THE HEFFERNAN GROUP
P.O. Box 1180
Surrey Hills North VIC 3127
Phone 03 9836 5021
info@theheffernangroup.com.au
www.theheffernangroup.com.au

# MATHS METHODS 3 & 4 TRIAL EXAMINATION 2 SOLUTIONS 2023

## SECTION A – Multiple-choice answers

| | | | | | | | |
|---|---|---|---|---|---|---|---|
| **1.** | A | **6.** | A | **11.** | E | **16.** | A |
| **2.** | A | **7.** | C | **12.** | D | **17.** | A |
| **3.** | E | **8.** | D | **13.** | B | **18.** | D |
| **4.** | D | **9.** | C | **14.** | C | **19.** | D |
| **5.** | B | **10.** | C | **15.** | E | **20.** | B |

## SECTION A – Multiple-choice solutions

**Question 1**

$$f(x)=2\tan\left(\frac{2\pi x}{3}\right)$$

$$\begin{aligned}\text{period}&=\frac{\pi}{n} \text{ where } n=\frac{2\pi}{3}\\&=\pi\div\frac{2\pi}{3}\\&=\frac{3}{2}\end{aligned}$$

The answer is A.

**Question 2**

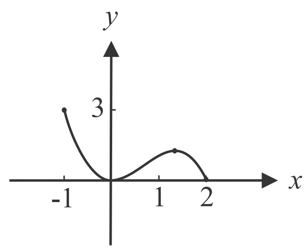
$$\begin{aligned}\text{average rate of change of } f &= \frac{f(\pi)-f\left(\frac{\pi}{3}\right)}{\pi-\frac{\pi}{3}}\\&=\frac{3}{4\pi}\end{aligned}$$

If you don't use the fraction template on your CAS, then make sure you bracket the numerator and bracket the denominator when entering the function into your CAS.

The answer is A.

**Question 3**

Sketch the graph of $g$, noting that $g$ has a restricted domain.




The maximum value of $g$ is 3. Note that this maximum occurs at the left endpoint of the graph, not at the turning point.
The answer is E.

**Question 4**

For a binomial distribution, $\mathrm{E}(X) = np$ and $\mathrm{Var}(X) = np(1-p)$. (formula sheet)
Solve $np = 12$ and $np(1-p) = 9.12$ using CAS.
$n = 50$ and $p = 0.24$
The answer is D.

**Question 5**

$$\begin{aligned} y &= 2\log_e(x) - \log_e(x+1) + 1 \\ &= \log_e(x^2) - \log_e(x+1) + \log_e(e) \\ &= \log_e\left(\frac{ex^2}{x+1}\right) \end{aligned}$$

The answer is B.

**Question 6**

Method 1
There will be infinitely many solutions when the two equations are the same. (i.e. we have two identical lines on top of each other and therefore they will intersect at an infinite number of points)

$$ax + 4y = a + 6 \quad [1]$$
$$x + ay = -2 \quad [2]$$

Rearranging [1] gives $4y = -ax + a + 6$

$$y = -\frac{a}{4}x + \frac{a+6}{4} \quad [3]$$

Rearranging [2] gives $y = -\frac{x}{a} - \frac{2}{a}$ [4]

Equating the coefficients of the $x$ terms and equating the constant terms of equations [3] and [4] gives

$$-\frac{a}{4} = -\frac{1}{a} \quad \text{AND} \quad \frac{a+6}{4} = -\frac{2}{a}$$
$$a = \pm 2 \qquad\qquad a = -4 \text{ or } a = -2$$

The value of $a$ which satisfies **both** equations is $a = -2$.
The answer is A.

<u>Method 2</u>
The system of equations

$$ax + 4y = a + 6$$
$$x + ay = -2$$

can be expressed as the matrix equation

$$\begin{bmatrix} a & 4 \\ 1 & a \end{bmatrix}\begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} a+6 \\ -2 \end{bmatrix}$$

There will be infinitely many solutions or no solutions when

$$a^2 - 4 = 0$$
$$a = \pm 2$$

When $a = 2$, the equations are

$$2x + 4y = 8 \qquad [1]$$
$$x + 2y = -2 \qquad [2]$$
$$[1] \div 2 \quad x + 2y = 4 \qquad [3]$$

Equations [2] and [3] have no solution (ie they are parallel lines).

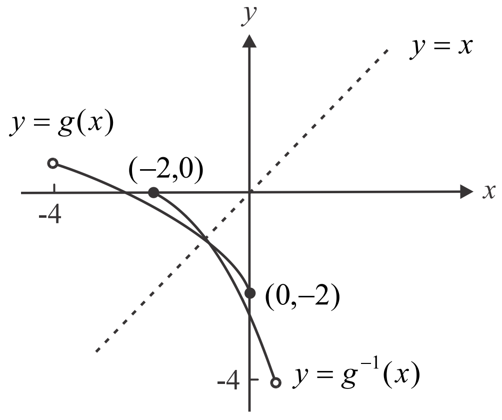
When $a = -2$, the equations are

$$-2x + 4y = 4 \qquad [1]$$
$$x - 2y = -2 \qquad [2]$$
$$[1] \div -2 \quad x - 2y = -2 \qquad [3]$$

Equations [2] and [3] are the same equation and hence have infinitely many solutions.
So we require $a = -2$ only.
The answer is A.

**Question 7**

The graph of $y = g^{-1}(x)$ is a reflection in the line $y = x$ of the graph of $y = g(x)$.
The point $(0,-2)$ becomes $(-2,0)$, so we can reject options A and B.
Note that the graph of *g* and the graph of its inverse, $g^{-1}$, intersect on the line $y = x$ so we can reject options D and E.




