The Mathematical Association of Victoria

Trial Examination 2023

MATHEMATICAL METHODS

Trial Written Examination 1 - SOLUTIONS

Question 1

a. $\frac{d}{dx}\left(x^3 \sin(2x)\right)$

$= x^3 \times 2\cos(2x) + \sin(2x) \times 3x^2$

$= 2x^3 \cos(2x) + 3x^2 \sin(2x)$ **1A**

b. $f(x) = \frac{e^{2x}}{2x+1}$

$f'(x) = \frac{(2x+1)2e^{2x} - e^{2x} \times 2}{(2x+1)^2}$ **1M**

$f'(2) = \frac{10e^4 - 2e^4}{5^2}$

$= \frac{8e^4}{25}$ **1A**

Question 2

$f:(-2,\infty) \to R, f(x) = \frac{1}{x+2}, \; g:(3,\infty) \to R, g(x) = \frac{1}{x-3}$

a. $f \circ g$ exists because the range of g, R^+, is a subset of the domain of f, $(-2,\infty)$. **1A**

b. $(f \circ g)(x) = \frac{1}{\frac{1}{x-3} + 2}$ **1M**

$= \frac{x-3}{1+2(x-3)}$

$= \frac{x-3}{2x-5}$

$= \frac{x - \frac{5}{2} - \frac{1}{2}}{2x-5}$

$= \frac{2x-5}{2(2x-5)} - \frac{1}{4x-10}$

$= \frac{1}{2} - \frac{1}{4x-10}$ **1M** *show that*

OR

$$2x-5\overline{)x-3}\quad \frac{1}{2}$$

$$x-\frac{5}{2}$$

$$-\frac{1}{2}$$

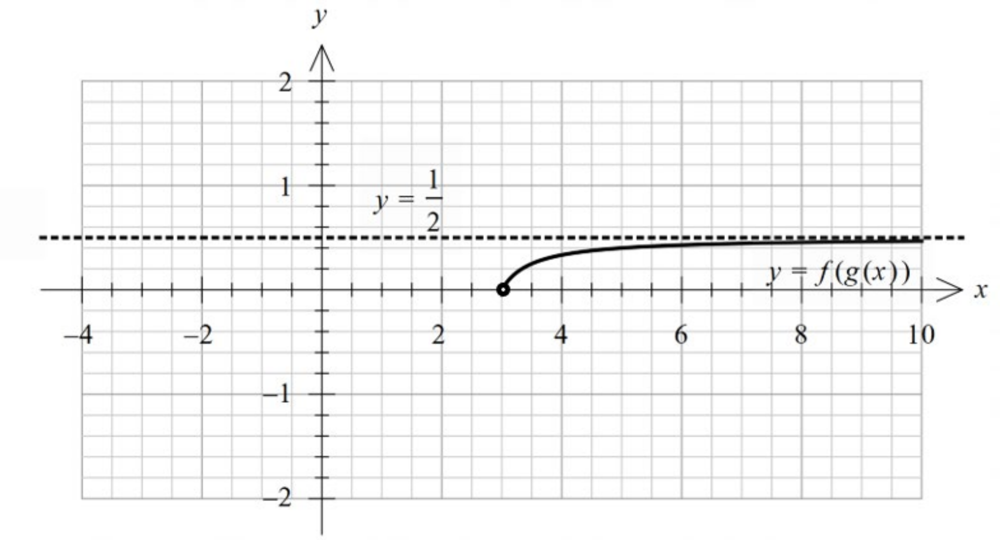
$$\frac{x-3}{2x-5}=\frac{1}{2}-\frac{\frac{1}{2}}{2x-5}$$

$$=\frac{1}{2}-\frac{1}{4x-10}$$ **1M** *show that*

c. The domain of $f \circ g$ is the same as the domain of g.

