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YEAR 12 *Trial Exam Paper*
2023
MATHEMATICAL METHODS
Written examination 2

Reading time: 15 minutes
Writing time: 2 hours

STUDENT NAME:

QUESTION AND ANSWER BOOK

Structure of book

Section	*Number of questions*	*Number of questions to be answered*	*Number of marks*
A	20	20	20
B	5	5	60
			Total 80

- Students are permitted to bring into the examination room: pens, pencils, highlighters, erasers, sharpeners, rulers, a protractor, set squares, aids for curve sketching, one bound reference, one approved technology (calculator or software) and, if desired, one scientific calculator. Calculator memory DOES NOT need to be cleared. For approved computer-based CAS, full functionality may be used.
- Students are NOT permitted to bring into the examination room: blank sheets of paper and/or correction fluid/tape.

Materials supplied
- Question and answer book of 23 pages
- Formula sheet
- Answer sheet for multiple-choice questions

Instructions
- Write your **name** in the space provided above on this page and on the answer sheet for multiple-choice questions.
- Unless otherwise indicated, the diagrams in this book are **not** drawn to scale.
- All written responses must be in English.

At the end of the examination
- Place the answer sheet for multiple-choice questions inside the front cover of this book.
- You may keep the formula sheet.

Students are NOT permitted to bring mobile phones and/or any other unauthorised electronic devices into the examination room.

SECTION A – Multiple-choice questions

Instructions for Section A

Answer **all** questions in pencil on the answer sheet provided for multiple-choice questions.

Choose the response that is **correct** for the question.

A correct answer scores 1; an incorrect answer scores 0.

Marks will **not** be deducted for incorrect answers.

No marks will be given if more than one answer is completed for any question.

Unless otherwise indicated, the diagrams in this book are **not** drawn to scale.

Question 1

Let $f: R \to R,\ f(x) = 2\sin(2x) - 1$.

The period and range, respectively, of f are

A. 2π and $[-3,1]$

B. $\frac{\pi}{2}$ and $[-3,-1]$

C. π and $[-3,1]$

D. 2π and $[-3,-1]$

E. π and $[-2,2]$

Question 2

Let $f: R\setminus\{-1\} \to R,\ f(x) = \frac{2}{(x+1)^2}$.

The average rate of change of f from $x = 0$ to $x = 4$ is

A. –4

B. –0.48

C. –0.032

D. 0.4

E. 1.6

Question 3

Let X be a normally distributed random variable with a mean of 7.

If $\Pr(X > 9) = a$, then which one of the following statements is true?

A. $\Pr(X < 7) = a$

B. $\Pr(X < 9) = a$

C. $\Pr(7 \le X \le 9) = 1 - a$

D. $\Pr(X \ge 5) = 1 - a$

E. $\Pr(5 \le X < 9) = \frac{1}{2} - a$

Question 4

Two fair six-sided dice are rolled. Let Y be the product of the two numbers facing up.

The probability that Y is prime is

A. $\frac{1}{6}$

B. $\frac{7}{36}$

C. $\frac{13}{36}$

D. $\frac{1}{2}$

E. $\frac{2}{9}$

Question 5

Let f and g be two functions such that $f(x) = x^2 + x$ and $g(x) = 1 - 2x$.

The function $f(x + g(x))$ is

A. $2 - 3x - x^2$

B. $2 - x^2$

C. $(1 - x)(x - 2)$

D. $1 - 4x - 2x^2$

E. $(x - 1)(x - 2)$

Question 6

If $\int_3^6 f(x-2)\,dx = 5$, then $\int_1^4 \left(3f(x)-2x\right)dx$ is

A. -30

B. 9

C. -10

D. 0

E. $\frac{15}{2}$

Question 7

The maximum possible number of intersections between a parabola and the graph of a quartic is

A. 1

B. 2

C. 3

D. 4

E. 5

Question 8

A system of linear equations is as follows:

$y = 2x - 5$

$2x + y - z = -5$

$4x - 2y = 10$

Which one of the following correctly describes a general solution to the system of linear equations given above?

