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MATHS METHODS UNITS 3 & 4 TRIAL EXAMINATION 1 SOLUTIONS 2023

Question 1 (3 marks)

a.

$$y = (x^2 + 3x + 2)^5$$

$$\frac{dy}{dx} = 5(x^2 + 3x + 2)^4(2x + 3) \quad \text{(chain rule)}$$

(1 mark)

b.

$$h(x) = \frac{\tan(x)}{3x}$$

$$h'(x) = \frac{3x \times \sec^2(x) - 3\tan(x)}{(3x)^2} \quad \text{(quotient rule)}$$

(1 mark)

$$h'\left(\frac{\pi}{3}\right) = \frac{\pi \times \left(\frac{1}{\cos^2\left(\frac{\pi}{3}\right)}\right) - 3\tan\left(\frac{\pi}{3}\right)}{\left(3 \times \frac{\pi}{3}\right)^2}$$

$$= \frac{\pi \times \frac{1}{\left(\frac{1}{2}\right)^2} - 3\sqrt{3}}{\pi^2}$$

$$= \frac{4\pi - 3\sqrt{3}}{\pi^2}$$

(1 mark)

Question 2 (3 marks)

a.

$$\int \frac{1}{3x+2}\,dx = \frac{1}{3}\log_e(3x+2) + c$$

(Note that "+c" is required here.)

(1 mark)

b. Using the rule $\int (ax+b)^n\,dx = \frac{1}{a(n+1)}(ax+b)^{n+1}+c, \quad n \neq -1$ from the formula sheet ,

$$\begin{aligned} g'(x) &= \frac{1}{(2x-1)^2} \\ &= (2x-1)^{-2} \\ g(x) &= \int (2x-1)^{-2}\,dx \\ &= \frac{1}{2\times -1}(2x-1)^{-1}+c \\ &= \frac{-1}{2(2x-1)}+c \end{aligned}$$

(1 mark)

Given $g\left(\frac{1}{4}\right)=5$

then $5 = \dfrac{-1}{2\left(\frac{1}{2}-1\right)}+c$

$c = 4$

So $g(x) = \dfrac{-1}{2(2x-1)}+4$

(1 mark)

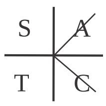
Question 3 (3 marks)

Method 1

$\sqrt{3}-\cos(2x) = \cos(2x) \qquad x \in R$

$\sqrt{3} = 2\cos(2x)$

$\cos(2x) = \frac{\sqrt{3}}{2}$



Cosine is positive in the first and fourth quadrants and the base angle is $\frac{\pi}{6}$.

(1 mark)

1st quadrant solution:

$2x = \frac{\pi}{6}+2k\pi, \quad k \in Z$

$x = \frac{\pi}{12}+k\pi, \quad k \in Z$

(1 mark)

4th quadrant solution:

$2x = \left(2\pi-\frac{\pi}{6}\right)+2k\pi, \quad k \in Z$

$= \frac{11\pi}{6}+2k\pi$

$x = \frac{11\pi}{12}+k\pi, \quad k \in Z$

(1 mark)

Method 2

$\sqrt{3}-\cos(2x) = \cos(2x), \qquad x \in R$

$\sqrt{3} = 2\cos(2x)$

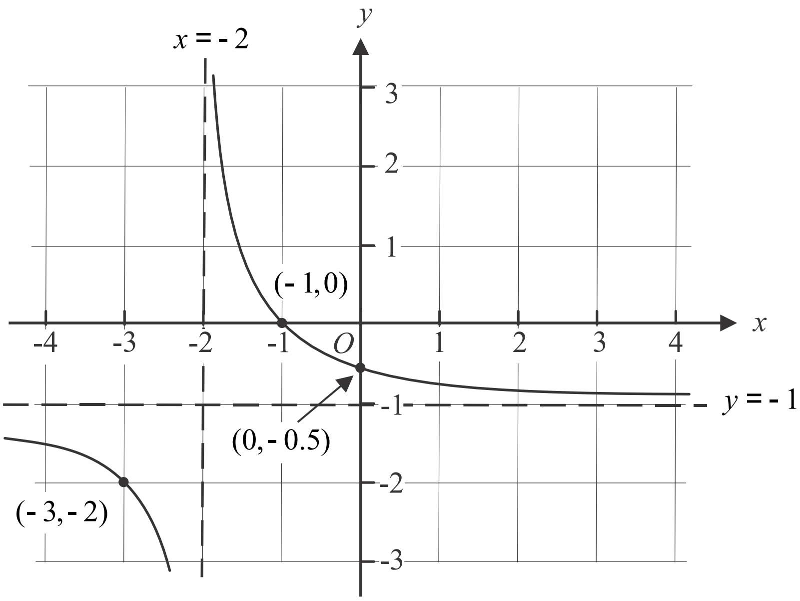
$\cos(2x) = \frac{\sqrt{3}}{2}$ (base angle is $\frac{\pi}{6}$) **(1 mark)**

