VICTORIAN CURRICULUM AND ASSESSMENT AUTHORITY

Victorian Certificate of Education 2022

SUPERVISOR TO ATTACH PROCESSING LABEL HERE

STUDENT NUMBER | | | | | | | | | Letter |

MATHEMATICAL METHODS

Written examination 2

Thursday 3 November 2022

Reading time: 11.45 am to 12.00 noon (15 minutes)
Writing time: 12.00 noon to 2.00 pm (2 hours)

QUESTION AND ANSWER BOOK

Structure of book

Section	*Number of questions*	*Number of questions to be answered*	*Number of marks*
A	20	20	20
B	5	5	60
			Total 80

- Students are permitted to bring into the examination room: pens, pencils, highlighters, erasers, sharpeners, rulers, a protractor, set squares, aids for curve sketching, one bound reference, one approved technology (calculator or software) and, if desired, one scientific calculator. Calculator memory DOES NOT need to be cleared. For approved computer-based CAS, full functionality may be used.
- Students are NOT permitted to bring into the examination room: blank sheets of paper and/or correction fluid/tape.

Materials supplied
- Question and answer book of 25 pages
- Formula sheet
- Answer sheet for multiple-choice questions

Instructions
- Write your **student number** in the space provided above on this page.
- Check that your **name** and **student number** as printed on your answer sheet for multiple-choice questions are correct, **and** sign your name in the space provided to verify this.
- Unless otherwise indicated, the diagrams in this book are **not** drawn to scale.
- All written responses must be in English.

At the end of the examination
- Place the answer sheet for multiple-choice questions inside the front cover of this book.
- You may keep the formula sheet.

Students are NOT permitted to bring mobile phones and/or any other unauthorised electronic devices into the examination room.

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SECTION A – Multiple-choice questions

Instructions for Section A

Answer **all** questions in pencil on the answer sheet provided for multiple-choice questions.
Choose the response that is **correct** for the question.
A correct answer scores 1; an incorrect answer scores 0.
Marks will **not** be deducted for incorrect answers.
No marks will be given if more than one answer is completed for any question.
Unless otherwise indicated, the diagrams in this book are **not** drawn to scale.

Question 1

The period of the function $f(x) = 3\cos(2x + \pi)$ is

A. 2π

B. π

C. $\frac{2\pi}{3}$

D. 2

E. 3

Question 2

The graph of $y = \frac{1}{(x+3)^2} + 4$ has a horizontal asymptote with the equation

A. $y = 4$

B. $y = 3$

C. $y = 0$

D. $x = -2$

E. $x = -3$

Question 3

The gradient of the graph of $y = e^{3x}$ at the point where the graph crosses the vertical axis is equal to

A. 0

B. $\frac{1}{e}$

C. 1

D. e

E. 3

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Question 4

Which one of the following functions is not continuous over the interval $x \in [0, 5]$?

A. $f(x) = \dfrac{1}{(x+3)^2}$

B. $f(x) = \sqrt{x+3}$

C. $f(x) = x^{\frac{1}{3}}$

D. $f(x) = \tan\left(\dfrac{x}{3}\right)$

E. $f(x) = \sin^2\left(\dfrac{x}{3}\right)$

Question 5

The largest value of a such that the function $f: (-\infty, a] \to R$, $f(x) = x^2 + 3x - 10$, where f is one-to-one, is

A. -12.25

B. -5

C. -1.5

D. 0

E. 2

Question 6

Which of the pairs of functions below are **not** inverse functions?

A. $\begin{cases} f(x) = 5x + 3 & x \in R \\ g(x) = \dfrac{x-3}{5} & x \in R \end{cases}$

B. $\begin{cases} f(x) = \dfrac{2}{3}x + 2 & x \in R \\ g(x) = \dfrac{3}{2}x - 3 & x \in R \end{cases}$

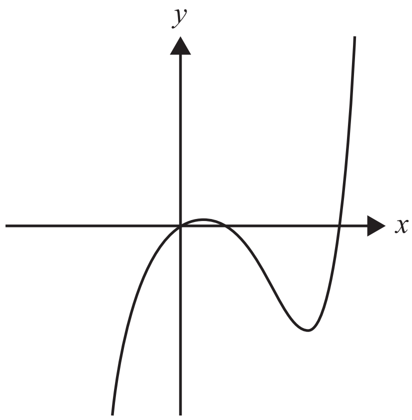
C. $\begin{cases} f(x) = x^2 & x < 0 \\ g(x) = \sqrt{x} & x > 0 \end{cases}$