The answer is C.

**Question 8**

Method 1

For option A, as $x \to 0^-$, $x+1 \to 1$, and at $x=0$, $f(x)=1$, so option A is continuous.

For option B, as $x \to 0^-$, $x \to 0$, and at $x=0$, $f(x)=0$, so option B is continuous.

For option C, at $x=0, f(x)=e^0=1$ and as $x \to 0^+$, $1-x \to 1$, so option C is continuous.

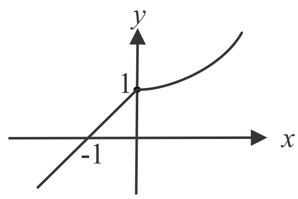
For option D, as $x \to 0^-$, $x^2 \to 0$, and at $x=0$, $f(x)=1$ so option D is **not** continuous.

Checking option E, at $x=0, f(x)=0$, and as $x \to 0^+, \sqrt{x} \to 0$ so option E is continuous.

The answer is D.

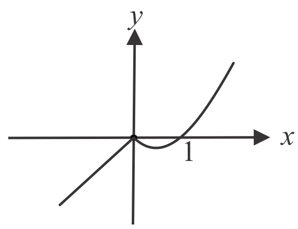
Method 2

The graph of option A, shown below, is continuous.



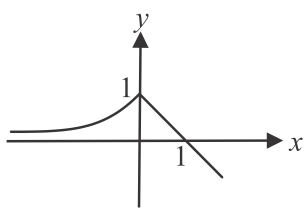

$$f(x)=\begin{cases} x+1 & x<0 \\ x^2+1 & x \geq 0 \end{cases}$$

The graph of option B, shown below, is continuous.



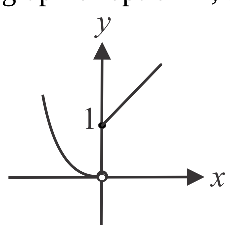

$$f(x)=\begin{cases} x & x<0 \\ x(x-1) & x \geq 0 \end{cases}$$

The graph of option C, shown below, is continuous.



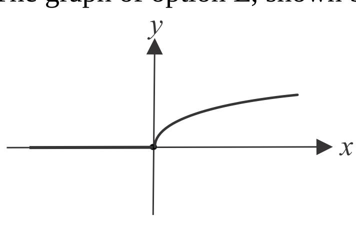

$$f(x)=\begin{cases} e^x & x \leq 0 \\ 1-x & x>0 \end{cases}$$

The graph of option D, shown below, is **not** continuous.




$$f(x)=\begin{cases} x^2 & x<0 \\ x+1 & x \geq 0 \end{cases}$$

The graph of option E, shown below, is continuous.




$$f(x)=\begin{cases} 0 & x \leq 0 \\ \sqrt{x} & x>0 \end{cases}$$

The answer is D.

**Question 9**

<u>Method 1</u> – using CAS

Try solving with $x = k$.

$$\text{solve}\left(\begin{cases} x = k \\ 2x - z = 1, \\ y + z = 3 \end{cases} \{x, y, z\}\right)$$

$$x = k \text{ and } y = -2(k-2) \text{ and } z = 2k - 1$$

This solution is not the same as option A so reject option A.

Try solving with $y = k$.

$$\text{solve}\left(\begin{cases} y = k \\ 2x - z = 1, \\ y + z = 3 \end{cases} \{x, y, z\}\right)$$

$$x = -0.5(k-4) \text{ and } y = k \text{ and } z = -(k-3)$$

This solution is not the same as option B so reject option B.

This solution **is** the same as option C so option C is correct.

The answer is C.

<u>Method 2</u>

$$2x - z = 1 \qquad [1]$$

$$y + z = 3 \qquad [2]$$

$$[1] \text{ gives } z = 2x - 1$$

$$[1] + [2] \text{ gives } 2x + y = 4$$

$$y = 4 - 2x$$

If $x = k, \quad y = 4 - 2k, \quad z = 2k - 1$

Option A is incorrect.

$$2x - z = 1 \qquad [1]$$

$$y + z = 3 \qquad [2]$$

$$[2] \text{ gives } z = 3 - y$$

$$[1] + [2] \text{ gives } 2x + y = 4$$

$$x = \frac{4 - y}{2}$$

If $y = k, \quad x = \frac{4-k}{2}, \quad z = 3 - k$

So option B is incorrect but option C **is** correct

The answer is C.

If you have time and want to check options D and E:

$$2x - z = 1 \qquad [1]$$

$$y + z = 3 \qquad [2]$$

$$[1] \text{ gives } x = \frac{z+1}{2}$$

$$[2] \text{ gives } y = 3 - z$$

If $z = k, \quad x = \frac{k+1}{2}, \quad y = 3 - k$.

Option D is incorrect because $y = k + 3$.

Option E is incorrect because $x = \frac{k-1}{2}$.

The answer is C.

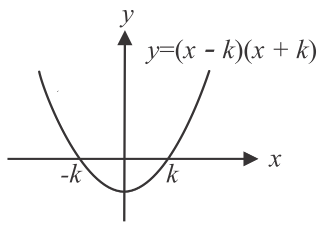
**Question 10**

$$2a + 4a + 3a + a = 1$$
$$a = 0.1$$
$$\text{var}(X) = \text{E}(X^2) - [\text{E}(X)]^2 \qquad \text{(formula sheet)}$$
$$= (-1)^2 \times 0.2 + 1^2 \times 0.3 + 3^2 \times 0.1 - (-1 \times 0.2 + 1 \times 0.3 + 3 \times 0.1)^2$$
$$= 1.24$$

The answer is C.

