A\|B)=\Pr(A\cap B$ ¿ $\frac{¿}{\Pr(B)}$			
mean	$\mu=E(X$ ¿	Variance	$\mathrm{var}(X$ ¿ $=\sigma^2=E(X^2$ ¿ $-\mu^2$

Probability distribution		Mean	Variance
discrete	$\Pr(X=x$ ¿ $=p(x)$	$u=\Sigma\, xp(x)$	$\sigma^2=E\left((X-\mu)^2\right)=E(X)^2-\mu^2$
continuous	$\Pr(a<X<b$ ¿ $=\int_a^b f(x)\,dx$	$\mu=\int_{-\infty}^{\infty} xf(x)\,dx$	$\sigma^2=\int_{-\infty}^{\infty}(x-\mu)^2 f(x)\,dx$

Sample proportions

$\hat{P}=\frac{X}{n}$		mean	$E\left(\hat{P}\right)=p$
Standard deviation	$sd\left(\hat{P}\right)=\sqrt{\frac{p(1-p)}{n}}$	approximate confidence interval	$\left(\hat{p}-z\sqrt{\frac{\hat{p}(1-\hat{p})}{n}},\hat{p}+z\sqrt{\frac{\hat{p}(1-\hat{p})}{n}}\right)$

VCE® Mathematical Methods

Unit 3 & 4 Trial Written Examination

EXAMINATION 1

ADVICE FOR TEACHERS

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Solution Pathway

Below are sample answers. Please consider the merit of alternative responses.

Question 1 (3 marks)

Let $f: R \to R, f(x) = \frac{\cos(x)}{\sin(2x)}$

a. Find *f'*(x) **1 mark**

Quotient rule.

Let $u = \cos(x), \frac{du}{dx} = -\sin(x)$

Let $v = \sin(2x), \frac{dv}{dx} = 2\cos(2x)$

$$f'(x) = \frac{v\frac{du}{dx} - u\frac{dv}{dx}}{v^2}$$

$$¿ \frac{\sin(2x) \times -\sin(x) - \cos(x) \times 2\cos(2x)}{\sin^2(2x)}$$

$$¿ \frac{-\sin(2x)\sin(x) - \cos(x)2\cos(2x)}{\sin^2(2x)}$$

- **1 mark** for correct derivative (any mathematically correct form acceptable)

b. Let $g: R \to R¿0\}, g(x) = 3\log_e(x)^2$. For what value of x is $g'(x) = 1$

2 marks

$g(x) = 3\log_e(x)^2$

$¿6\log_e(x).$

$g'(xh) = \frac{6}{x}$

At $g'(x) = 1$,

$\frac{6}{x} = 1$

$x=6$

Alternatively:

$g(x)=3\log_e(x)^2$

$Let\ y=3\log_e(x)^2$

Let $u=x^2$, $y=3\log_e(u)$

$\frac{du}{dx}=2x, \frac{dy}{du}=\frac{3}{u}$

$\frac{dy}{dx}=\frac{dy}{du}\times\frac{du}{dx}$

¿ $2x\times\frac{3}{u}$

¿ $2x\times\frac{3}{x^2}$

¿ $\frac{6}{x}$

Then as per above.

- **1 mark** for correct derivative of $g(x)$.
- **1 mark** for correct value of x for when $g'(x) = 1$

Question 2 (3 marks)

Given $x \in [0, 8\pi]$,solve the equation:

$$\left(\sin\left(\frac{x}{2}\right)\right)^2 = 2\sin\left(\frac{x}{2}\right) - 1$$

$$\left(\sin\left(\frac{x}{2}\right)\right)^2 - 2\sin\left(\frac{x}{2}\right) + 1 = 0$$

$$Let\, k = \sin\left(\frac{x}{2}\right)$$

$$\therefore k^2 - 2k + 1 = 0$$

$$(k-1)^2 = 0$$

$$k = 1$$

$$\therefore \sin\left(\frac{x}{2}\right) = 1$$

$$\therefore \frac{x}{2} = \frac{\pi}{2}$$

$$x = \pi$$

Check for other solutions within the domain [0, 8π].

$$Period = \frac{2\pi}{n} = \frac{2\pi}{\frac{1}{2}} = 4\pi$$

$$x = \pi + 4\pi$$

$$¿ 5\pi$$

No more solutions since adding 4π to 5π will yield 9π, which is outside the domain.

$$x = \{\pi, 5\pi\}$$

- **1 mark** for a valid approach to solving equation.
- **1 mark** for a valid consideration of the domain
- **1 mark** for correct values of *x*.

