insight
▸innovative ▸engaging ▸evolving

YEAR 12 *Trial Exam Paper*

2022

MATHEMATICAL METHODS

Written examination 1

Worked solutions

This book presents:

- worked solutions
- mark allocations
- tips.

Question 1a.

Worked solution

$$\frac{dy}{dx}=1\times(2-x)^3+x\times-3(2-x)^2$$
$$=(2-x)^3-3x(2-x)^2$$
$$=(2-x-3x)(2-x)^2$$
$$=(2-4x)(2-x)^2$$
$$=2(1-2x)(2-x)^2$$

Mark allocation: 2 marks

- 1 answer mark for correctly applying the chain rule to find the derivative: $\frac{d}{dx}(2-x)^3=-3(2-x)^2$
- 1 answer mark for correctly applying the product rule to find $\frac{dy}{dx}=(2-x)^3-3x(2-x)^2$ or equivalent

Tip

- *You can always take extra time and solve the problem in smaller steps when applying the product or quotient rules. If you forgot to apply the chain rule when calculating the derivative of $(2-x)^3$ then you should definitely do this!*

Question 1b.

Worked solution

$$f'(x)=2e^{-\frac{1}{2}(x-1)}\times\frac{d}{dx}\left(-\frac{1}{2}(x-1)\right)$$
$$=2e^{-\frac{1}{2}(x-1)}\times-\frac{1}{2}$$
$$=-e^{-\frac{1}{2}(x-1)}$$
$$f'(1)=-e^{-\frac{1}{2}(1-1)}$$
$$=-e^0$$
$$=-1$$

Mark allocation: 2 marks

- 1 answer mark for finding the correct derivative: $f'(x)=-e^{-\frac{1}{2}(x-1)}$
- 1 answer mark for correctly evaluating the derivative at $x=1$: $f'(1)=-1$

Question 2a.

Worked solution

$y = \left(\sin(x)+1\right)^{-1}$

$\dfrac{d}{dx}\left(\sin(x)+1\right) = \cos(x)$

Apply the chain rule.

$\dfrac{dy}{dx} = -1\cdot\left(\sin(x)+1\right)^{-2}\times\cos(x)$

$= \dfrac{-\cos(x)}{\left(\sin(x)+1\right)^{2}}$

Alternatively, apply the quotient rule.

Mark allocation: 1 mark

- 1 method mark for using a correct method for calculating the derivative

Tip

- *You should clearly show your method in ‘Show that …’ questions. If you are in two minds about whether to include showing an extra step in your working out, then always include it.*

Question 2b.

Worked solution

Use the result from **part a.**

$$\frac{d}{dx}\left(\frac{1}{\sin(x)+1}\right)=\frac{-\cos(x)}{\left(\sin(x)+1\right)^2}$$

$$\frac{\cos(x)}{\left(\sin(x)+1\right)^2}=-\frac{d}{dx}\left(\frac{1}{\sin(x)+1}\right)$$

$$f(x)=\int\frac{\cos(x)}{\left(\sin(x)+1\right)^2}dx$$

$$=\int-\frac{d}{dx}\left(\frac{1}{\sin(x)+1}\right)dx$$

$$=-\frac{1}{\sin(x)+1}+c$$

$$f\left(\frac{\pi}{2}\right)=0=-\frac{1}{\sin\left(\frac{\pi}{2}\right)+1}+c$$

$$=-\frac{1}{2}+c$$

$$c=\frac{1}{2}$$

$$f(x)=-\frac{1}{\sin(x)+1}+\frac{1}{2}$$

Mark allocation: 3 marks

- 1 answer mark for finding an antiderivative of $\frac{\cos(x)}{\left(\sin(x)+1\right)^2}$ using the result from **part a.**
- 1 answer mark for finding that $c=\frac{1}{2}$
- 1 answer mark for finding the solution: $f(x)=-\frac{1}{\sin(x)+1}+\frac{1}{2}$

Question 3a.

Worked solution

$$\Pr(\text{success}) = \Pr(\text{success on six-sided}) + \Pr(\text{success on four-sided}) - \Pr(\text{success on both})$$

$$= \frac{3}{6} + \frac{1}{4} - \frac{3}{6} \times \frac{1}{4} = \frac{5}{8}$$

Mark allocation: 1 mark

- 1 answer mark for finding the correct answer: $\frac{5}{8}$

Question 3b.

Worked solution

$$\Pr(\text{exceptional success} \mid \text{success}) = \frac{\Pr(\text{exceptional success})}{\Pr(\text{success})}$$

$$= \frac{\frac{1}{2} \times \frac{1}{4}}{\frac{5}{8}}$$

$$= \frac{\frac{1}{8}}{\frac{5}{8}}$$

$$= \frac{1}{5}$$

Mark allocation: 2 marks

- 1 answer mark for applying the conditional probability rule: $\Pr(A \mid B) = \frac{\Pr(A \cap B)}{\Pr(B)}$
- 1 answer mark for finding the correct answer: $\frac{1}{5}$

Tip

- *You could also use a tree diagram to assist with this question and the previous one.*

Question 3c.