D. $\begin{cases} f(x) = \dfrac{1}{x} & x \neq 0 \\ g(x) = \dfrac{1}{x} & x \neq 0 \end{cases}$

E. $\begin{cases} f(x) = \log_e(x) + 1 & x > 0 \\ g(x) = e^{x-1} & x \in R \end{cases}$

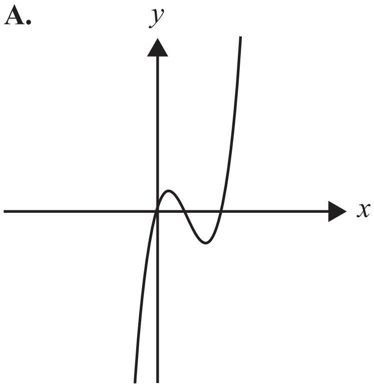
Question 7

The graph of $y = f(x)$ is shown below.



The graph of $y = f'(x)$, the first derivative of $f(x)$ with respect to x, could be

A.



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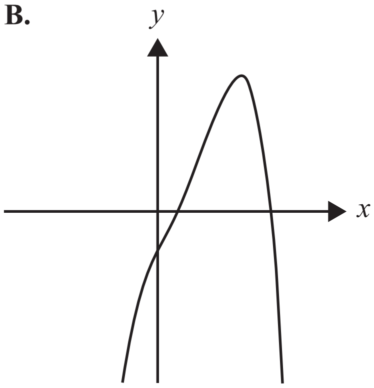
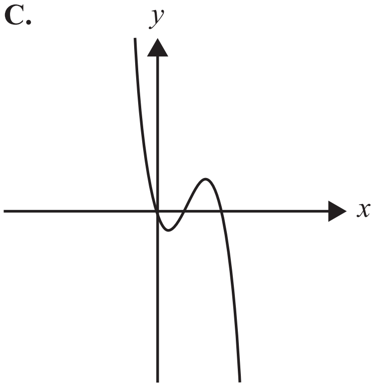
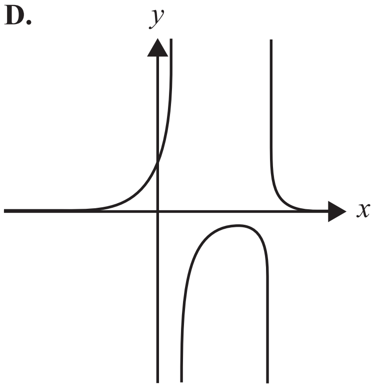


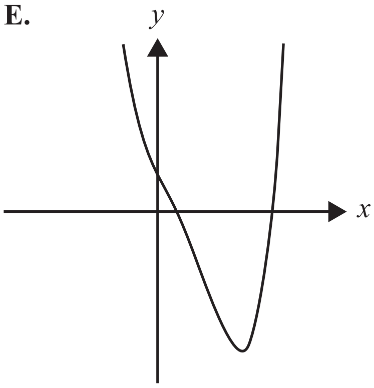
C.



D.



E.



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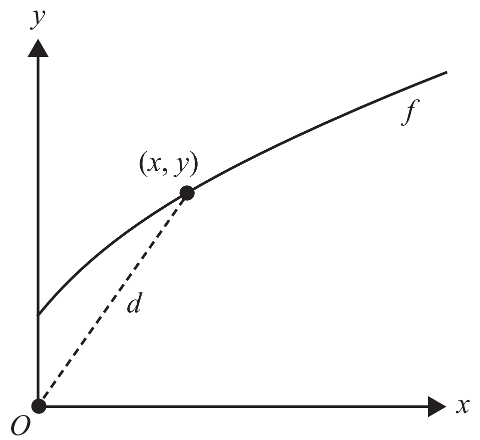
Question 8

If $\int_0^b f(x)\,dx = 10$ and $\int_0^a f(x)\,dx = -4$, where $0 < a < b$, then $\int_a^b f(x)\,dx$ is equal to

A. -6
B. -4
C. 0
D. 10
E. 14

Question 9



Let $f:[0, \infty) \to R,\ f(x) = \sqrt{2x+1}$.

The shortest distance, d, from the origin to the point (x, y) on the graph of f is given by

A. $d = x^2 + 2x + 1$
B. $d = x^2 + \sqrt{2x+1}$
C. $d = \sqrt{x^2 - 2x + 1}$
D. $d = x + 1$
E. $d = 2x + 1$

Question 10

An organisation randomly surveyed 1000 Australian adults and found that 55% of those surveyed were happy with their level of physical activity.