Question 3 (7 marks)

Consider $f(x)=\frac{a}{\sqrt{b-x}}, a, b \in R^{+}$¿¿

a. Determine the equation of $f^{-1}(x)$ and state its domain. **2 marks**

Let $y = f(x)$

Substitute x and y and rearrange to make y the subject.

$$x=\frac{a}{\sqrt{b-y}}$$

$$x^2=\frac{a^2}{b-y}$$

$$b-y=\frac{a^2}{x^2}$$

$$y=b-\frac{a^2}{x^2}$$

$$\therefore f^{-1}(x)=b-\frac{a^2}{x^2}$$

Domain of $f^{-1}(x)=$ range of $f(x)$

Range of $f(x) = (0, \infty)$

Domain $f^{-1}(x) = (0, \infty)$

- **1 mark** for correct equation for $f^{-1}(x)$
- **1 mark** for correct domain of $f^{-1}(x)$.

b. If $f^{-1}(1) = 2$ and $f(5) = 2$, find the values of *a* and *b*. **2 marks**

$f^{-1}(1) = 2$

$\therefore 2 = b - a^2 1.$

$f(5) = 2$

$\therefore 2 = \frac{a}{\sqrt{b-5}} 2.$

From $1., b = a^2 + 2$

$Substitute\, a^2 + 2\, for\, b \in 2.$

$2 = \frac{a}{\sqrt{a^2+2-5}}$

$2 = \frac{a}{\sqrt{a^2-3}}$

$4 = \frac{a^2}{a^2-3}$

$4a^2 - 12 = a^2$

$3a^2 = 12$

$a^2 = 4$

$a = 2$¿

$Since\, b = a^2 + 2, b = 6$

$a = 2, b = 6$

- **1 mark** for correct value of *a*.
- **1 mark** for correct value of *b*.

c. Sketch the graph of $f(x)$ on the axes provided. Include the coordinate of all intercepts and equations of any asymptotes. Hint: The graph of $f^{-1}(x)$ may be of assistance. **3 marks**

Consider $f^{-1}(x) = 6 - \frac{4}{x^2}, x > 0$

Truncus reflected about the x-axis, dilated by a factor of 4 parallel to y-axis and translated 6 units up.

Horizontal asymptote at $y = 6$. Vertical asymptote at $x = 0$.

No y-intercept.

x-intercept at $f^{-1}(x) = 0$

$$6 - \frac{4}{x^2} = 0$$

$$\frac{4}{x^2} = 6$$

$$x^2 = \frac{2}{3}$$

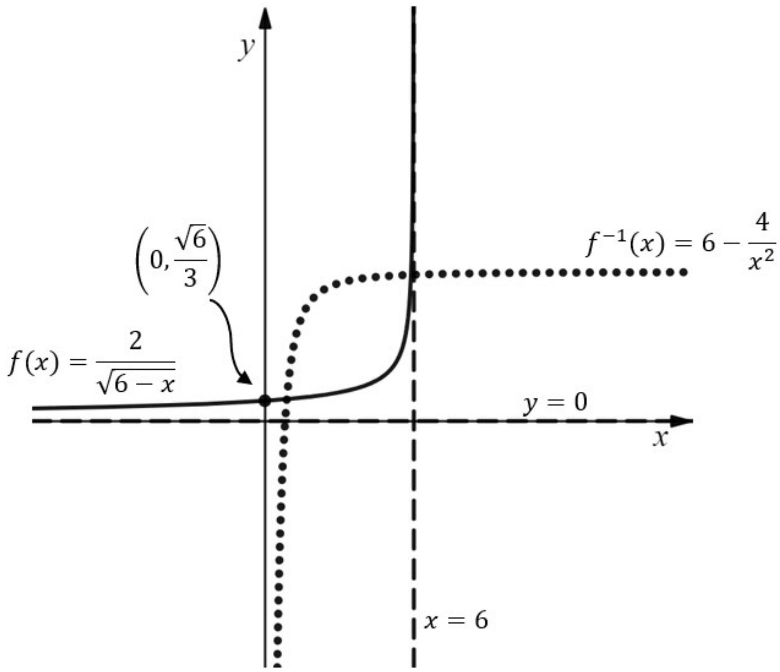
$$x = \frac{\sqrt{2}}{\sqrt{3}} = \frac{\sqrt{6}}{3} \ (since\ x > 0)$$