Shape with open circle at (3, 0) **1A**

Asymptote with equation **1A**



Question 3

a. $g(x)=\int \frac{1}{(3x+1)^2}\,dx$

$$=\int (3x+1)^{-2}\,dx$$

$$=\frac{(3x+1)^{-1}}{3\times -1}+c$$

$$=\frac{(3x+1)^{-1}}{-3}+c$$

$$g(x)=-\frac{1}{3(3x+1)}+c$$ **1M**

given $g(0)=3$

$$3=-\frac{1}{3}+c$$

$$c=\frac{10}{3}$$

$g(x) = -\frac{1}{3(3x+1)} + \frac{10}{3}$ **1A**

b. $\int_{-\frac{\pi}{18}}^{\frac{\pi}{4}} \left(-\frac{1}{2}\sin(6x)\right) dx$

$= \left[\frac{1}{12}\cos(6x)\right]_{-\frac{\pi}{18}}^{\frac{\pi}{4}}$

$= \frac{1}{12}\left(\cos\left(\frac{3\pi}{2}\right) - \cos\left(-\frac{\pi}{3}\right)\right)$ **1M**

$= \frac{1}{12}\left(0 - \frac{1}{2}\right)$

$= -\frac{1}{24}$ **1A**

Question 4

a.i. As the y-coordinates of the points of inflection are 1 and the y-coordinates of the maximum turning points are 4, the graph of $y = \cos(x)$ has been dilated by a factor of $4 - 1 = 3$ from the x-axis (amplitude is 3) and translated 1 unit up. So $y_1 = 3\cos(x) + 1$.

Hence, $A = 3$ and $c = 1$. **1A**

a.ii. Solve $nx + \frac{1}{4} = 0$ for $x = -1$

$n = \frac{1}{4}$ **1A**

OR

As the turning point is at $(-1, 4)$, $y_1 = 3\cos(x) + 1$ has been translated 1 unit left. $f(x)$ can be written in the form $f(x) = A\cos\left(\frac{1}{4}(x+1)\right) + c$.

$n = \frac{1}{4}$ **1A**

Period $= \frac{2\pi}{\frac{1}{4}} = 8\pi$, so a point of inflection will be at $x = -1 + \frac{\text{period}}{4} = -1 + 2\pi$

Hence the general solution for the x-coordinates of the points of inflection is $x = -1 + 2\pi + 4\pi k$, $k \in Z$. **1A**

OR

Solve $3\cos\left(\frac{1}{4}x + \frac{1}{4}\right) + 1 = 1$

$3\cos\left(\frac{1}{4}x + \frac{1}{4}\right) = 0$

$\cos\left(\frac{1}{4}x + \frac{1}{4}\right) = 0$

$\frac{1}{4}x + \frac{1}{4} = \cos^{-1}(0)$

$\frac{1}{4}x+\frac{1}{4}=\frac{\pi}{2},...$

$x=-1+2\pi$, …

Hence the general solution for the x-coordinates of the points of inflection is $x=-1+2\pi+4\pi k$, $k\in Z$. **1A**

b. $f(x)=3\cos\left(\frac{1}{4}x+\frac{1}{4}\right)+1$ to $g(x)=\cos\left(\frac{1}{4}x+\frac{1}{4}\right)+1$

- Dilate by a factor $\frac{1}{3}$ of from the x-axis
- Translate $\frac{2}{3}$ of a unit up

(Dilate by a factor $\frac{1}{A}$ from the x-axis and translate $c-\frac{c}{A}$ units up) **1H**

OR

- Translate 1 unit down
- Dilate by $\frac{1}{3}$ from x-axis
- Translate 1 unit up

(Translate c units down, dilate by a factor of $\frac{1}{A}$ from the x-axis, translate c units up) **1H**