$2x = 2k\pi \pm \frac{\pi}{6}, \quad k \in Z$ **(1 mark)**

$x = k\pi \pm \frac{\pi}{12}, \quad k \in Z$ **(1 mark)**

Question 4 (5 marks)

a.



<u>x-intercepts</u> occur when $y = 0$

$$0 = \frac{1}{x+2} - 1$$
$$1 = \frac{1}{x+2}$$
$$x + 2 = 1$$
$$x = -1$$

<u>y-intercepts</u> occur when $x = 0$

$$y = \frac{1}{2} - 1$$
$$= -\frac{1}{2}$$

(1 mark) – correct asymptotes with equations **(1 mark) –** correct intercepts
(1 mark) – correct branches/shape

b. $$f(x) = \frac{1}{x+2} - 1$$

Let $$y = \frac{1}{x+2} - 1$$

Swap x and y for inverse.

$$x = \frac{1}{y+2} - 1$$
$$x + 1 = \frac{1}{y+2}$$
$$(x+1)(y+2) = 1$$
$$y + 2 = \frac{1}{x+1}$$
$$y = \frac{1}{x+1} - 2$$

So $f^{-1}(x) = \frac{1}{x+1} - 2$ **(1 mark)**

$r_f = R \setminus \{-1\}$ from part **a**.
$d_{f^{-1}} = r_f$
$= R \setminus \{-1\}$ **1 mark)**

Question 5 (3 marks)

a. Method 1

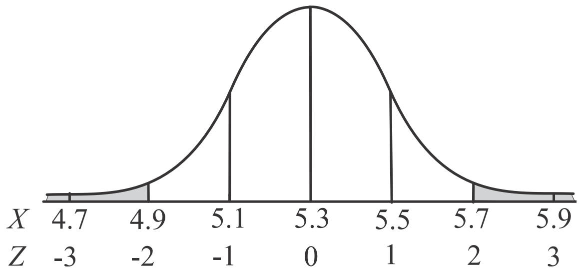
Draw a diagram.

$\Pr(X < 4.9) = \Pr(Z < -2)$

$= \Pr(Z > 2)$

by symmetry

So $a = 2$. **(1 mark)**



Method 2

$x = 4.9$

so $z = \dfrac{4.9 - 5.3}{0.2}$ using the rule $z = \dfrac{x - \mu}{\sigma}$

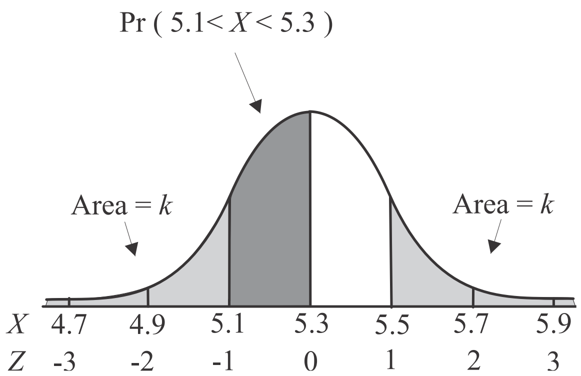
$= -2$

$\Pr(X < 4.9) = \Pr(Z < -2)$

$= \Pr(Z > 2)$ by symmetry

So $a = 2$. **(1 mark)**

b.



Note that since $\Pr(Z > 1) = k$, then $\Pr(Z < -1) = k$ by symmetry.