An approximate 95% confidence interval for the percentage of Australian adults who were happy with their level of physical activity is closest to

A. (4.1, 6.9)
B. (50.9, 59.1)
C. (52.4, 57.6)
D. (51.9, 58.1)
E. (45.2, 64.8)

Question 11

If $\frac{d}{dx}\left(x \cdot \sin(x)\right) = \sin(x) + x \cdot \cos(x)$, then $\frac{1}{k}\int x\cos(x)\,dx$ is equal to

A. $k\left(x \cdot \sin(x) - \int \sin(x)\,dx\right) + c$

B. $\frac{1}{k}x \cdot \sin(x) - \int \sin(x)\,dx + c$

C. $\frac{1}{k}\left(x \cdot \sin(x) - \int \sin(x)\,dx\right) + c$

D. $\frac{1}{k}\left(x \cdot \sin(x) - \sin(x)\right) + c$

E. $\frac{1}{k}\left(\int x \cdot \sin(x)\,dx - \int \sin(x)\,dx\right) + c$

Question 12

A bag contains three red pens and x black pens. Two pens are randomly drawn from the bag without replacement. The probability of drawing a pen of each colour is equal to

A. $\frac{6x}{(2+x)(3+x)}$

B. $\frac{3x}{(2+x)(3+x)}$

C. $\frac{x}{2+x}$

D. $\frac{3+x}{(2+x)(3+x)}$

E. $\frac{3+x}{5+2x}$

Question 13

The function $f(x) = \log_e\left(\frac{x+a}{x-a}\right)$, where a is a positive real constant, has the maximal domain

A. $[-a, a]$

B. $(-a, a)$

C. $R \setminus [-a, a]$

D. $R \setminus (-a, a)$

E. R

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Question 14

A continuous random variable, X, has a probability density function given by

$$f(x)=\begin{cases}\frac{2}{9}xe^{-\frac{1}{9}x^2} & x \geq 0 \\ 0 & x < 0\end{cases}$$

The expected value of X, correct to three decimal places, is

A. 1.000
B. 2.659
C. 3.730
D. 6.341
E. 9.000

Question 15

The maximal domain of the function with rule $f(x)=\sqrt{x^2-2x-3}$ is given by

A. $(-\infty, \infty)$
B. $(-\infty, -3) \cup (1, \infty)$
C. $(-1, 3)$
D. $[-3, 1]$
E. $(-\infty, -1] \cup [3, \infty)$

Question 16

The function $f(x)=\frac{1}{3}x^3+mx^2+nx+p$, for $m, n, p \in R$, has turning points at $x=-3$ and $x=1$ and passes through the point (3, 4).

The values of m, n and p respectively are

A. $m=0, \quad n=-\frac{7}{3}, \quad p=2$
B. $m=1, \quad n=-3, \quad p=-5$
C. $m=-1, \quad n=-3, \quad p=13$
D. $m=\frac{5}{4}, \quad n=\frac{3}{2}, \quad p=-\frac{83}{4}$
E. $m=\frac{5}{2}, \quad n=6, \quad p=-\frac{91}{2}$

Question 17

A function g is continuous on the domain $x \in [a, b]$ and has the following properties:

- The average rate of change of g between $x = a$ and $x = b$ is positive.
- The instantaneous rate of change of g at $x = \frac{a+b}{2}$ is negative.

Therefore, on the interval $x \in [a, b]$, the function must be

A. many-to-one.

B. one-to-many.

C. one-to-one.

D. strictly decreasing.

E. strictly increasing.

Question 18

If X is a binomial random variable where $n = 20$, $p = 0.88$ and $\Pr(X \geq 16 \mid X \geq a) = 0.9175$, correct to four decimal places, then a is equal to

A. 11

B. 12

C. 13

D. 14

E. 15

Question 19

A box is formed from a rectangular sheet of cardboard, which has a width of a units and a length of b units, by first cutting out squares of side length x units from each corner and then folding upwards to form a container with an open top.