Question 5

a. $f(x)=x^3+2x^2-x+1$

$f'(x)=3x^2+4x-1=0$ **1M**

$x=\frac{-4\pm\sqrt{16+12}}{6}$

$x=\frac{-4\pm 2\sqrt{7}}{6}$

$x=\frac{-2\pm\sqrt{7}}{3}$ **1A**

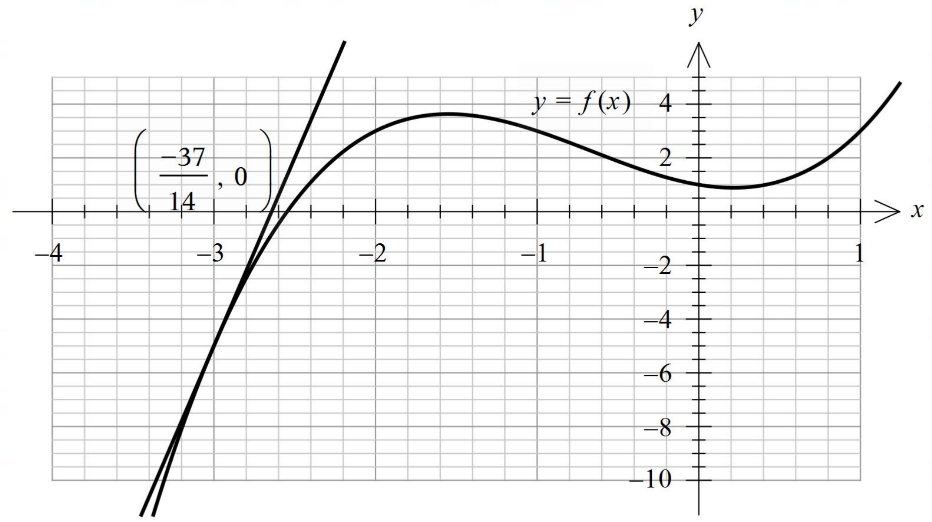
b. $x_1=-3-\frac{f(-3)}{f'(-3)}$ **1M**

$x_1=-3-\frac{-5}{14}$

$x_1=-2\frac{9}{14}=-\frac{37}{14}$ **1A**

c. Tangent line with the coordinates of the x-intercept **1A**

Must touch at $(-3,-5)$.



Question 6

a. $v(t)=(t-8)^{\frac{2}{3}}+1$

$a=v'(t)=\frac{2}{3}(t-8)^{-\frac{1}{3}}$

$\frac{2}{3}(t-8)^{-\frac{1}{3}}=1$ **1M**

$(t-8)^{-\frac{1}{3}}=\frac{3}{2}$

$t-8=\left(\frac{3}{2}\right)^{-3}$

$t-8=\left(\frac{2}{3}\right)^{3}$

$t=\left(\frac{2}{3}\right)^{3}+8$

$t=\frac{8}{27}+8$

$t=\frac{224}{27}=8\frac{8}{27}$ **1A**

b. $v(t)=(t-8)^{\frac{2}{3}}+1$

Average rate of change $=\frac{v(35)-v(0)}{35-0}$ **1M**

$=\frac{(35-8)^{\frac{2}{3}}+1-(-8)^{\frac{2}{3}}-1}{35}$

$=\frac{(27)^{\frac{2}{3}}-(-8)^{\frac{2}{3}}}{35}$

$=\frac{9-4}{35}$

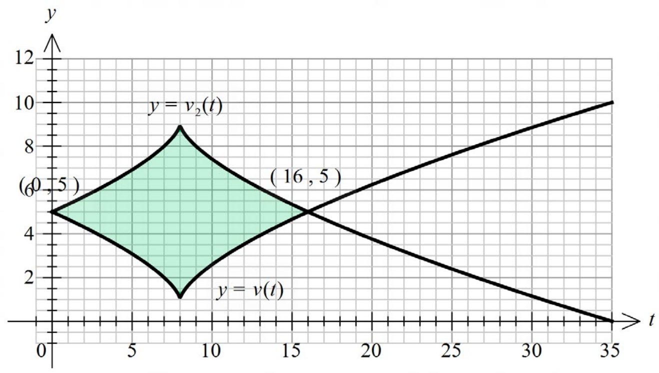
$= \frac{5}{35} = \frac{1}{7}$ **1A**

c. The graphs intersect at $t = 0$ and $t = 16$.

The reflection of the graph of v in the x-axis gives the endpoint at $(0, -5)$.

A translation of 10 units up gives the endpoint as (0, 5).

So the graphs intersect at $t = 0$ and 8 units to the right of $t = 8$.



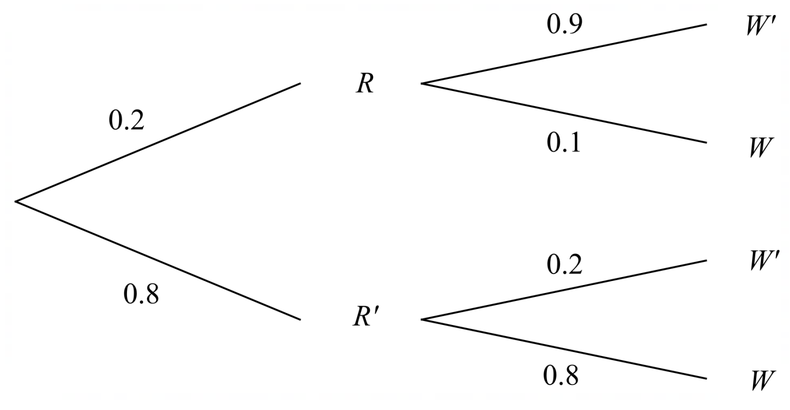
The bounded area $= \int_0^{16} \left(v_2(t) - v(t)\right)dt$ or $2\int_0^{16} \left(5 - v(t)\right)dt$ or $4\int_0^{8} \left(5 - v(t)\right)dt$ (other forms)