A. $x = k,\ \ y = 2k - 5,\ \ z = -4k,$ for all $k \in R$

B. $x = k,\ \ y = 2k - 5,\ \ z = 4k,$ for all $k \in R$

C. $x = k,\ \ y = 2k - 5,\ \ z = -4k + 10,$ for all $k \in R$

D. $x = k,\ \ y = 2k - 5,\ \ z = 4k - 10,$ for all $k \in R$

E. $x = k,\ \ y = 2k - 5,\ \ z = 10,$ for all $k \in R$

Question 9

The graph of the function g is obtained from the graph of the function f with rule $f(x)=\sqrt{x-2}$ by a dilation of factor 2 from the x-axis, a translation 3 units in the negative x direction and a reflection in the y-axis, in that order.

The rule for g is

A. $2\sqrt{1-x}$

B. $-\sqrt{\dfrac{x-1}{2}}$

C. $-2\sqrt{x+1}$

D. $2\sqrt{-x-5}$

E. $-\sqrt{\dfrac{x}{2}+1}$

Question 10

Let $f:(-\infty,a]\to R,\ f(x)=x^3-8x^2+16x-3.$

The maximum value of a for which f will have an inverse function, correct to two decimal places, is

A. 4.79

B. 0.21

C. 1.33

D. 3

E. 4

Question 11

A survey is conducted, and a 95% confidence interval is created.

The 95% confidence interval would be **guaranteed** to be narrower if which one of the following were increased?

A. $N,$ the population size

B. $n,$ the sample size

C. $p,$ the population proportion

D. $\hat{p},$ the sample proportion

E. $\mathrm{E}\left(\hat{P}\right)$

Question 12

The sum of the solutions to $3\tan(2x)=\sqrt{3}$ over the interval $[0,2\pi]$ is equal to

A. $\frac{\pi}{12}$

B. $\frac{2\pi}{3}$

C. $\frac{4\pi}{3}$

D. $\frac{10\pi}{3}$

E. 3π

Question 13

The left-endpoint estimate for the area under the graph of $y=f(x)$ is

$Area \approx \frac{x_n - x_0}{n}\left[f(x_0)+f(x_1)+...+f(x_{n-1})\right]$, where n is the number of left-endpoint rectangles.

The algorithm below, described in pseudocode, estimates the value of a definite integral using left-endpoint rectangles.

```
Inputs:   f(x), the function to integrate
          a, the lower terminal of integration
          b, the upper terminal of integration
          n, the number of left-endpoint rectangles to use

Define  leftrectangles(f(x),a,b,n)
      w ← (b - a) ÷ n
      sum ← 0
      x ← a
      i ← 1
     While i < n - 1 Do
          [          ]
     EndWhile
     area ← w × sum
     Return area
```

Which one of the following options would be most appropriate to fill the empty box?

A.
```
sum ← sum + f(x)

x ← x + 1

i ← i + w
```

B.
```
sum ← sum + f(x)

x ← x + w

i ← i + 1
```

C.
```
sum ← sum + f(a)

x ← x + w

a ← a + 1
```

D.
```
sum ← sum + f(x)

x ← w + 1

i ← i + 1
```

E.
```
sum ← sum + f(a)

x ← x + 1

i ← i + 1
```

Question 14

Which one of the following is the inverse function of $h:(-\infty, 1] \to R,\ h(x) = 4 - \sqrt{1-x}$?

A. $h^{-1}:(-\infty, 1] \to R,\ h^{-1}(x) = 1 - (4-x)^2$

B. $h^{-1}:(-\infty, 4] \to R,\ h^{-1}(x) = (4-x)^2 - 1$

C. $h^{-1}:[4, \infty) \to R,\ h^{-1}(x) = 1 - (x-4)^2$

D. $h^{-1}:[4, \infty) \to R,\ h^{-1}(x) = (4-x)^2 + 1$

E. $h^{-1}:(-\infty, 4] \to R,\ h^{-1}(x) = -(x-4)^2 + 1$

Question 15

A restaurant finds that 75% of their customers order an entrée. Let $\hat{P}$ be the random variable that represents the proportion of the restaurant's customers who order an entrée when random samples of five customers are taken.