$\Pr(X > 5.1 | X < 5.3)$ (conditional probability)

$= \dfrac{\Pr(5.1 < X < 5.3)}{\Pr(X < 5.3)}$ **(1 mark)**

$= \dfrac{0.5 - k}{0.5}$ Note that $\Pr(X < 5.3) = 0.5$

$= (0.5 - k) \div \dfrac{1}{2}$

$= (0.5 - k) \times 2$

$= 1 - 2k$

(1 mark)

Question 6 (4 marks)

a. Since f is a probability density function, then

$$\int_1^4 k\sqrt{x}\,dx = 1$$

$$k\int_1^4 x^{\frac{1}{2}}\,dx = 1$$

(1 mark)

$$k\left[\frac{2}{3}x^{\frac{3}{2}}\right]_1^4 = 1$$

$$\frac{2k}{3}\left(4^{\frac{3}{2}} - 1^{\frac{3}{2}}\right) = 1 \qquad \text{Note } 4^{\frac{3}{2}} = \left(\sqrt{4}\right)^3 = 8$$

$$\frac{2k}{3}(8-1) = 1$$

$$14k = 3$$

$$k = \frac{3}{14}$$

as required

(1 mark)

b.

$$\begin{aligned}\mathrm{E}(X) &= \int_1^4 x\,f(x)dx\\ &= \int_1^4 x \times \frac{3}{14}\sqrt{x}\,dx\\ &= \frac{3}{14}\int_1^4 x^{\frac{3}{2}}\,dx\end{aligned}$$

(1 mark)

$$\begin{aligned}&= \frac{3}{14}\left[\frac{2}{5}x^{\frac{5}{2}}\right]_1^4\\ &= \frac{3}{14} \times \frac{2}{5}\left(4^{\frac{5}{2}} - 1^{\frac{5}{2}}\right)\\ &= \frac{3}{35}(32-1)\\ &= \frac{93}{35}\\ &= 2\frac{23}{35}\end{aligned}$$

(1 mark)

Question 7 (3 marks)

Method 1

OP runs perpendicular to $y=\frac{1}{2}x+3$ (in order to make the shortest distance) and therefore has a gradient of – 2. *OP* therefore has the equation $y=-2x$. **(1 mark)**

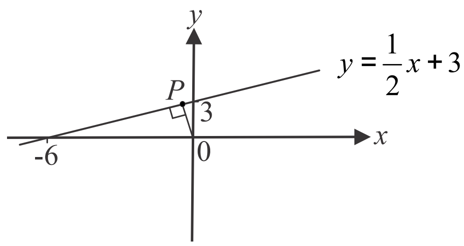
Solving $y=\frac{1}{2}x+3$ and $y=-2x$ simultaneously gives

$$-2x=\frac{1}{2}x+3$$

$$-\frac{5}{2}x=3$$

$$x=-\frac{6}{5}$$ **(1 mark)**

$$y=\frac{12}{5}$$



P is the point $\left(-\frac{6}{5},\frac{12}{5}\right)$. **(1 mark)**

Method 2 – using the distance formula

We have the two points $O(0,0)$ and $P\left(x,\left(\frac{1}{2}x+3\right)\right)$.

Let $D=$ distance from *O* to *P*

$$=\sqrt{(x-0)^2+\left(\frac{1}{2}x+3-0\right)^2}$$

$$=\sqrt{x^2+\frac{1}{4}x^2+3x+9}$$

$$=\sqrt{\frac{5}{4}x^2+3x+9}$$

$$=\left(\frac{5}{4}x^2+3x+9\right)^{\frac{1}{2}}$$ **(1 mark)**

$$\frac{dD}{dx}=\frac{1}{2}\left(\frac{5}{4}x^2+3x+9\right)^{-\frac{1}{2}}\times\left(\frac{5}{2}x+3\right)$$

$$=\frac{\frac{5}{2}x+3}{2\sqrt{\frac{5}{4}x^2+3x+9}}$$

$\frac{dD}{dx}=0$ for minimum **(1 mark)**

So $\frac{5}{2}x+3=0$ (note the denominator of $\frac{dD}{dx}$ cannot equal zero)

$$x=-\frac{6}{5}$$

Substitute $x=-\frac{6}{5}$ into $y=\frac{1}{2}x+3$

gives $y=\frac{12}{5}$

P is the point $\left(-\frac{6}{5},\frac{12}{5}\right)$.