$= \int_0^{16} \left(-(t-8)^{\frac{2}{3}} + 9 - \left((t-8)^{\frac{2}{3}} + 1 \right) \right) dt$ or $2\int_0^{16} \left(5 - \left((t-8)^{\frac{2}{3}} + 1 \right) \right) dt$ or $4\int_0^{8} \left(5 - \left((t-8)^{\frac{2}{3}} + 1 \right) \right) dt$

$= \int_0^{16} \left(-2(t-8)^{\frac{2}{3}} + 8 \right) dt$ or $2\int_0^{16} \left(4 - (t-8)^{\frac{2}{3}} \right) dt$ or $4\int_0^{8} \left(4 - (t-8)^{\frac{2}{3}} \right) dt$ **1A**

Question 7

a.



$\Pr(R \mid W)$

$= \frac{\Pr(R \cap W)}{\Pr(W)}$

$= \frac{0.2 \times 0.1}{0.2 \times 0.1 + 0.8 \times 0.8}$ **1M**

$= \frac{0.02}{0.02 + 0.64}$

$=\dfrac{0.02}{0.66}=\dfrac{1}{33}$ **1A**

b. $f(t)=\begin{cases}\dfrac{3}{5}\sqrt{t} & 0\le t\le 1\\ \dfrac{3e}{5}e^{-t} & 1<t<\infty\\ 0 & t<0\end{cases}$

$\Pr(T\le 1)=\dfrac{3}{5}\int_0^1\left(t^{\frac{1}{2}}\right)dt$

$=\dfrac{3}{5}\times\dfrac{2}{3}\left[t^{\frac{3}{2}}\right]_0^1$

$=\dfrac{3}{5}\times\dfrac{2}{3}(1-0)$

$=\dfrac{2}{5}$ **1M** *Show that*

c. $\dfrac{3}{5}\int_0^1\left(t^{\frac{1}{2}}\right)dt+\dfrac{3}{5}e\int_1^m\left(e^{-t}\right)dt=\dfrac{1}{2}$

$\dfrac{3}{5}e\int_1^m\left(e^{-t}\right)dt=\dfrac{1}{2}-\dfrac{2}{5}=\dfrac{1}{10}$ **1M**

$-\dfrac{3}{5}e\left[e^{-t}\right]_1^m=\dfrac{1}{10}$

$e^{-m}-e^{-1}=-\dfrac{1}{6e}$ **1M**

$e^{-m}=\dfrac{5}{6e}$

$m=-\log_e\left(\dfrac{5}{6e}\right)=1-\log_e\left(\dfrac{5}{6}\right)=1+\log_e\left(\dfrac{6}{5}\right)$ **1A** (any form)

OR

$\dfrac{3}{5}e\int_m^\infty\left(e^{-t}\right)dt=\dfrac{1}{2}$ **1M**

$-\dfrac{3}{5}e\left[e^{-t}\right]_m^\infty=\dfrac{1}{2}$

$e^{-\infty}-e^{-m}=-\dfrac{5}{6e}$ **1M**

$-e^{-m}=-\dfrac{5}{6e}$

$m=-\log_e\left(\dfrac{5}{6e}\right)=1-\log_e\left(\dfrac{5}{6}\right)=1+\log_e\left(\dfrac{6}{5}\right)$ **1A** (any form)

Question 8

$g(x)=a\log_2\left(x+b\right)$, $g(2)=6$, $g(6)=9$

$a\log_2(2+b)=6 \ldots(1)$

$a\log_2(6+b)=9 \ldots(2)$

(2) $\div$ (1)