**Question 11**

The maximal domain occurs for values of $x$ such that
$\sqrt{(x-k)(x+k)} > 0$
i.e. $(x-k)(x+k) > 0$
Solve this inequality for $x$, given $k > 0$, by drawing the graph of $y = (x-k)(x+k)$.
So $x < -k$ or $x > k$.
Alternatively, $x \in (-\infty, -k) \cup (k, \infty)$.
In the solutions, this appears as $x \in R \backslash [-k, k]$.
The answer is E.

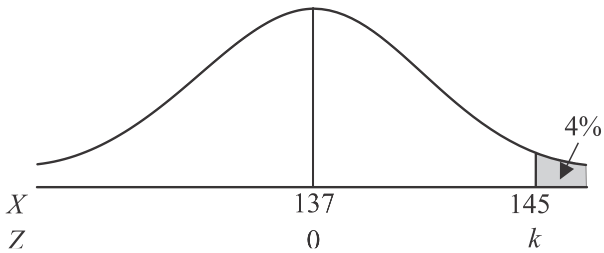



**Question 12**

Using the Inverse Normal function on your CAS,

$$\Pr(Z < k) = 0.96$$
$$k = 1.7506... \qquad \text{(invNorm (0.96,0,1))}$$
$$z = \frac{x - \bar{x}}{\sigma}$$
$$\text{solve } 1.7506... = \frac{145 - 137}{\sigma} \text{ for } \sigma$$
$$\sigma = 4.5696...$$

The closest answer is 4.6 cm.
The answer is D.




**Question 13**

Stationary points occur when $g'(x) = 0$.

$$g'(x) = 3ax^2 + 2bx - c$$

We want **no** stationary points i.e. we want there to be no solutions to the equation

$$3ax^2 + 2bx - c = 0$$

Because this is a quadratic equation, there will be no solutions when

$$\Delta = (2b)^2 - 4 \times 3a \times -c < 0 \qquad \text{(using the discriminant)}$$
$$4b^2 + 12ac < 0$$
$$12ac < -4b^2$$
$$a < -\frac{b^2}{3c}$$

Note that $c$ is a positive constant so the inequality sign doesn't change in that last step.
The answer is B.

**Question 14**

Initially,

`xcurrent = 1`

After one iteration of the while loop,

`xnext = 1 – f(1) ÷ df(1) = 7/3`

After two iterations of the while loop,

`xnext = 7/3 – f(7/3) ÷ df(7/3) = 1.8616…`

Since

`xnext – xcurrent = 1.8616… – 7/3 = -0.47…and -0.47… <`$-10^{-4}$

then proceed to the **`Else`** instruction.

The second last instruction before the end of the while loop is

`xcurrent ← xnext`

So `xcurrent = 1.8616…`

The last instruction is

**`Return`** `f(xcurrent)`

So the required output is

`f(1.8616…) = -1.4522…`

The closest answer is `-1.452`

The answer is C.

**Question 15**

$$\begin{aligned}&\Pr(G,G)+\Pr(Y,Y)\\&=\frac{x}{x+y}\times\frac{x-1}{x+y-1}+\frac{y}{x+y}\times\frac{y-1}{x+y-1}\\&=\frac{x(x-1)}{(x+y)(x+y-1)}+\frac{y(y-1)}{(x+y)(x+y-1)}\\&=\frac{x(x-1)+y(y-1)}{(x+y)(x+y-1)}\end{aligned}$$

The answer is E.

**Question 16**

Let $X$ be the number of retirees in the samples of four who would like to find part-time work.

$X \sim \text{Bi}(4, p)$

Given $\Pr(\hat{P}=1)=\frac{1}{81}$,

then $\Pr(X=4)=\frac{1}{81}$ since $\hat{P}=\frac{X}{n}$

$$X = 4\times 1$$
$$= 4$$

Solve ${}^4C_4 p^4(1-p)^0=\frac{1}{81}$ for $p$

$$p=\frac{1}{3}$$

$\Pr(\hat{P}>0.5)=\Pr(X>2)$

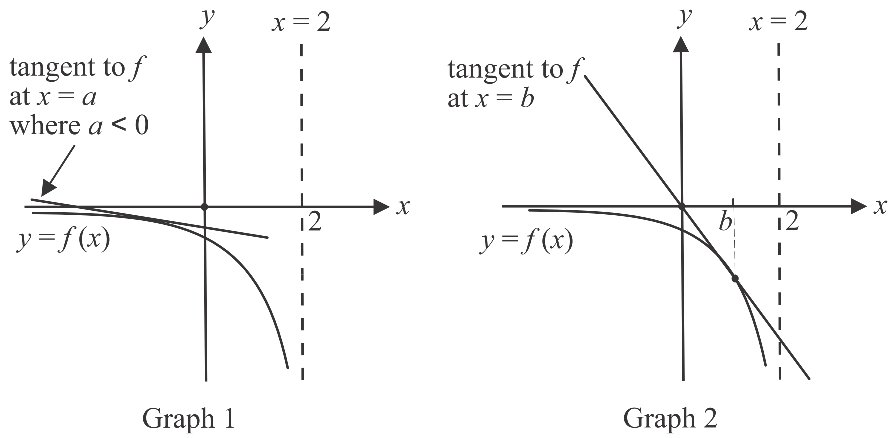
$=\Pr(X\geq 3)$ $\quad$ $\text{binomCdf}\left(4,\frac{1}{3},3,4\right)$

$=0.1111...$

$=0.1111$ (correct to 4 decimal places)

The answer is A.

**Question 17**




The tangents to $f$ at $x=a$ where $a\leq 0$, all have negative vertical axis (ie $y$-axis) intercepts. An example is shown on Graph 1 above.