$$x\text{-}intercept\ at\ \left(\frac{\sqrt{6}}{3}, 0\right)$$

$\therefore f(x)$ *has a vertical asymptote at* $x = 6 \wedge$ *horizintal asymptote are* $y = 0$.

$\therefore f(x)$ has a y-intercept at $\left(0. \frac{\sqrt{6}}{3}\right)$

$f(x)$ is a reflection of $f^{-1}(x)$ about $y = x$.



- **1 mark** for correct position and shape of graph.
- **1 mark** for correct *y*-intercept.
- **1 mark** for asymptotes shown with correct equations.

Question 4 (3 marks)

Consider $f(x) = 2\cos(2x)$

Determine the area bound by $f(x)$, the x-axis, $x = 0$ and $x = 3\pi/4$.

Need to determine how the relevant area is sectioned within the specified domain.

No horizontal translation of the graph so section between $x = 0$ and the first x-intercept is above the x-axis.

x-intercepts at $f(x) = 0$

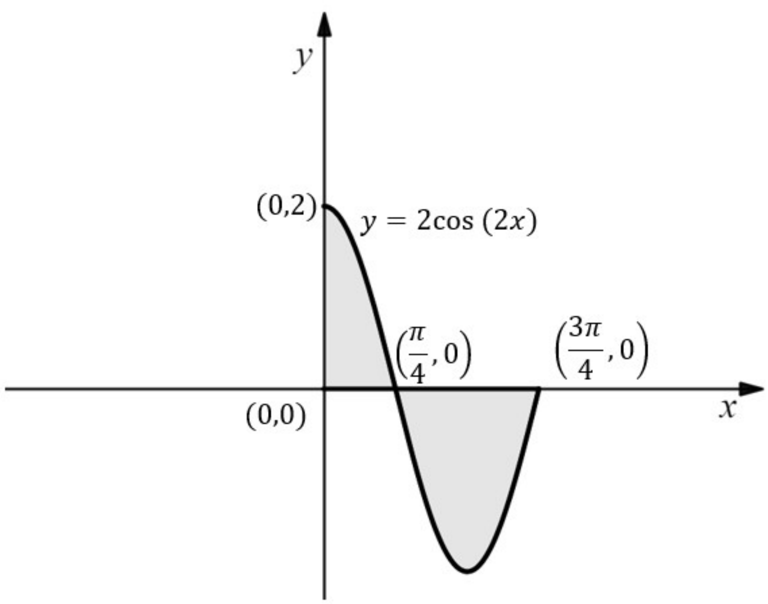
$2\cos(2x) = 0$

$\cos(2x) = 0$

$x = \frac{\pi}{4}, \frac{5\pi}{4}$ etc

$\therefore$ *The* section between $x = \frac{\pi}{4}$ and $x = \frac{3\pi}{4}$ is below the x-axis.

The graph below shows the relevant sections.



$$\text{Area} = \int_0^{\frac{\pi}{4}} 2\cos(2x)\,dx + \int_{\frac{3\pi}{4}}^{\frac{\pi}{4}} 2\cos(2x)\,dx$$

Note: terminals reversed in second definite integral to make area positive.