(1 mark)

Question 8 (7 marks)

a. <u>x-intercepts</u> occur when $y=0$

$$0=3-e^{kx}$$
$$e^{kx}=3$$
$$\log_e(3)=kx$$
$$x=\frac{1}{k}\log_e(3) \text{ as required}$$

(1 mark)

b. $$\text{average value}=\frac{1}{\left(\frac{1}{k}\log_e(3)-0\right)}\int_0^{\frac{1}{k}\log_e(3)}(3-e^{kx})\,dx$$ **(1 mark)**

$$=\frac{k}{\log_e(3)}\left[3x-\frac{1}{k}e^{kx}\right]_0^{\frac{1}{k}\log_e(3)}$$ **(1 mark)**

$$=\frac{k}{\log_e(3)}\left\{\left(\frac{3}{k}\log_e(3)-\frac{1}{k}e^{\log_e(3)}\right)-\left(0-\frac{1}{k}e^0\right)\right\}$$

$$=\frac{k}{\log_e(3)}\left(\frac{3}{k}\log_e(3)-\frac{3}{k}+\frac{1}{k}\right)$$ Note that $e^{\log_e(3)}=3.$ **(1 mark)**

$$=\frac{k}{\log_e(3)}\left(\frac{3}{k}\log_e(3)-\frac{2}{k}\right)$$

$$=3-\frac{2}{\log_e(3)}$$

(1 mark)

c. $$\text{average rate of change}=\frac{f(1)-f(0)}{1-0}$$
$$=3-e^k-(3-e^0)$$
$$=3-e^k-3+1$$
$$=1-e^k$$

(1 mark)

We require $1-e^k<0$
$$-e^k<-1$$
$$e^k>1$$
$$k>0$$

(1 mark)

Question 9 (9 marks)

a.

$$f(x) = x\sin(\pi x)$$

$$f'(x) = x \times \pi\cos(\pi x) + \sin(\pi x) \qquad \text{(product rule)}$$

(1 mark)

$$f'\left(\frac{1}{4}\right) = \frac{\pi}{4}\cos\left(\frac{\pi}{4}\right) + \sin\left(\frac{\pi}{4}\right)$$

$$= \frac{\pi}{4\sqrt{2}} + \frac{1}{\sqrt{2}}$$

$$= \frac{\pi + 4}{4\sqrt{2}}$$

$$= \frac{\sqrt{2}(\pi + 4)}{8}$$

(1 mark)

b. Area required = $\int_0^1 x\sin(\pi x)\,dx$

Rearranging $\frac{d}{dx}(x\cos(\pi x)) = \cos(\pi x) - \pi x\sin(\pi x)$

gives $\pi x\sin(\pi x) = \cos(\pi x) - \frac{d}{dx}(x\cos(\pi x))$

so $$\pi\int_0^1 x\sin(\pi x)dx = \int_0^1 \cos(\pi x)\,dx - \int_0^1 \frac{d}{dx}(x\cos(\pi x))\,dx$$

$$= \left[\frac{1}{\pi}\sin(\pi x)\right]_0^1 - \left[x\cos(\pi x)\right]_0^1$$

(1 mark)

$$= \frac{1}{\pi}(\sin(\pi) - \sin(0)) - (\cos(\pi) - 0)$$

$$= \frac{1}{\pi}(0 - 0) - (-1)$$

$$= 1$$

So $$\int_0^1 x\sin(\pi x)\,dx = \frac{1}{\pi}$$

Area = $\frac{1}{\pi}$ square units **(1 mark)**

c. Required area $= \int_0^1 f(x)\,dx - \int_1^2 f(x)\,dx + \int_2^3 f(x)\,dx$

From part **b.**, $\int_0^1 f(x)\,dx = \frac{1}{\pi}$.

Note that $\int_1^2 f(x-1)\,dx = \int_0^1 f(x)\,dx$ because the graph of $y = f(x)$ has been translated 1 unit right to become the graph of $y = f(x-1)$.