The value of $\Pr(\hat{P} \geq 0.75)$ is closest to

A. 0

B. 0.7383

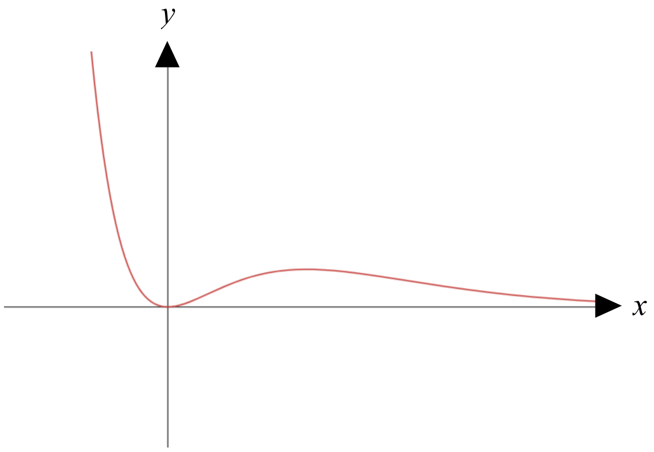
C. 0.3164

D. 0.2373

E. 0.6328

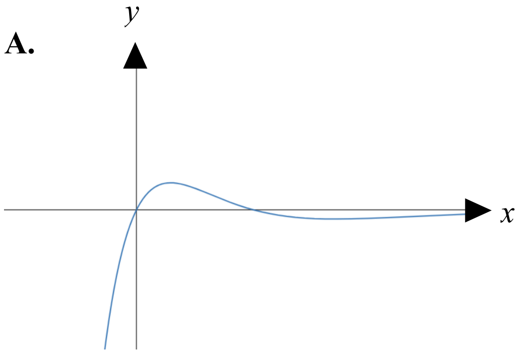
Question 16

Part of the graph of $y = f'(x)$ is shown below.

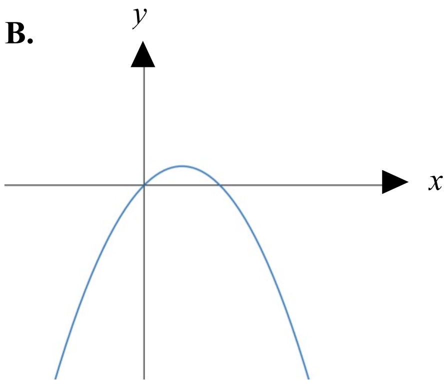


The corresponding part of the graph of $y = f(x)$ could be represented by

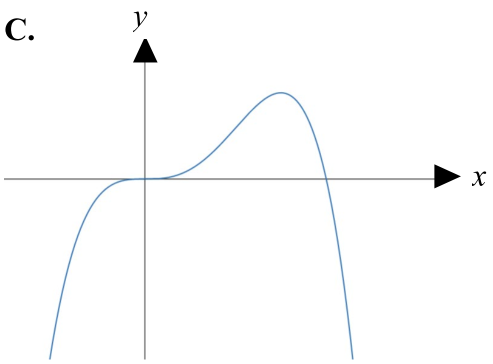
A.



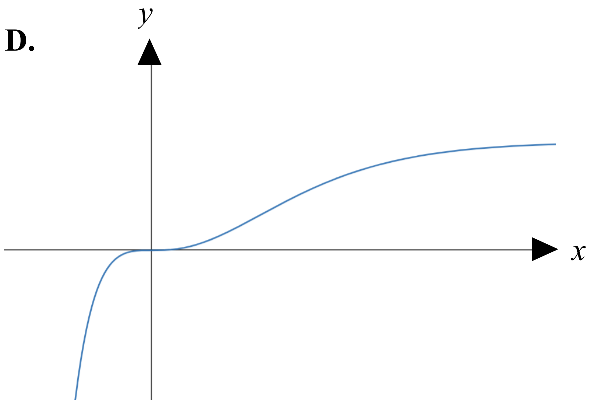
B.