The maximum volume of the box occurs when x is equal to

A. $\dfrac{a - b + \sqrt{a^2 - ab + b^2}}{6}$

B. $\dfrac{a + b + \sqrt{a^2 - ab + b^2}}{6}$

C. $\dfrac{a - b - \sqrt{a^2 - ab + b^2}}{6}$

D. $\dfrac{a + b - \sqrt{a^2 - ab + b^2}}{6}$

E. $\dfrac{a + b - \sqrt{a^2 - 2ab + b^2}}{6}$

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Question 20

A soccer player kicks a ball with an angle of elevation of $\theta°$, where θ is a normally distributed random variable with a mean of 42° and a standard deviation of 8°.

The horizontal distance that the ball travels before landing is given by the function $d = 50\sin(2\theta)$.

The probability that the ball travels more than 40 m horizontally before landing is closest to

A. 0.969

B. 0.937

C. 0.226

D. 0.149

E. 0.027

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END OF SECTION A
TURN OVER

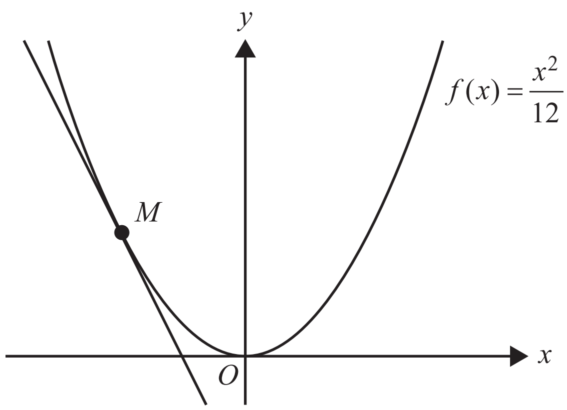
SECTION B

Instructions for Section B

Answer **all** questions in the spaces provided.
In all questions where a numerical answer is required, an exact value must be given unless otherwise specified.
In questions where more than one mark is available, appropriate working **must** be shown.
Unless otherwise indicated, the diagrams in this book are **not** drawn to scale.

Question 1 (11 marks)

The diagram below shows part of the graph of $y = f(x)$, where $f(x) = \frac{x^2}{12}$.



a. State the equation of the axis of symmetry of the graph of f. 1 mark

b. State the derivative of f with respect to x. 1 mark

The tangent to f at point M has gradient -2.

c. Find the equation of the tangent to f at point M. 2 marks

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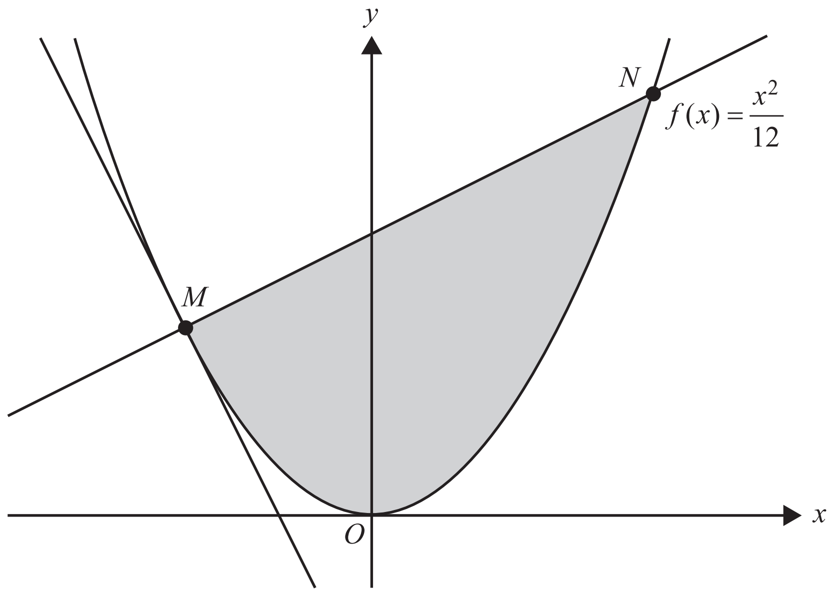
The diagram below shows part of the graph of $y = f(x)$, the tangent to f at point M and the line perpendicular to the tangent at point M.