Let the tangent to $f$ at $x=b$, pass through the origin, i.e. (0,0) as shown on Graph 2 above.

The point on $f$ where $x=b$ is $\left(b, (f(b)\right)=\left(b,\frac{1}{b-2}\right)$.

The gradient of the tangent to $f$ at $x=b$ therefore has a gradient of

$\left(\frac{1}{b-2}-0\right)\div(b-0)=\frac{1}{b(b-2)}$.

Also, the gradient of the tangent to $f$ at $x=b$ is given by $f'(b)$ where

$f'(x)=\frac{-1}{(x-2)^2}$ and so $f'(b)=\frac{-1}{(b-2)^2}$.

Solve $\frac{1}{b(b-2)}=\frac{-1}{(b-2)^2}$ for $b$. So $b=1$.

So tangents with a negative $y$-axis intercept occur for $a\in(-\infty,1)$.

The answer is A.

**Question 18**

$f(x) = x(e^x - 1)$

Let $y = x(e^x - 1)$

After a dilation by a factor of 2 from the *x*-axis we have

$$\frac{y}{2} = x(e^x - 1)$$
$$y = 2x(e^x - 1)$$

After a reflection in the *x*-axis we have

$$-y = 2x(e^x - 1)$$
$$y = -2x(e^x - 1)$$

After a translation of 1 unit in the negative *x* direction we have

$$y = -2(x+1)(e^{x+1} - 1)$$

After a translation of 2 units in the positive *y* direction we have

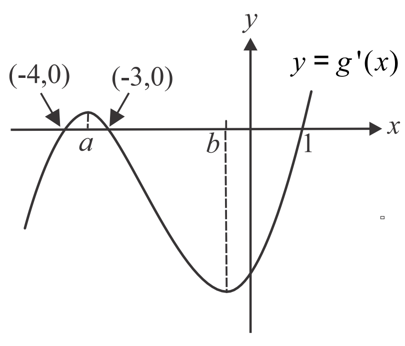
$$y - 2 = -2(x+1)(e^{x+1} - 1)$$
$$y = 2 - 2(x+1)(e^{x+1} - 1)$$

So $h(x) = 2 - 2(x+1)(e^{x+1} - 1)$

The answer is D.

**Question 19**

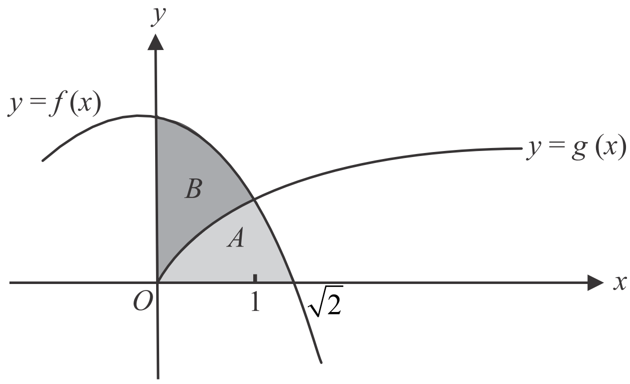
The graph of the gradient function $g'$ will be a cubic graph. It must be a positive cubic graph (i.e. the coefficient of the $x^3$ term must be positive) because the maximum turning point at $x = a$ ,is to the left of the minimum turning point at $x = b$ since $b > a$ .
A possible graph is shown below.




The turning points (stationary points) of the graph of *g* occur when
$g'(x) = 0$ i.e. when $x = -4$, $x = -3$ and $x = 1$.
Around $x = -4$, $g'(x) < 0$ for $x < -4$ and $g'(x) > 0$ for $x > -4$ so there is a minimum turning point on the graph of *g* when $x = -4$.
Around $x = -3$, $g'(x) > 0$ for $x < -3$ and $g'(x) < 0$ for $x > -3$ so there is a maximum turning point on the graph of *g* when $x = -3$.
Around $x = 1$, $g'(x) < 0$ for $x < 1$ and $g'(x) > 0$ for $x > 1$ so there is a minimum turning point on the graph of *g* when $x = 1$.
So minimum turning points occur on the graph of *g* when $x = -4$ and $x = 1$ only.
The answer is D.

**Question 20**

Start by finding the $x$-value of the point of intersection of $f$ and $g$. i.e. solve $f(x) = g(x)$ for $x$ and $x = 1$.
The $x$-intercept of the graph of $f$ occurs when $f(x) = 0$ so solve $f(x) = 0$ for $x$ and $x = \sqrt{2}$ .




$$A = \int_0^1 g(x)\,dx + \int_1^{\sqrt{2}} f(x)\,dx$$
$$= \frac{4\sqrt{2}}{3} - 1$$

$$B = \int_0^1 f(x)\,dx - \int_0^1 g(x)\,dx$$
$$= 1$$

So $\quad A:B$

$$\frac{4\sqrt{2}}{3} - 1 : 1$$

The answer is B.

## SECTION B

**Question 1** (14 marks)

**a.** Stationary points (and hence turning points) exist when $f'(x)=0$
Define $f(x)$ on your CAS.
Solve $f'(x)=0$ for $x$

$$x=-1 \text{ or } x=1$$

$$f(-1)=4 \quad f(1)=0$$

$A$ is the point $(-1,4)$. **(1 mark)**
$B$ is the point $(1,0)$. **(1 mark)**

**b.** distance $AB=\sqrt{(1--1)^2+(0-4)^2}$

$=2\sqrt{5}$ units

**(1 mark)**

**c.** gradient $=\dfrac{4-0}{-1-1}=-2$ (consistent with graph) **(1 mark)**

Through $(-1,4)$, $y-4=-2(x--1)$

$y=-2x+2$

or through $(1,0)$, $y-0=-2(x-1)$

$y=-2x+2$

**(1 mark)**

**d.** The graphs of $y=x^3-3x+2$ and $y=-2x+2$ both have a $y$-intercept at $(0,2)$.

area $=\int_{-1}^{0}\left(f(x)-(-2x+2)\right)dx+\int_{0}^{1}\left((-2x+2)-f(x)\right)dx$

**(1 mark)** **(1 mark)**

$=\dfrac{1}{2}$ square unit

**(1 mark)**

**e.** $f(x)=x^3-3x+2$

$g(x)=x^3+(k-4)x+2k$

We require $k-4=-3$ AND $2k=2$

So $k=1$.