¿ $\left[\sin(2x)\right]_{0}^{\frac{\pi}{4}} + \left[\sin(2x)\right]_{\frac{3\pi}{4}}^{\frac{\pi}{4}}$

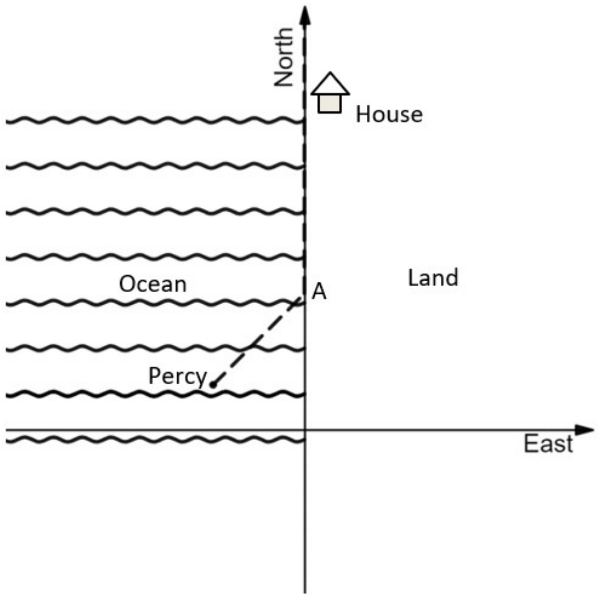
¿ $(1-0)+(1--1)$

¿ 3 units squared

- **1 mark** for correct terminals.
- **1 mark** for correct equation for area involving definite integrals.
- **1 mark** for correct area.

Question 5 (6 marks)

Percy was swimming at the beach and suddenly remembered he had an appointment. His location at that moment was 4 km West and 10 km south of his house, as shown in the diagram below.



Percy needed to get home as quickly as possible. He decided to swim in a straight line towards the beach, then run north along the beach towards home. His average swimming speed is 3 km/hr and his running speed along the sandy beach is 4 km/hr.

Let *A* be the point where he reaches the beach and begins his run. Let *x* be the distance of *A* north of his original position.

a. If *L* is the total distance Percy travelled to get home, develop an equation for *L* in terms of *x*.

2 marks

Let L_1 be the straight-line distance to point *A* from his initial location. L_1 can be determined using Pythagoras' Theorem.

$$L_1^2 = x^2 + 4^2$$

$$L_1 = \sqrt{x^2 + 16}$$

Let L_2 be the distance from point *A* to the house.

$L_2 = 10 - x$

$$\therefore L = L_1 + L_2$$

$$¿\sqrt{x^2 + 16} + 10 - x$$

- **1 mark** for correct expressions for the component distances.
- **1 mark** for a correct expression for the total distance.

b. Hence, develop an equation for *T*, the total time taken for Percy to reach the house.

1 mark

$$\text{speed} = \frac{\text{distance travelled}}{\text{time}}$$

$$\text{time} = \frac{\text{distance travelled}}{\text{speed}}$$

$$\therefore T = \frac{\sqrt{x^2+16}}{3} + \frac{10-x}{4}$$

- **1 mark** for a correct equation for the time taken.

c. Determine the shortest time that Percy needs to get home. Express your answer in the form $\frac{a+b\sqrt{d}}{d}, a, b, c, d \in Z 3 marks$

Solve $\frac{dT}{dx} = 0$

$$T = \frac{\sqrt{x^2+16}}{3} + \frac{10-x}{4}$$

Let $y = \frac{\sqrt{x^2+16}}{3}$

Let $u = x^2 + 16$ $y = \frac{\sqrt{u}}{3} = \frac{1}{3}u^{\frac{1}{2}}$

$$\frac{du}{dx} = 2x \quad \frac{dy}{du} = \frac{1}{6}u^{\frac{-1}{2}} = \frac{1}{6\sqrt{u}}$$