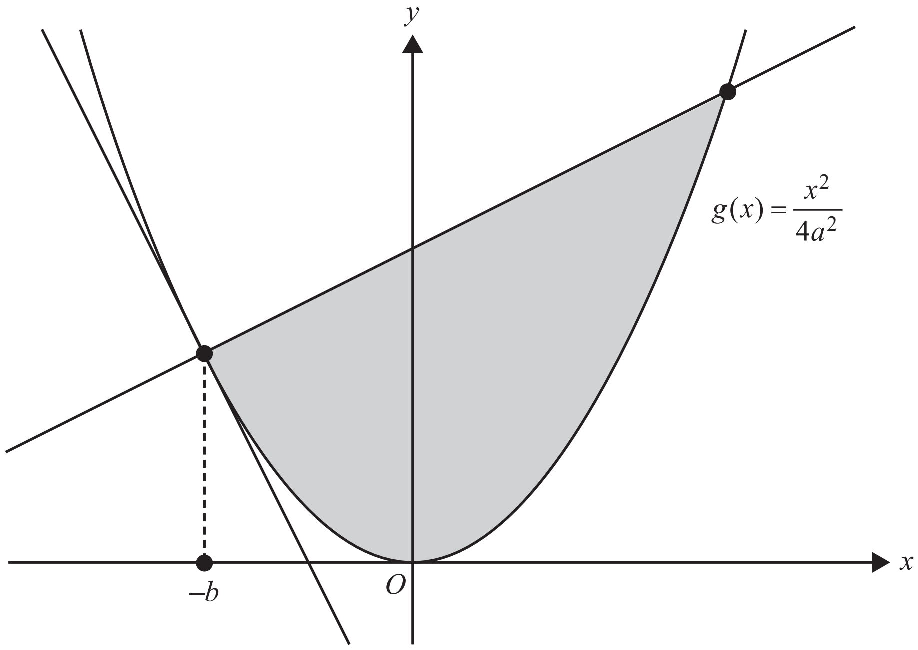
d. i. Find the equation of the line perpendicular to the tangent passing through point M. 1 mark

ii. The line perpendicular to the tangent at point M also cuts f at point N, as shown in the diagram above.

Find the area enclosed by this line and the curve $y = f(x)$. 2 marks

e. Another parabola is defined by the rule $g(x) = \dfrac{x^2}{4a^2}$, where $a > 0$.

A tangent to g and the line perpendicular to the tangent at $x = -b$, where $b > 0$, are shown below.



Find the value of b, in terms of a, such that the shaded area is a minimum. 4 marks

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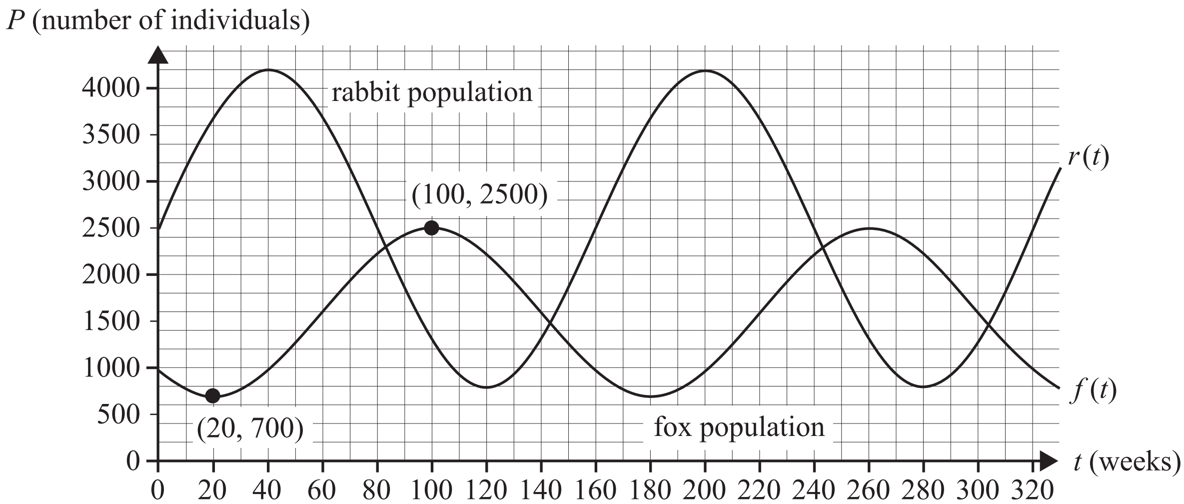
Question 2 (16 marks)

On a remote island, there are only two species of animals: foxes and rabbits. The foxes are the predators and the rabbits are their prey.

The populations of foxes and rabbits increase and decrease in a periodic pattern, with the period of both populations being the same, as shown in the graph below, for all $t \geq 0$, where time t is measured in weeks.

One point of minimum fox population, (20, 700), and one point of maximum fox population, (100, 2500), are also shown on the graph.