**(1 mark)**

**f.** Using CAS, solve $g'(x)=0$ for $x$.

$x=\pm\dfrac{\sqrt{3(4-k)}}{3}$

**(1 mark)**

**g.** Stationary points occur when $g'(x)=0$ i.e. when $x=\pm\dfrac{\sqrt{3(4-k)}}{3}$.

When $\dfrac{\sqrt{3(4-k)}}{3}=-\dfrac{\sqrt{3(4-k)}}{3}$

i.e. when $k=4$ there is only one stationary point.

**(1 mark)**

**h.** $x$-intercepts occur when $g(x)=0$

Solve $g(x)=0$ for $x$

$x=1\pm\sqrt{1-k},\ x=-2$

We need 3 distinct solutions, i.e. 3 $x$–intercepts.
So if any two of the solutions above are equal then we won't have 3 $x$–intercepts.
To find such cases:
Solve $1+\sqrt{1-k}=-2$ for $k$
False i.e. no solution
Solve $1-\sqrt{1-k}=-2$ for $k$

$k=-8$

Solve $1+\sqrt{1-k}=1-\sqrt{1-k}$ for $k$

$k=1$

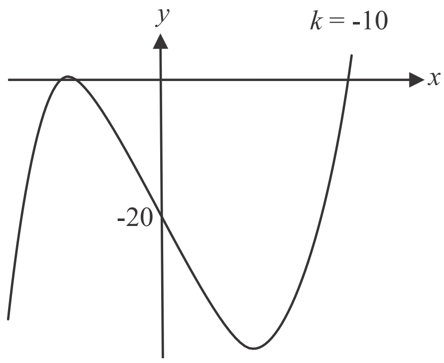
So if $k=-8$ or $k=1$ we <u>won't</u> have 3 $x$–intercepts.

**(1 mark)**

<u>Method 1</u> – algebraically
Also, from earlier, we saw that the $x$–intercepts occur when $x=1\pm\sqrt{1-k}$.
For any $x$ values to exist, $1-k\geq 0$ i.e. $k\leq 1$ but $k=1$ was ruled out above. **(1 mark)**
So combining all the limitations we have on the values of $k$ for 3 $x$–intercepts to exist, we have $k\in(-\infty,1)\setminus\{-8\}$.

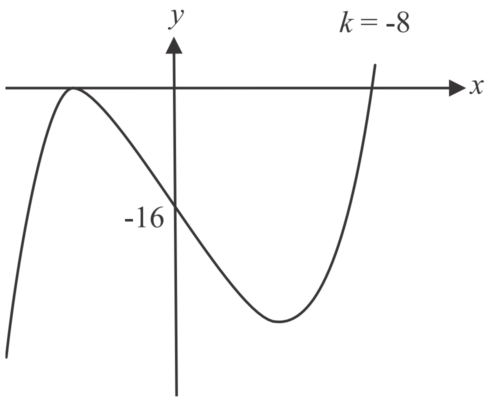
**(1 mark)**

<u>Method 2</u> – graphically
Using a slider, set different values of $k$.
For example when $k=-10$, there are 3 $x$–intercepts as shown below.

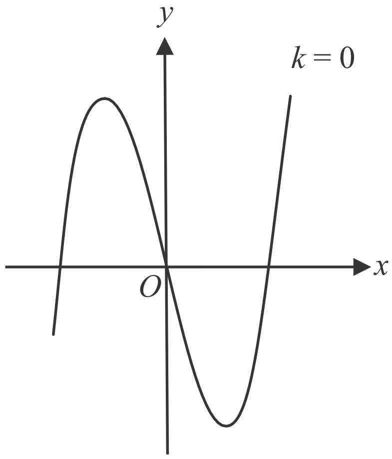



When $k=-8$, there are 2 $x$–intercepts (we know there wouldn't be 3!) as shown below.

**(1 mark)**

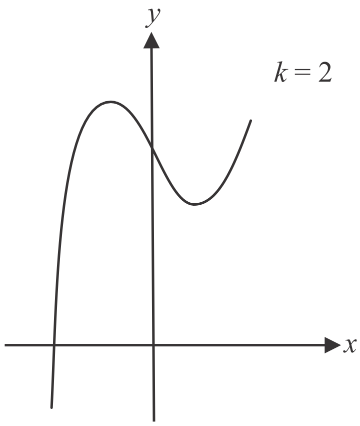


When $k = 0$, there are 3 *x*–intercepts as shown below.




When $k = 1$, there are 2 *x*–intercepts (again, we knew that there weren't going to be 3!). This case is actually the graph of $y = f(x)$.

When $k = 2$, there is 1 *x*–intercept as shown below.




If you continue to draw graphs where $k > 1$, they will have just one *x*–intercept.
In summary, for 3 *x*–intercepts to exist, we require $k \in (-\infty, 1) \setminus \{-8\}$.