$$\frac{dy}{dx} = \frac{dy}{du} \times \frac{du}{dx}$$

$$¿ \frac{1}{6\sqrt{u}} \times 2x$$

$$¿ \frac{x}{3\sqrt{x^2+16}}$$

$$\therefore \frac{dT}{dx} = \frac{x}{3\sqrt{x^2+16}} - \frac{1}{4}$$

$$\frac{x}{3\sqrt{x^2+16}}-\frac{1}{4}=0$$

$$\frac{x}{3\sqrt{x^2+16}}=\frac{1}{4}$$

$$4x=3\sqrt{x^2+16}$$

$$16x^2=9\left(x^2+16\right)$$

$$16x^2=9x^2+144$$

$$7x^2=144$$

$$x^2=\frac{144}{7}$$

$$x=\frac{12}{\sqrt{7}}=\frac{12\sqrt{7}}{7}$$

$$\textit{Point A located on the beach } \frac{12\sqrt{7}}{7} \textit{ km north of his original position.}$$

$$T=\frac{\sqrt{x^2+16}}{3}+\frac{10-x}{4}$$

$$¿\frac{\sqrt{\left(\frac{12\sqrt{7}}{7}\right)^2+16}}{3}+\frac{10-\frac{12\sqrt{7}}{7}}{4}$$

$$¿\frac{\sqrt{\frac{144}{7}+16}}{3}+\frac{\frac{70}{7}-\frac{12\sqrt{7}}{7}}{4}$$

$$¿\frac{\sqrt{\frac{144}{7}+\frac{112}{7}}}{3}+\frac{\frac{70-12\sqrt{7}}{7}}{4}$$

$$¿\frac{\sqrt{\frac{256}{7}}}{3}+\frac{\frac{70-12\sqrt{7}}{7}}{4}$$

¿ $\frac{16\sqrt{\frac{1}{7}}}{3}+\frac{\frac{70-12\sqrt{7}}{7}}{4}$

¿ $\frac{16}{3\sqrt{7}}+\frac{70-12\sqrt{7}}{28}$

¿ $\frac{16\sqrt{7}}{21}+\frac{35-6\sqrt{7}}{14}$

¿ $\frac{32\sqrt{7}}{42}+\frac{105-18\sqrt{7}}{42}$

¿ $\frac{105+14\sqrt{7}}{42}$

¿ $\frac{15+2\sqrt{7}}{6}$ *hours*

- **1 mark** for correct derivative.
- **1 mark** for correct value of *x*.
- **1 mark** for correct time expressed in the required form.

Question 6 (5 marks)

A game of ‘Doubles’ involve a player rolling two regular six-sided dice. If the number rolled on one die is double the number rolled on the other, the player wins. All other combinations result in a loss.

a. What is the probability that a player wins a particular game of Doubles? **2 marks**

The winning roles are:

(1,2), (2,1), (2,4), (4,2), (3,6) and (6, 3). Thus, there are six ways to win.

The total number of possible roles is 36.

$$Pr(\text{Win}) = \frac{6}{36} = \frac{1}{6}$$

- **1 mark** for determination of all winning combinations.
- **1 mark** for correct probability.

b. In a particular game of Doubles, one of the rolls was greater than 3. What is the probability that the player won the game? **1 mark**

This is a conditional probability problem. Let the die that produced a number greater than 3 be ‘Roll 1’.

$$Pr(\text{Win}|\text{Roll }1<4) = \frac{Pr(\text{Win} \cap \text{Roll }1>3)}{Pr(\text{Roll }1>3)}$$

$$¿ \frac{\frac{2}{27}}{\frac{1}{2}}$$

$$¿ \frac{4}{27}$$

- **1 mark** for correct probability.

c. A game of doubles costs $1 to play. How much should the prize payment be so that the game is fair? A fair game is one where the average amount won or lost is $0. **2 marks**

For a game to be fair, expected value of prize amount is 0. Let *X* be the random variable that corresponds to the amount won or lost in a game. Let *P* be the prize amount and *C* be the cost to play.

$$E(X)=C\times Pr(losing)+P\times Pr\ (winning)$$

$$0=-1\times\frac{5}{6}+P\times\frac{1}{6}$$

$$P\times\frac{1}{6}=\frac{5}{6}$$

$$P=\$5$$

- **1 mark** for an Expected value equation or equivalent.
- **1 mark** for the correct prize amount.