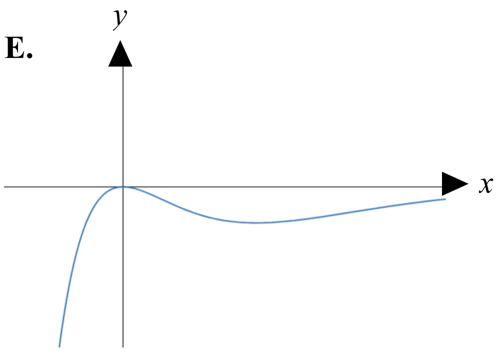
C.



D.



E.



Question 17

Newton's method is being used to estimate the x-intercept of $f(x) = e^x - x^2$ with an initial estimate of $x_0 = 0$.

The value of x_2 is closest to

A. 0

B. −1

C. −0.704

D. −0.703

E. −0.733

Question 18

For two differentiable functions, f and g, the derivative of $f(2x) \times g(x^2)$ is

A. $4x\,f'(2x)\,g'(x^2)$

B. $x^2 f(2x)\,g'(x^2) + 2x\,f'(2x)\,g(x^2)$

C. $2f(2x)\,g'(x^2) + 2x\,f'(2x)\,g(x^2)$

D. $2x\,f(2x)\,g'(x^2) + 2\,f'(2x)\,g(x^2)$

E. $\dfrac{2f'(2x)\,g(x^2) - 2x\,f(2x)\,g'(x^2)}{\left[g(x^2)\right]^2}$

Question 19

The tangent to $y = e^{2x+1}$ at $x = a$ passes through the origin. The value of a is

A. e

B. $\dfrac{1}{2}$

C. e^2

D. $2e^2$

E. 1

SECTION A – continued
TURN OVER

Question 20

The continuous random variable, X, has a probability density function given by

$$f(x)=\begin{cases} ae^{-x} & 0\le x\le k \\ 0 & \text{elsewhere} \end{cases}$$

If $E(X)=1-\log_e(2)$ then the values of a and k, respectively, are

A. 1 and 1

B. 2 and $\log_e(2)$

C. $\log_e(2)$ and 2

D. 2 and 2

E. $\log_e(2)$ and $\log_e(2)$

END OF SECTION A

SECTION B

Instructions for Section B

Answer **all** questions in the spaces provided.

In all questions where a numerical answer is required, an exact value must be given unless otherwise specified.

In questions where more than one mark is available, appropriate working **must** be shown.

Unless otherwise indicated, the diagrams in this book are **not** drawn to scale.

Question 1 (11 marks)

Consider the function $f: R \to R,\ f(x) = x\cos(x)$.

a. State the derivative of *f*.

1 mark

b. Show that $f'(2n\pi) = 1$ for all $n \in Z$.

1 mark

c. Explain why $f(x) \le x$ for $x \ge 0$.

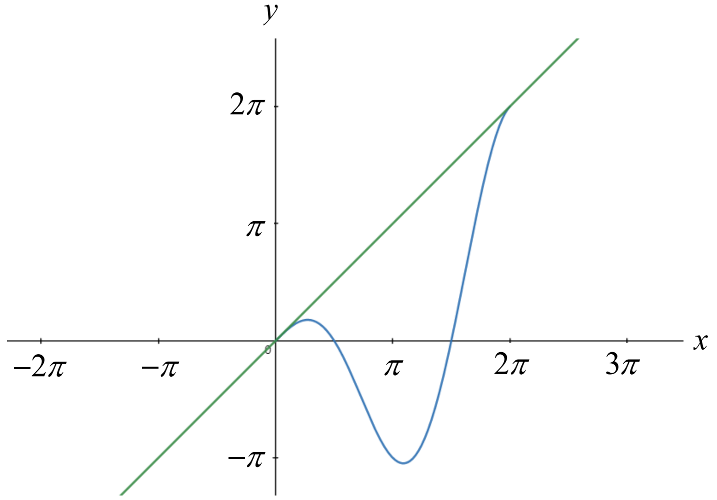
1 mark

d. Find the average value of f over the interval $x \in \left[0, \frac{5\pi}{2}\right]$.