The graph has been drawn to scale.



The population of rabbits can be modelled by the rule $r(t) = 1700\sin\left(\frac{\pi t}{80}\right) + 2500$.

a. **i.** State the initial population of rabbits. 1 mark

ii. State the minimum and maximum population of rabbits. 1 mark

iii. State the number of weeks between maximum populations of rabbits. 1 mark

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The population of foxes can be modelled by the rule $f(t) = a\sin\left(b(t-60)\right) + 1600$.

b. Show that $a = 900$ and $b = \frac{\pi}{80}$. 2 marks

c. Find the maximum combined population of foxes and rabbits. Give your answer correct to the nearest whole number. 1 mark

d. What is the number of weeks between the periods when the combined population of foxes and rabbits is a maximum? 1 mark

The population of foxes is better modelled by the transformation of $y = \sin(t)$ under Q given by

$$Q: R^2 \to R^2,\ Q\left(\begin{bmatrix} t \\ y \end{bmatrix}\right) = \begin{bmatrix} \frac{90}{\pi} & 0 \\ 0 & 900 \end{bmatrix}\begin{bmatrix} t \\ y \end{bmatrix} + \begin{bmatrix} 60 \\ 1600 \end{bmatrix}$$

e. Find the average population during the first 300 weeks for the combined population of foxes and rabbits, where the population of foxes is modelled by the transformation of $y = \sin(t)$ under the transformation Q. Give your answer correct to the nearest whole number. 4 marks

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Over a longer period of time, it is found that the increase and decrease in the population of rabbits gets smaller and smaller.

The population of rabbits over a longer period of time can be modelled by the rule

$$s(t) = 1700 \cdot e^{-0.003t} \cdot \sin\left(\frac{\pi t}{80}\right) + 2500, \qquad \text{for all } t \geq 0$$

f. Find the average rate of change between the first two times when the population of rabbits is at a maximum. Give your answer correct to one decimal place. 2 marks

g. Find the time, where $t > 40$, in weeks, when the rate of change of the rabbit population is at its greatest positive value. Give your answer correct to the nearest whole number. 2 marks

h. Over time, the rabbit population approaches a particular value.

State this value. 1 mark

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Question 3 (14 marks)

Mika is flipping a coin. The unbiased coin has a probability of $\frac{1}{2}$ of landing on heads and $\frac{1}{2}$ of landing on tails.

Let X be the binomial random variable representing the number of times that the coin lands on heads.

Mika flips the coin five times.

a. **i.** Find $\Pr(X = 5)$. 1 mark

ii. Find $\Pr(X \geq 2)$. 1 mark

iii. Find $\Pr(X \geq 2 \mid X < 5)$, correct to three decimal places. 2 marks

iv. Find the expected value and the standard deviation for X. 2 marks

The height reached by each of Mika's coin flips is given by a continuous random variable, H, with the probability density function

$$f(h) = \begin{cases} ah^2 + bh + c & 1.5 \le h \le 3 \\ 0 & \text{elsewhere} \end{cases}$$

where h is the vertical height reached by the coin flip, in metres, between the coin and the floor, and a, b and c are real constants.

b. **i.** State the value of the definite integral $\int_{1.5}^{3} f(h)\,dh$. 1 mark

ii. Given that $\Pr(H \le 2) = 0.35$ and $\Pr(H \ge 2.5) = 0.25$, find the values of a, b and c. 3 marks

iii. The ceiling of Mika's room is 3 m above the floor. The minimum distance between the coin and the ceiling is a continuous random variable, D, with probability density function g.

The function g is a transformation of the function f given by $g(d) = f(rd + s)$, where d is the minimum distance between the coin and the ceiling, and r and s are real constants.

Find the values of r and s. 1 mark

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c. Mika's sister Bella also has a coin. On each flip, Bella's coin has a probability of p of landing on heads and $(1 - p)$ of landing on tails, where p is a constant value between 0 and 1.

Bella flips her coin 25 times in order to estimate p.