$\dfrac{\log_2(6+b)}{\log_2(2+b)}=\dfrac{3}{2}$ **1M**

$\log_2(6+b)=\dfrac{3}{2}\log_2(2+b)$

$6+b=(2+b)^{\frac{3}{2}}$ **1M**

Square both sides

$(6+b)^2=(2+b)^3$

$36+12b+b^2=b^3+6b^2+12b+8$

$b^3+5b^2-28=0$

When $b=2$, $b^3+5b^2-28=8+20-28=0$

$(b-2)(b^2+7b+14)=0$

$b^2+7b+14=0$

$\Delta=49-56=-7$, no real solution

$b=2$ **1A**

$a\log_2(2+2)=6$

$2a=6$

$a=3$ **1H**

Question 9

a. $h:(-\infty,a)\to R, h(x)=\dfrac{1}{(x-a)^2}$

Let $y=\dfrac{1}{(x-a)^2}$

Inverse swap *x* and *y*

$x=\dfrac{1}{(y-a)^2}$

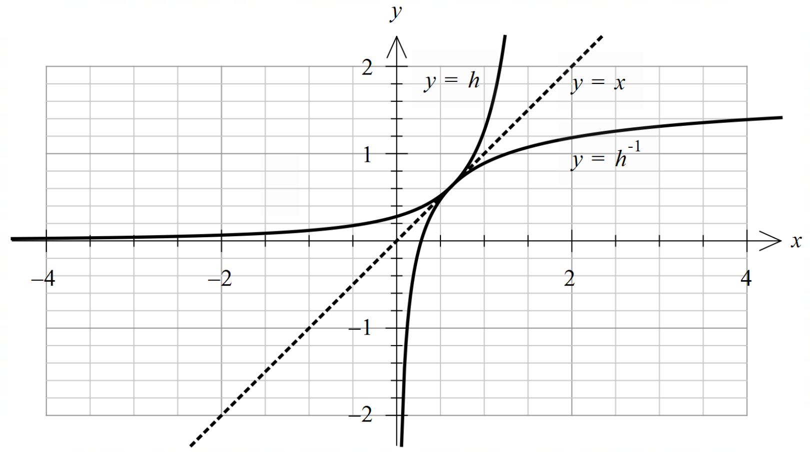
$(y-a)^2=\dfrac{1}{x}$

$y=-\sqrt{\dfrac{1}{x}}+a$, $y\neq\sqrt{\dfrac{1}{x}}+a$ as the domain of *h* is $(-\infty,a)$

$h^{-1}(x)=-\sqrt{\dfrac{1}{x}}+a$ **1A**

b. The graphs will touch along the line $y=x$.

The gradient equals 1.



$h'(x)=\dfrac{-2}{(x-a)^3}=1$ **1M**

$-2=(x-a)^3$

$x=\sqrt[3]{-2}+a$

$h^{-1\prime}(x)=\dfrac{1}{2x^{\frac{3}{2}}}=1$

$x=\left(\dfrac{1}{2}\right)^{\frac{2}{3}}$

$\sqrt[3]{-2}+a=\left(\dfrac{1}{2}\right)^{\frac{2}{3}}$ **1M**

$a=2^{-\frac{2}{3}}+2^{\frac{1}{3}}$

$=\dfrac{1}{2^{\frac{2}{3}}}+2^{\frac{1}{3}}\times\dfrac{2^{\frac{2}{3}}}{2^{\frac{2}{3}}}$

$=\dfrac{1}{2^{\frac{2}{3}}}+\dfrac{2}{2^{\frac{2}{3}}}$

$=\dfrac{3}{2^{\frac{2}{3}}}$ **1A**

OR

The point of intersection is at $\left(\sqrt[3]{-2}+a,\sqrt[3]{-2}+a\right)$.

Substitute into $h(x)=\dfrac{1}{(x-a)^2}$.

$\sqrt[3]{-2}+a=\dfrac{1}{(\sqrt[3]{-2}+a-a)^2}$ **1M**

$\sqrt[3]{-2}+a=\dfrac{1}{(\sqrt[3]{-2})^2}$

$a = \frac{1}{(\sqrt[3]{-2})^2} - \sqrt[3]{-2}$

$= \frac{1}{2^{\frac{2}{3}}} + 2^{\frac{1}{3}}$

$= \frac{1}{2^{\frac{2}{3}}} + 2^{\frac{1}{3}} \times \frac{2^{\frac{2}{3}}}{2^{\frac{2}{3}}}$

$= \frac{1}{2^{\frac{2}{3}}} + \frac{2}{2^{\frac{2}{3}}}$

$= \frac{3}{2^{\frac{2}{3}}}$ or $\frac{3}{4^{\frac{1}{3}}}$ **1A**

END OF SOLUTIONS