Express your answer in the form $1 - \frac{a}{b\pi}$, where *a* and *b* are positive integers.

2 marks

Now consider the function $g:[0,2\pi]\to R,\ g(x)=x\cos(x)$. The graph of $y=g(x)$ is shown below, along with the line $y=x$.



e. State the coordinates of the minimum turning point of $y=g(x)$. Give your values correct to three decimal places.

1 mark

f. Find the average rate of change of g over the interval $x\in[0,2\pi]$.

1 mark

g. Find the area of the region bound by the graph of $y=g(x)$ and the line $y=x$.

1 mark

h. Find the minimum distance between the graph of $y=g(x)$ and the point $(\pi,0)$. Give your answer correct to three decimal places.

3 marks

Question 2 (11 marks)

A patient in a hospital is administered a drug. The number of milligrams of the drug in a patient's system can be modelled by $d(t) = 500t\,e^{-0.8t}$, where t is the number of hours after the drug is administered.

a. Find the time it takes for the drug to reach a maximum amount in the patient's system.

Give your answer in minutes.

1 mark

b. Find the maximum amount of the drug in the patient's system, in milligrams, correct to one decimal place.

1 mark

c. What was the average amount of the drug in the patient's system over the first three hours of them taking the drug? Give your answer to the nearest milligram.

2 marks

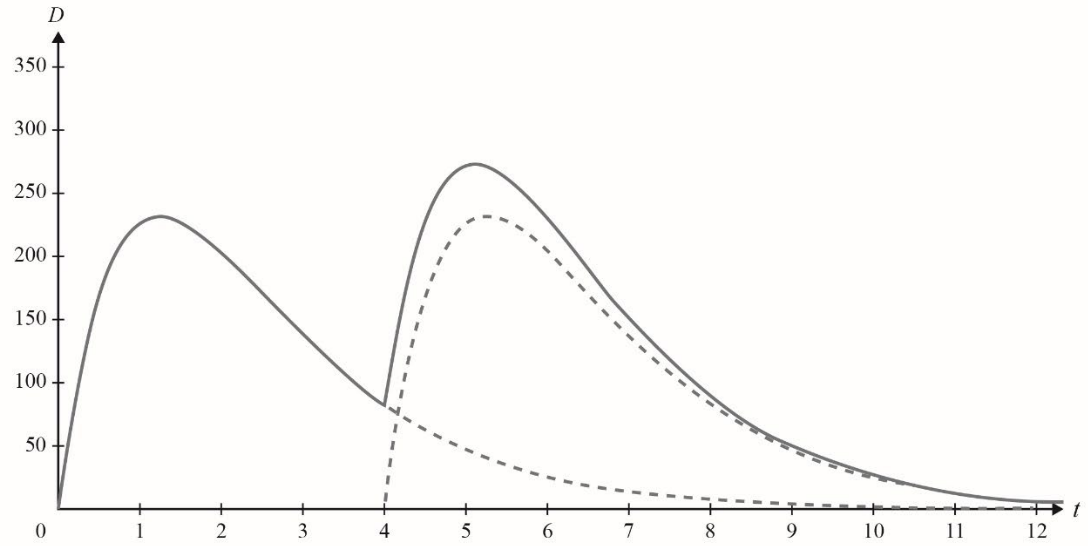
d. According to the model, the drug is **leaving** the patient's system at a rate of 50 mg/h for what value(s) of t? Give your answer(s) correct to three decimal places.

2 marks

The patient's doctor will administer another dose of the drug, which is expected to be absorbed in the patient's system according to the same model as the first dose. This means the amount of the drug in the patient's system will be the sum of the remaining amount of the first dose, and the amount from the new dose.

The second dose will be administered four hours after the first dose.

A graph of the total amount of the drug in the patient's system is shown below, where the grey curves are the graphs of each individual dose contributing to the total amount.



e. How many milligrams of the first dose is still in the patient's system when the second dose is administered? Give your answer to two decimal places.

1 mark

---

f. What is the maximum number of milligrams of the drug in the patient's system after the second dose is given? Round your answer to the nearest milligram.