**(1 mark)**

**Question 2** (9 marks)

**a.** **i.** $h$ is a composite function and it exists because

$r_g \subseteq d_f$

i.e. $R \subseteq R$

**(1 mark)**

**ii.** $D = d_{f \circ g}$

$= d_g$

$= (0, \infty)$ (or $R^+$)

**(1 mark)**

**iii.** Method 1

$h(x) = (f \circ g)(x)$

$= [\log_e(x)]^2 - 1$ (Place your brackets carefully)

Do a quick sketch on your CAS.

The minimum turning point of the graph occurs at the point $(1, -1)$.

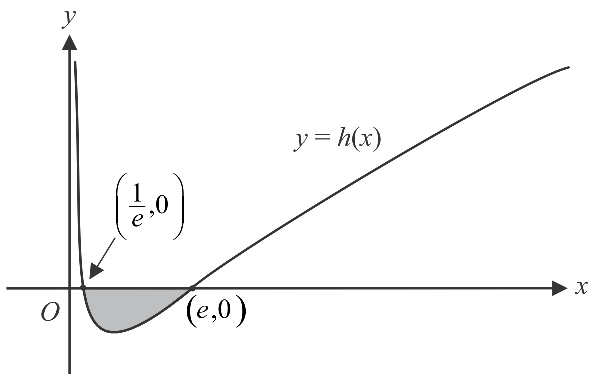
$r_h = [-1, \infty)$ **(1 mark)**

Method 2 – tracking the domains and ranges

$R^+ \to$ [ $g$ ] —$R$— [ $f$ ] $\to [-1, \infty)$

$r_h = [-1, \infty)$ **(1 mark)**

**b.** Solve $h(x) = 0$ for $x$ to find the $x$–intercepts of the graph, $x = e^{-1}$ or $x = e$.




$$\text{area} = -\int_{e^{-1}}^{e} h(x)\,dx$$

**(1 mark)** for the integrand and the $-1$

**(1 mark)** for terminals

$= \frac{4}{e}$ square units **(1 mark)**

(Note that since the area required lies below the $x$–axis, we must multiply the integral by $-1$.)

**c.** Using CAS, $h'(e) = \frac{2}{e}$

$\tan(\theta) = \frac{2}{e}$

$\theta = \tan^{-1}\left(\frac{2}{e}\right)$

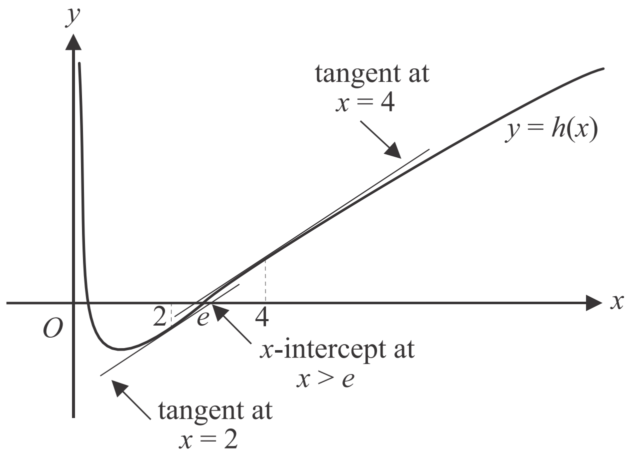
$= 36.34...°$

$\theta = 36°$ (to nearest degree) **(1 mark)**

**d.** Solve $h'(x)=\log_e(2)$ for $x$

$x=2$ or $x=4$ **(1 mark)**

The two parallel tangents at $x=2$ and $x=4$ are shown on the diagram below.




From the graph we see that the tangent which has an $x$–intercept greater than $e$, is the tangent to $h$ at the point where $x=2$.

Using CAS, the equation of this tangent is $y=x\log_e(2)+[\log_e(2)]^2-2\log_e(2)-1$

So the $y$-intercept of this tangent is $[\log_e(2)]^2-2\log_e(2)-1$.

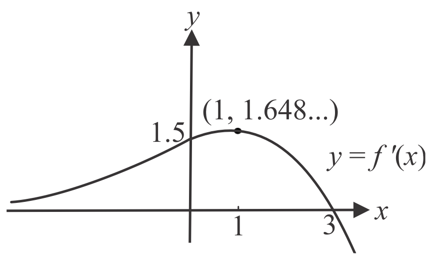
**(1 mark)**

**Question 3** (12 marks)

**a.** Define *f* on your CAS.

$f'(x) = \left(\frac{3}{2} - \frac{x}{2}\right)e^{\frac{x}{2}}$ **(1 mark)**

**b.** Sketch the graph of the gradient function of $f$ ie $y = f'(x)$, on your CAS.




The graph of $y = f'(x)$ has a maximum turning point at $(1, 1.648...)$.
You can confirm this if you wish by solving

$$\frac{d}{dx}(f'(x)) = 0 \text{ for } x$$
$$x = 1$$
$$f'(1) = 1.648...$$

The graph of $y = f'(x)$ is strictly increasing for $x \in (-\infty, 1]$. **(1 mark)**
Note that the $x$ coordinate of the turning point is included.

**c.** **i.** Solve $f'(x) = 0$ for $x$ using CAS

$x = 3$

So the maximum turning point on the graph of *f* occurs at $x = 3$.
If $g$ has an inverse then $g$ must be a 1:1 function so $a = 3$.

**(1 mark)**

**ii.**

$$d_{g^{-1}} = r_g$$
Since $d_g = (-\infty, 3]$
then $r_g = (0,\ f(3)]$
$$= (0,\ 2e^{\frac{3}{2}} + 1]$$
So $d_{g^{-1}} = (0,\ 2e^{\frac{3}{2}} + 1]$.