So
$$\begin{aligned}\int_1^2 f(x)\,dx &= -3\int_1^2 f(x-1)\,dx \\ &= -3\int_0^1 f(x)\,dx \\ &= -3\times\frac{1}{\pi} \\ &= -\frac{3}{\pi}\end{aligned}$$

(1 mark)

Also
$$\begin{aligned}\int_1^3 f(x)\,dx &= 2\int_0^1 f(x)dx \\ &= 2\times\frac{1}{\pi} \\ &= \frac{2}{\pi}\end{aligned}$$

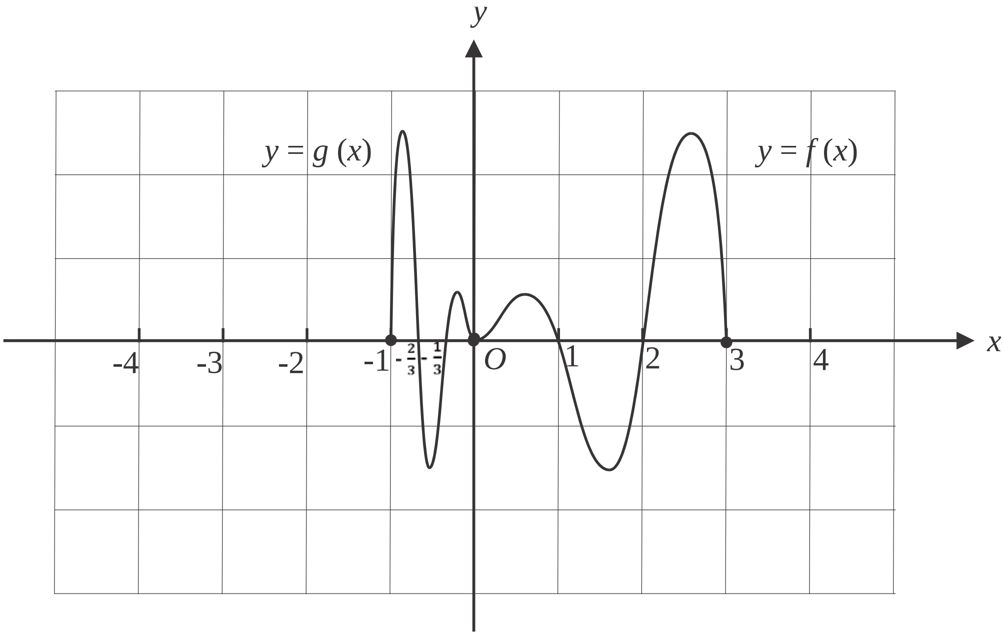
(1 mark)

So $\int_2^3 f(x)\,dx + \int_1^2 f(x)\,dx = \frac{2}{\pi}$
$$\begin{aligned}\int_2^3 f(x)\,dx &= \frac{2}{\pi} + \frac{3}{\pi} \\ &= \frac{5}{\pi}\end{aligned}$$

So area between the graph of *f* and the *x*-axis is $\frac{1}{\pi} + \frac{3}{\pi} + \frac{5}{\pi} = \frac{9}{\pi}$.

(1 mark)

d. **i.** <u>Method 1</u> – sketch the graph of *g*



$d_g = [-1, 0]$ **(1 mark)**

<u>Method 2</u>

$d_f = [0, 3]$

If the graph of *f* is dilated by a factor of $\frac{1}{3}$ from the *y*-axis (i.e. compressed horizontally) then the new domain will be $x \in [0,1]$.
If this graph is then reflected in the *y*-axis to become the graph of *g*, then $d_g = [-1, 0]$.

(1 mark)

ii. $f(x) = x\sin(\pi x)$
Let $y = x\sin(\pi x)$
After a dilation by a factor of $\frac{1}{3}$ from the *y*-axis, replace *x* with $\frac{x}{1/3} = 3x$
and we obtain $y = 3x\sin(3\pi x)$.

After a reflection in the *y*-axis, replace *x* with $-x$

and we obtain $y = -3x\sin(-3\pi x)$.

So $g(x) = -3x\sin(-3\pi x)$. **(1 mark)**
The equivalent answer of $g(x) = 3x\sin(3\pi x)$ is also acceptable.