1 mark

The doctor wanted to give the second dose earlier than four hours after the first dose, but needed to be sure that the total number of milligrams of the drug in the patient's system would not exceed 330 mg.

g. What is the earliest time after the first dose that the doctor could have administered the second dose? Give your answer in hours and minutes, rounded to the nearest minute.

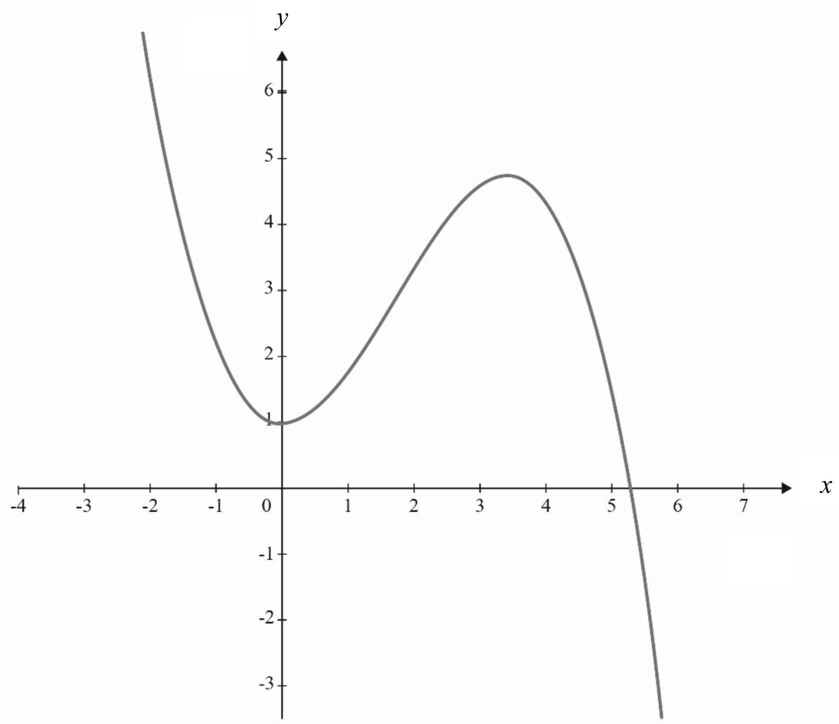
3 marks

Question 3 (12 marks)

The function f is defined as follows:

$$f : R \rightarrow R,\ f(x) = 1 + x^2 - \frac{1}{5}x^3$$

The graph of $y = f(x)$ is shown on the set of axes below.



a. State the equation of the tangent to f at $x = 4$.

1 mark

b. Sketch the graph of the tangent to f at $x = 4$ on the graph provided above, labelling the point of tangency and the x-intercept with their coordinates.

2 marks

Newton's method is used to find an approximate x-intercept of f, with an initial estimate of $x_0 = 4$.

c. Find the value of x_1.

1 mark

d. Find the horizontal distance between x_3 and the x-intercept of f, correct to four decimal places.

1 mark

e. Explain why Newton's method will not converge to the x-intercept of f for an initial estimate of $x_0 = 0$ or $x_0 = \frac{10}{3}$.

1 mark

The area under the graph of f between $x = 0$ and $x = 4$ is to be estimated using the trapezium rule approximation, with trapeziums of horizontal width 1.

f. Find the approximation for the area under f between $x = 0$ and $x = 4$, using this trapezium rule approximation.

1 mark

g. Hence, determine the difference between this trapezium rule approximation and the actual area that is being approximated. Give your answer correct to two decimal places.

2 marks

h. On the graph provided on page 16, sketch the trapezium used to estimate the area under f between $x = 3$ and $x = 4$. Coordinate points on the graph do not need to be labelled.