Let $\hat{P}$ be the random variable representing the proportion of times that Bella's coin lands on heads in her sample.

i. Is the random variable $\hat{P}$ discrete or continuous? Justify your answer. 1 mark

ii. If $\hat{p} = 0.4$, find an approximate 95% confidence interval for p, correct to three decimal places. 1 mark

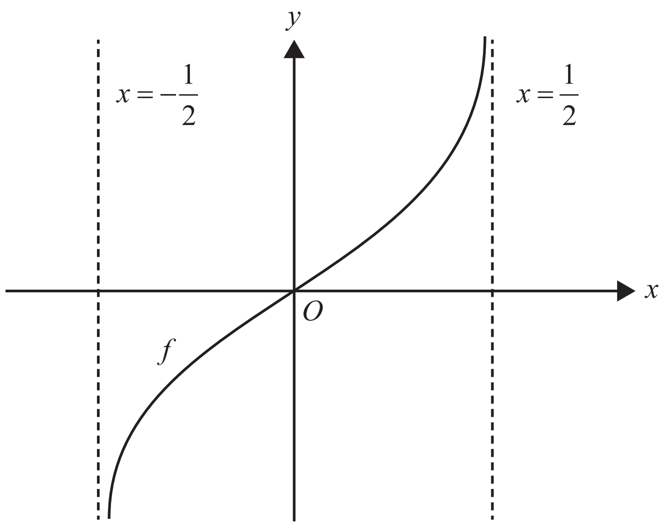
iii. Bella knows that she can decrease the width of a 95% confidence interval by using a larger sample of coin flips.

If $\hat{p} = 0.4$, how many coin flips would be required to halve the width of the confidence interval found in **part c.ii.**? 1 mark

Question 4 (10 marks)

Consider the function f, where $f: \left(-\frac{1}{2}, \frac{1}{2}\right) \to R$, $f(x) = \log_e\left(x + \frac{1}{2}\right) - \log_e\left(\frac{1}{2} - x\right)$.

Part of the graph of $y = f(x)$ is shown below.



a. State the range of $f(x)$. 1 mark

b. **i.** Find $f'(0)$. 2 marks

ii. State the maximal domain over which f is strictly increasing. 1 mark

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c. Show that $f(x) + f(-x) = 0$. 1 mark

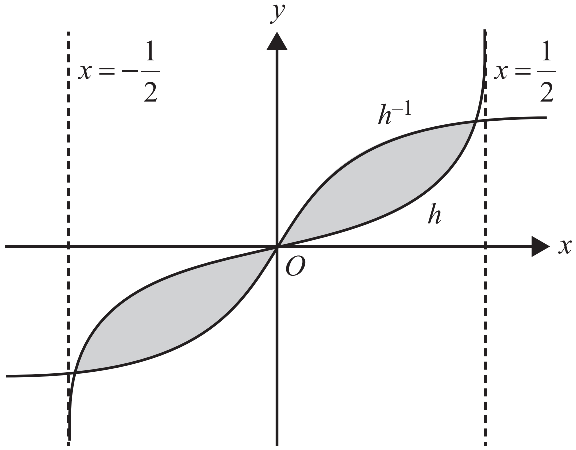
d. Find the domain and the rule of f^{-1}, the inverse of f. 3 marks

e. Let h be the function $h:\left(-\frac{1}{2}, \frac{1}{2}\right) \to R$, $h(x) = \frac{1}{k}\left(\log_e\left(x+\frac{1}{2}\right) - \log_e\left(\frac{1}{2}-x\right)\right)$, where $k \in R$ and $k > 0$.

The inverse function of h is defined by $h^{-1}: R \to R$, $h^{-1}(x) = \frac{e^{kx}-1}{2(e^{kx}+1)}$.

The area of the regions bound by the functions h and h^{-1} can be expressed as a function, $A(k)$.

The graph below shows the relevant area shaded.



You are not required to find or define $A(k)$.

i. Determine the range of values of k such that $A(k) > 0$. 1 mark

ii.

This question has been redacted following the findings of the Independent Review into the VCAA's Examination-Setting Policies, Processes and Procedures for the VCE.

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Question 5 (9 marks)

Consider the composite function $g(x) = f(\sin(2x))$, where the function $f(x)$ is an unknown but differentiable function for all values of x.