Question 7 (4 marks)

X is a random variable representing the number of mosquito bites a hiker may receive while traversing the Swamp of Despair. The probability density function of X is provided below.

$$f(x)=ax(b-x),0<x<b$$

a. Assuming a hiker cannot receive more than 10 mosquito bites, determine the value of the parameters a and b in $f(x)$. **2 marks**

Since the maximum number of mosquito bites is 10, $b = 10$.

For $f(x)$ to be a pdf, area under $f(x)$ must equal one.

$$Area=\int_0^{10} ax(10-x)\,dx$$

$$1=\int_0^{10} 10ax-ax^2\,dx$$

¿ $\left[5ax^2-\frac{1}{3}ax^3\right]_0^{10}$

¿ $\left(500a-\frac{1000a}{3}\right)-0$

¿ $\frac{1500a}{3}-\frac{1000a}{3}$

¿ $\frac{500a}{3}$

$$500a=3$$

$$a=\frac{3}{500}$$

$$\therefore a=\frac{3}{500},b=10$$

- **1 mark** for a correct definite integral.
- **1 mark** for correct parameter values.

b. Hence what is the probability that a hiker will receive not more than two mosquito bites when crossing the swamp? **2 marks**

Evaluate:

$$Pr(X \leq 2) = \int_0^2 \frac{3}{500} x(10-x)\,dx$$

¿ $\left[\frac{15x^2 - x^3}{500}\right]_0^2$

¿ $\left(\frac{60-8}{500}\right) - 0$

¿ $\frac{52}{500}$

¿ $\frac{13}{125}$

- **1 mark** for a valid definite integral.
- **1 mark** for correct probability.

Question 8 (3 marks)

50 people from the city of Wombat were surveyed regarding their favourite breed of dog. A certain proportion, less than half, of the respondents indicated that they preferred cocker spaniels. Let $\hat{p}$ be the sample proportion of people who prefer cocker spaniels.

a. The standard deviation of $\hat{p}$ is $\sqrt{42}/100$, how many of the 50 people surveyed preferred cocker spaniels? **2 marks**

$$sd(\hat{p})=\sqrt{\frac{\hat{p}(1-\hat{p})}{n}}$$

$$\frac{\sqrt{42}}{100}=\sqrt{\frac{\hat{p}(1-\hat{p})}{50}}$$

$$\frac{42}{10000}=\frac{\hat{p}(1-\hat{p})}{50}$$

$$\frac{21}{100}=\hat{p}-\hat{p}^2$$

$$\hat{p}^2-\hat{p}+\frac{21}{100}=0$$

$$100\hat{p}^2-100\hat{p}+21=0$$

$$(10\hat{p}-7)(10\hat{p}-3)=0$$

$$\hat{p}=\frac{7}{10}\text{ or }\frac{3}{10}$$

$\hat{p}=\frac{3}{10}$, since it is known that less than half prefer cocker spaniels.

- **1 mark** for correctly relating given information to equation for standard deviation of $\hat{p}$.
- **1 mark** for correct value of $\hat{p}$.

b. If the standard deviation of $\hat{p}$ was to be halved, how many people would need to be surveyed?

1 mark

$$sd(\hat{p})=\frac{\sqrt{42}}{200}$$

$$sd(\hat{p})=\sqrt{\frac{\hat{p}(1-\hat{p})}{n}}$$

$$\frac{\sqrt{42}}{200}=\sqrt{\frac{\widehat{\frac{3}{10}}\left(1-\frac{3}{10}\right)}{n}}$$

$$\frac{42}{40000}=\frac{\frac{21}{100}}{n}$$

$$\frac{21}{20000}=\frac{21}{100n}$$

$$\frac{1}{200}=\frac{1}{n}$$

$$n=200$$

200 people would need to be surveyed to halve the standard deviation of $\hat{p}$.