1 mark

The algorithm below, described in pseudocode, estimates the solution to an equation in the form $f(x)=0$, using the bisection method.

```
Inputs:   f, the rule of a function in x
          n, the number of iterations to run
          a, the initial lower estimate
          b, the initial upper estimate

Define  bisect(f,n,a,b)
      If f(a) × f(b) > 0 Then
          Return "Initial estimates are invalid."
     i ← 0
     While i < n Do
          mid ← (a + b) ÷ 2
          If f(mid)=0 Then
               Return mid
          Else If f(a) × f(mid) > 0 Then
               a ← mid
          Else
               b ← mid
          i ← i + 1
     EndWhile
     Return mid
```

The pseudocode below is implemented as follows:

```
bisect(1+x^2-1/5x^3,3,5,6)
```

i. What are the initial lower and upper estimates used in the implementation as given?

1 mark

____

____

j. What value would be returned when the algorithm is implemented as given?

1 mark

____

____

Question 4 (15 marks)

A company produces and sells light globes. When left on continuously, the number of days, *X*, that it takes for a randomly selected light globe to burn out can be modelled by the following probability density function:

$$f(x)=\begin{cases}30\left(\dfrac{x}{1000}\right)^4(100-x) & 0<x<100\\ 0 & \text{elsewhere}\end{cases}$$

a. Find the probability that a randomly selected globe burns out before 70 days. Give your answer correct to four decimal places.

1 mark

b. Find the average number of days it takes for a light globe to burn out, correct to one decimal place.

1 mark

c. Find the value of a if 20% of globes burn out before a days. Give your answer correct to two decimal places.

1 mark

A globe is considered defective if it burns out in less than 50 days.

d. A random sample of 40 globes is taken. What is the probability, correct to four decimal places, that at least five are defective?

2 marks

Due to the problem with some globes burning out quickly, the company changes the way its globes are manufactured. It finds that the number of days, Y, that it takes for the new globes to burn out can be modelled by a normal distribution with a mean of 80 days and a standard deviation of 12.3 days.

e. What proportion of the new globes last for more than 100 days? Give your answer correct to four decimal places.

1 mark

f. 15% of the new globes burn out before b days. Find b, correct to two decimal places.

1 mark

While changing over from the old globes to the new globes, there is a short period of time when the company has globes of both types in its warehouse for distribution. On one particular day, the company sends out 400 of the old globes and 200 of the new globes.

g. What is the probability, correct to four decimal places, that a randomly selected one of these 600 globes will last for more than 90 days?

2 marks

h. One of these 600 globes is randomly selected and found to last for more than 90 days. What is the probability that it is one of the old globes? Give your answer correct to two decimal places.

2 marks

The company decides to carry out some market research before considering switching to more environmentally friendly light globes. In a sample of n customers, it finds that 40% of customers are willing to pay more money for environmentally friendly light globes. From this survey, a 90% confidence interval is created and the upper bound of the interval is 0.4963.

i. What is the lower bound of the confidence interval?

1 mark

j. Find a such that $\Pr(-a < Z < a) = 0.9$, where Z is the standard normal distribution. Give your answer correct to three decimal places.

1 mark

k. How many customers were sampled in the survey?

2 marks

Question 5 (11 marks)

Consider the function $f: R \to R,\ f(x) = ax(k - x^2)$, where a and k are both positive, real numbers.

a. Express a in terms of k if the graph of $y = f(x)$ passes through (1, 1).

1 mark

b. Show that the coordinates of the two stationary points of the graph of $y = f(x)$ occur at $x = \pm\frac{\sqrt{3k}}{3}$.

2 marks

c. Hence, find the values of a and k if $(1,1)$ is a maximum turning point.

2 marks

d. State the equation of the tangent to $y = f(x)$ at its positive x-intercept, in terms of a and k.

1 mark

e. Find the area, in terms of a and k, of the region bound by the graph of $y = f(x)$ and the tangent to $y = f(x)$ at its positive x-intercept.

2 marks

The graph of $y = f(x)$ is translated so that its minimum turning point is at the origin. The equation of this new graph is defined as the function g.

f. State the two translations that must be applied to the graph of $y = f(x)$ for it to become the graph of $y = g(x)$.

1 mark

g. Hence, or otherwise, find the area bound between the graph of $y = g(x)$ and the x-axis.

2 marks

END OF QUESTION AND ANSWER BOOK