Use the following table of values for f and f'.

x	$\frac{1}{2}$	$\frac{\sqrt{2}}{2}$	$\frac{\sqrt{3}}{2}$
$f(x)$	–2	5	3
$f'(x)$	7	0	$\frac{1}{9}$

a. Find the value of $g\left(\frac{\pi}{6}\right)$. 1 mark

The derivative of g with respect to x is given by $g'(x) = 2 \cdot \cos(2x) \cdot f'(\sin(2x))$.

b. Show that $g'\left(\frac{\pi}{6}\right) = \frac{1}{9}$. 1 mark

c. Find the equation of the tangent to g at $x = \frac{\pi}{6}$. 2 marks

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d. Find the average value of the derivative function $g'(x)$ between $x = \frac{\pi}{8}$ and $x = \frac{\pi}{6}$. 2 marks

e. Find **four** solutions to the equation $g'(x) = 0$ for the interval $x \in [0, \pi]$. 3 marks

END OF QUESTION AND ANSWER BOOK

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AND ASSESSMENT AUTHORITY

Victorian Certificate of Education
2022

MATHEMATICAL METHODS

Written examination 2

FORMULA SHEET

Instructions

This formula sheet is provided for your reference.
A question and answer book is provided with this formula sheet.

Students are NOT permitted to bring mobile phones and/or any other unauthorised electronic devices into the examination room.

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Mathematical Methods formulas

Mensuration

area of a trapezium	$\frac{1}{2}(a+b)h$	volume of a pyramid	$\frac{1}{3}Ah$
curved surface area of a cylinder	$2\pi rh$	volume of a sphere	$\frac{4}{3}\pi r^3$
volume of a cylinder	$\pi r^2 h$	area of a triangle	$\frac{1}{2}bc\sin(A)$
volume of a cone	$\frac{1}{3}\pi r^2 h$		

Calculus

$\frac{d}{dx}\left(x^n\right) = nx^{n-1}$	$\int x^n dx = \frac{1}{n+1}x^{n+1} + c, \ n \neq -1$
$\frac{d}{dx}\left((ax+b)^n\right) = an(ax+b)^{n-1}$	$\int (ax+b)^n dx = \frac{1}{a(n+1)}(ax+b)^{n+1} + c, n \neq -1$
$\frac{d}{dx}\left(e^{ax}\right) = ae^{ax}$	$\int e^{ax} dx = \frac{1}{a}e^{ax} + c$
$\frac{d}{dx}\left(\log_e(x)\right) = \frac{1}{x}$	$\int \frac{1}{x} dx = \log_e(x) + c, \ x > 0$
$\frac{d}{dx}\left(\sin(ax)\right) = a\ \cos(ax)$	$\int \sin(ax)dx = -\frac{1}{a}\cos(ax) + c$
$\frac{d}{dx}\left(\cos(ax)\right) = -a\ \sin(ax)$	$\int \cos(ax)dx = \frac{1}{a}\sin(ax) + c$
$\frac{d}{dx}\left(\tan(ax)\right) = \frac{a}{\cos^2(ax)} = a\ \sec^2(ax)$	

product rule	$\frac{d}{dx}(uv) = u\frac{dv}{dx} + v\frac{du}{dx}$	quotient rule	$\frac{d}{dx}\left(\frac{u}{v}\right) = \frac{v\frac{du}{dx} - u\frac{dv}{dx}}{v^2}$
chain rule	$\frac{dy}{dx} = \frac{dy}{du}\frac{du}{dx}$		

Probability

$\Pr(A) = 1 - \Pr(A')$		$\Pr(A \cup B) = \Pr(A) + \Pr(B) - \Pr(A \cap B)$	
$\Pr(A \mid B) = \dfrac{\Pr(A \cap B)}{\Pr(B)}$			
mean	$\mu = \mathrm{E}(X)$	variance	$\operatorname{var}(X) = \sigma^2 = \mathrm{E}((X-\mu)^2) = \mathrm{E}(X^2) - \mu^2$

Probability distribution		Mean	Variance
discrete	$\Pr(X = x) = p(x)$	$\mu = \sum x\, p(x)$	$\sigma^2 = \sum (x-\mu)^2\, p(x)$
continuous	$\Pr(a < X < b) = \int_a^b f(x)\,dx$	$\mu = \int_{-\infty}^{\infty} x\, f(x)\,dx$	$\sigma^2 = \int_{-\infty}^{\infty} (x-\mu)^2 f(x)\,dx$

Sample proportions

$\hat{P} = \dfrac{X}{n}$		mean	$\mathrm{E}(\hat{P}) = p$
standard deviation	$\operatorname{sd}(\hat{P}) = \sqrt{\dfrac{p(1-p)}{n}}$	approximate confidence interval	$\left(\hat{p} - z\sqrt{\dfrac{\hat{p}(1-\hat{p})}{n}},\ \hat{p} + z\sqrt{\dfrac{\hat{p}(1-\hat{p})}{n}}\right)$

END OF FORMULA SHEET