**(1 mark)**

Note: since we have had no direction in the question to express endpoints as an approximation, i.e. to a certain number of decimal places, then we must give the right endpoint of the interval as an exact value i.e. $2e^{\frac{3}{2}} + 1$.

**d.** area of five trapeziums

$= \frac{1}{2}[f(0) + 2f(1) + 2f(2) + 2f(3) + 2f(4) + f(5)] \times 1$ **(1 mark)**

$= 38.6021...$

$= 38.6$ square units (correct to one decimal place)

**(1 mark)**

**e.** **i.** From the graph, we see that the trapezium on the far left between $x=0$ and $x=1$ has its diagonal top edge just above the curve of $y=f(x)$.
So its area will be an overestimate.
So $p=0$ and $q=1$

**(1 mark)**

**ii.** From part **b**., $f\,'(x)$ is strictly increasing for $x\in(-\infty,1]$ i.e. the graph of *f* is concave up over this interval.
For $x\in[1,\infty)$, $f\,'(x)$ is strictly decreasing and the graph of *f* is concave down.
So there is a point of inflection at $x=1$.
So for $x\in[0,1]$, where the curve is concave up, the top side of the trapezium will be above the graph and the trapezium will overestimate the actual area.

**(1 mark)**

**f.**

$$x_0=5$$

$$x_1=x_0-\frac{f(x_0)}{f\,'(x_0)}$$

$$=5-\frac{f(5)}{f\,'(5)}$$

**(1 mark)**

$$=5-\frac{(5-5)e^{\frac{5}{2}}+1}{\left(\frac{3}{2}-\frac{5}{2}\right)e^{\frac{5}{2}}} \quad \text{(from part \textbf{a.})}$$

$$=5-\frac{1}{-e^{\frac{5}{2}}}$$

$$=5+e^{-\frac{5}{2}} \text{ as required}$$

**(1 mark)**

**g.** Solve $f(x)=0$ for $x$ using CAS
actual value of $x=5.078909...$
$=5.0789$ (correct to 4 decimal places)
From part **f.**, $x_0=5$ and $x_1=5.0820...$

$$x_2=x_1-\frac{f(x_1)}{f\,'(x_1)}$$

$$=5.0820...-\frac{f(5.0820...)}{f\,'(5.0820...)}$$

$$=5.078914...$$

**(1 mark)**

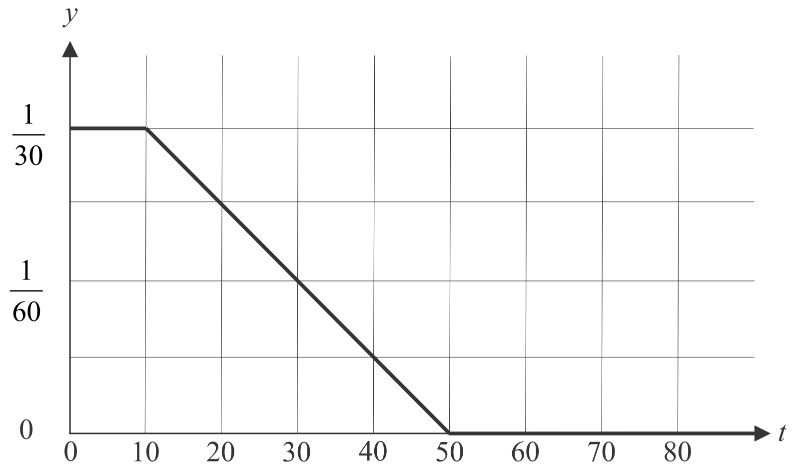
$$=5.0789 \text{ (to 4 decimal places)}$$

So $n=2$

**(1 mark)**

**Question 4** (15 marks)

**a.** **i.**




**(1 mark)** line for $0 \le t < 10$
**(1 mark)** line for $10 \le t \le 50$
**(1 mark)** line for $t > 50$

**ii.** Define $f(t)$ on your CAS.

$$\int_5^{30} f(t)\,dt = \frac{2}{3}$$

**(1 mark)** for definite integral
**(1 mark)** for correct answer

**iii.** Solve $\int_0^k f(t)\,dt = 0.9$ for $k$ **(1 mark)**

$k = 34.5080...$

$k = 34.5$ (correct to one decimal place)

**(1 mark)**

**iv.** $\text{Var}(T) = \text{E}(T^2) - \mu^2$ (formula sheet)

$$= \int_0^{50} t^2 f(t)\,dt - \left(\frac{155}{9}\right)^2$$

$$= \frac{11075}{81}$$

$$\text{sd}(T) = \sqrt{\frac{11075}{81}}$$

$= 11.6930...$

$= 11.7$ minutes (correct to 1 decimal place)

**(1 mark)**

Note – if your CAS won't calculate $\int_0^{50} t^2 f(t)\,dt$, try breaking the integral up into $\int_0^{10} t^2 f(t)\,dt + \int_{10}^{50} t^2 f(t)\,dt$ .