- **1 mark** for correct value of $\hat{p}$.

Question 9 (6 marks)

Consider $f(x)=ae^{n(x-h)}, a, h, n \in R^{+¿¿}$

a. Show that the equation for the area of the triangle formed by the *x*-axis and tangent and normal lines of *f*(*x*) at *x* = 0 is: **3 marks**

$$Area=\frac{a^3n^3+ae^{2nh}}{2ne^{3nh}}$$

Find equation of the tangent at *x* = 0

$f'(x)=ane^{n(x-h)}$

$f'(0)=ane^{-nh}=\frac{an}{e^{nh}}$

$f(0)=ae^{-nh}=\frac{a}{e^{nh}}=y_1$

$y-y_1=m(x-x¿¿1)¿$

$y-\frac{a}{e^{nh}}=\frac{an}{e^{nh}}x$

$y=\frac{an}{e^{nh}}x+\frac{a}{e^{nh}}$

Find *x*-intercept of tangent line.

$\frac{an}{e^{nh}}x+\frac{a}{e^{nh}}=0$

$\frac{an}{e^{nh}}x=\frac{-a}{e^{nh}}$

$anx=-a$

$x=\frac{-1}{n}$

Find equation of the normal at *x* = 0

$f'_N(0)=\frac{-1}{f'(0)}$

$¿-\frac{e^{nh}}{an}$

$$f(0)=ae^{-nh}=\frac{a}{e^{nh}}=y_1$$

$y-y_1=m(x-x$¿¿$1)$¿

$$y-\frac{a}{e^{nh}}=\frac{-e^{nh}}{an}x$$

$$y=\frac{-e^{nh}}{an}x+\frac{a}{e^{nh}}$$

Find *x*-intercept of normal line.

$$\frac{-e^{nh}}{an}x+\frac{a}{e^{nh}}=0$$

$$\frac{e^{nh}}{an}x=\frac{a}{e^{nh}}$$

$$xe^{nh}=\frac{a^2n}{e^{nh}}$$

$$x=\frac{a^2n}{e^{2nh}}$$

$$Base\,of\,triangle=\frac{1}{n}+\frac{a^2n}{e^{2nh}}$$

¿ $\dfrac{e^{2nh}+a^2n^2}{ne^{2nh}}$

$$Height\,of\,triangle=\frac{a}{e^{nh}}$$

$$\text{Area}=\frac{1}{2}bh$$

¿ $\dfrac{1}{2}\left(\dfrac{e^{2nh}+a^2n^2}{ne^{2nh}}\right)\dfrac{a}{e^{nh}}$

¿ $\dfrac{ae^{2nh}+a^3n^2}{2ne^{3nh}}$

¿ $\dfrac{a^3n^2+ae^{2nh}}{2ne^{3nh}}$

- **1 mark** for correct tangent and normal lines

- **1 mark** for correct x-intercepts of tangent and normal lines.
- **1 mark** for simplification of expression for area to that required.

b. If the area of the triangle is k units squared, and $h = 0$ in $f(x)$, develop an equation that will provide the value of n for any given value of a and k. Include any restrictions on a. **3 marks**

$$Area = \frac{a^3 n^2 + a e^{2nh}}{2ne^{3nh}}$$

$$k = \frac{a^3 n^2 + a}{2n}$$

$$2kn = a^3 n^2 + a$$

$$a^3 n^2 - 2kn + a = 0$$

Solve using quadratic formula.

$$n = \frac{2k \pm \sqrt{4k^2 - 4a^4}}{2a^3}$$

¿ $\frac{2k \pm 2\sqrt{k^2 - a^4}}{2a^3}$

¿ $\frac{k \pm \sqrt{k^2 - a^4}}{a^3}, a < \sqrt{k}$

- **1 mark** for an equation relating k to a and n.
- **1 mark** for a valid solution to the equation (n is the subject).
- **1 mark** for specification of conditions.