**b.** **i.** $0.72^4 = 0.26873...$
$= 0.269$ (to 3 decimal places)

**(1 mark)**

**ii.** Let $X$ be the number of the four shoppers entering the store who make a purchase.
$X \sim \text{Bi}(4, 0.72)$
$\Pr(X > 1) = \Pr(X \geq 2)$ binomCdf$(4, 0.72, 2, 4)$
$= 0.9306...$
$= 0.931$ (to 3 decimal places)

**(1 mark)**

**iii.** $\Pr(X = 3 | X > 1)$
$= \frac{\Pr(X = 3)}{\Pr(X \geq 2)}$ binomialPdf$(4, 0.72, 3)$ (for numerator) Use part **ii.** (for denominator)
$= \frac{0.4180...}{0.9306...}$ **(1 mark)** – recognition of conditional probability
$= 0.4491...$
$= 0.449$ (to 3 decimal places)

**(1 mark)**

**c.** **i.** $\text{sd}(\hat{P}) = \sqrt{\frac{p(1-p)}{n}}$ (formula sheet)
$= \sqrt{\frac{0.48 \times 0.52}{100}}$
$= 0.04995...$
$= 0.0500$ (to 4 decimal places)

**(1 mark)**

**ii.** There were 48 shoppers in the sample of 100 who paid for a bag.
Use CAS to find the confidence interval $(0.3821, 0.5779)$. **(1 mark)**
(using zInterval_1Prop 48,100,0.95)

**iii.** The distance between the sample estimate $\hat{p}$ (i.e. 0.48) and the endpoints of the confidence interval found in part **ii**. is the margin of error, *m*, where
$m = 1.96\sqrt{\frac{\hat{p}(1-\hat{p})}{n}}$
$= 1.96\sqrt{\frac{0.48 \times 0.52}{100}}$
$= 0.09792...$
We can reduce *m* by 50%, (and hence reduce the width of the confidence interval by 50%) by changing the sample size *n*.
Solve $\frac{0.09792...}{2} = 1.96\sqrt{\frac{0.48 \times 0.52}{n}}$ for $n$
$n = 400$

**(1 mark)**

**Question 5** (10 marks)

**a.** The maximum value of $h(t)$ is 30 cm when the weight is at the extreme left or right of its motion.
The minimum value of $h(t)$ is 20 cm when the weight is vertically below *P*.
Since *a* is the amplitude then $a=5$.

**(1 mark)**

**b.** The time it takes for one swing is the period of the function (i.e. the weight starts 30 cm above the ground, drops to 20 cm then returns to 30 cm above the ground before repeating the process).

$$\text{period} = \frac{2\pi}{\pi}$$
$$= 2 \text{ seconds}$$

The pendulum makes $60 \div 2 = 30$ swings per minute. **(1 mark)**

**c.** From part **b**., a swing takes 2 seconds (i.e. the period is 2 seconds).
Solve $h(t) = 27.5$ for $t \in [0,2]$

$$t = \frac{1}{3} \text{ or } t = \frac{5}{3}$$

Since $h(0) = 30$, the weight starts at 30 cm above the ground.

By $t = \frac{1}{3}$ the weight will be at 27.5 cm above the ground and it will be less than 27.5 cm above the ground up until $t = \frac{5}{3}$ when it will be back at 27.5 cm above the ground. So $h(t) \le 27.5$ for $\frac{1}{3} \le t \le \frac{5}{3}$. **(1 mark)**

The proportion of time during each swing when the weight is no more than 27.5 cm above the ground is

$$\left(\left(\frac{5}{3}-\frac{1}{3}\right) \div 2 \times 100\right)\%$$
$$= 66.6666...\%$$
$$= 67\% \text{ (to the nearest whole percent)}$$

**(1 mark)**

**d.** We require the average value of $h(t)$ for $t \in [0, 0.5]$.

$$\frac{1}{0.5-0}\int_0^{0.5} h(t)\,dt$$ **(1 mark)**

$$= 28.1830...$$
$$= 28.18 \text{ cm (correct to 2 decimal places)}$$ **(1 mark)**

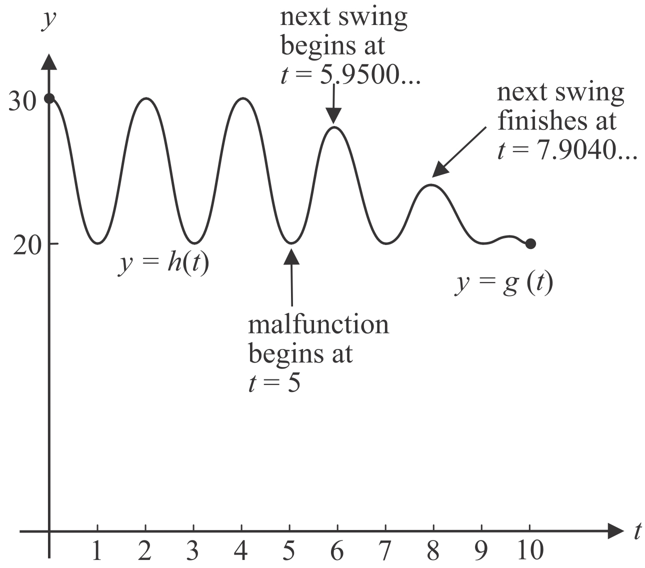
**e.** $h(5) = 20$

Now g(5) is undefined but since the join of *h* and *g* is described as 'continuous', then we require that $\lim_{t \to 5} [g(t)] = 20$

$$\lim_{t \to 5} [k + (10-t)(\cos(\pi t)+1)] = 20$$
$$k + (10-5)(\cos(5\pi)+1) = 20$$
$$k + 5(-1+1) = 20$$
$$k = 20$$

**(1 mark)**

**f.**




A swing is defined as the "movement of the pendulum from one side to the other". The malfunction occurs at $t=5$ and $h(5)=20$ i.e. when the weight is 20 cm above the ground, i.e. in the 'middle' of a swing.

The next swing starts when
$g'(t)=0, \quad 5<t<7$
$t=5.9500...$ **(1 mark)**

The next swing finishes when
$g'(t)=0, \quad 7<t<9$
$t=7.9040...$ **(1 mark)**

The difference in height is given by

$g(5.9500...)-g(7.9040...)$
$=3.9527...$
$=3.95$ cm (correct to two decimal places)

**(1 mark)**