Worked solution

$$\Pr\left(\hat{P} = 0\right) = \left(\frac{3}{8}\right)^2 = \frac{9}{64}$$

$$\Pr\left(\hat{P} > 0\right) = 1 - \frac{9}{64} = \frac{55}{64}$$

Mark allocation: 1 mark

- 1 answer mark for finding the correct answer: $\frac{55}{64}$

Question 4a.

Worked solution

$$e^{2x} - 5e^{x} + 4 = 0$$

$$\left(e^{x} - 1\right)\left(e^{x} - 4\right) = 0$$

$$e^{x} - 1 = 0 \quad \text{or} \quad e^{x} - 4 = 0$$

$$e^{x} = 1 \qquad e^{x} = 4$$

$$x = \log_{e}(1) = 0 \qquad x = \log_{e}(4) = 2\log_{e}(2)$$

$$x = 0,\ 2\log_{e}(2)$$

Mark allocation: 2 marks

- 1 method mark for factorising the quadratic as $\left(e^{x} - 1\right)\left(e^{x} - 4\right) = 0$, or an equivalent with a substitution
- 1 answer mark for finding $x = 0,\ 2\log_{e}(2)$, or equivalent

Tip

- *You could perform a substitution, such as $k = e^{x}$, to help you work with the expression.*

Question 4b.

Worked solution

The domain of g is $[0, \infty)$. From **part a.**, $e^{2x} - 5e^{x} + 4 = 0$ when $x = 0,\ 2\log_{e}(2)$.

The right-most solution gives $k = 2\log_{e}(2)$.

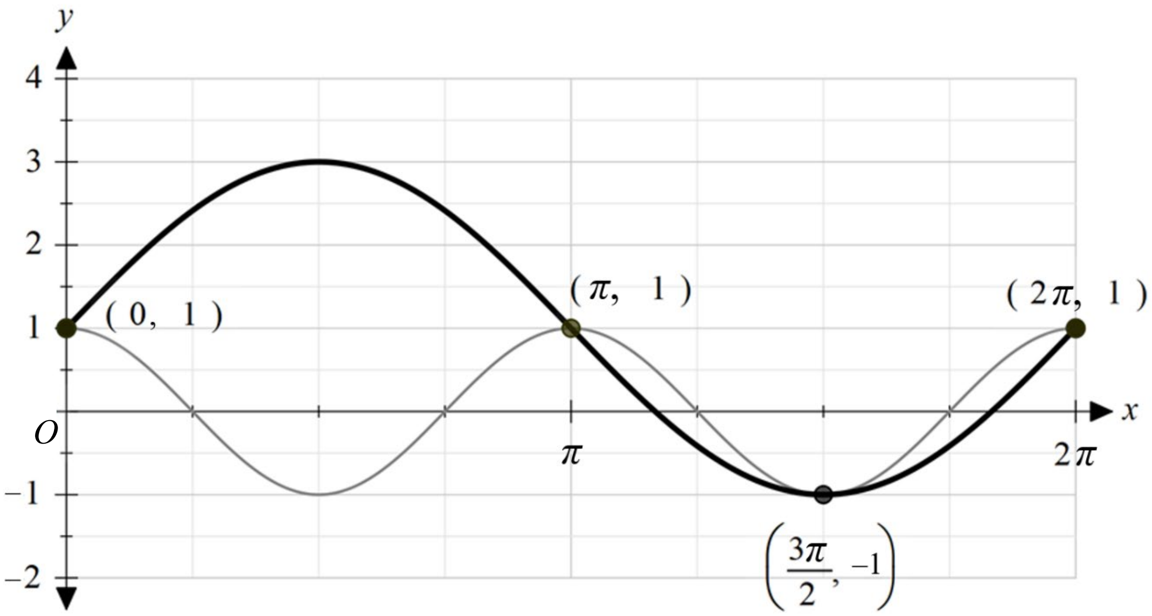
It can be confirmed that $f(x)$ is positive to the right of $x = 2\log_{e}(2)$ by substituting a point, such as $f\left(\log_{e}(5)\right) = e^{2\log_{e}(5)} - 5e^{\log_{e}(5)} + 4 = 4$.

Mark allocation: 1 mark

- 1 answer mark for finding the correct answer: $k = 2\log_{e}(2)$, or equivalent

Question 5a.

Worked solution



From the graph, the points of intersection occur at $x = 0,\ \pi,\ \frac{3\pi}{2}$ and 2π.

Mark allocation: 2 marks

- 1 answer mark for correctly drawing the graph passing through the points $(0, 1)$, $\left(\frac{\pi}{2}, 3\right)$, $(\pi, 1)$, $\left(\frac{3\pi}{2}, -1\right)$ and $(2\pi, 1)$
- 1 answer mark for correctly labelling the coordinates of all four points of intersection

Tip

- *You should identify the key points that your graph should pass through before drawing it. When planning the graph, you may wish to indicate the position of a point with a short line of the appropriate gradient instead of a point.*

Question 5b.

Worked solution

Because f has a cosine function and g has a sine function, a key first step is to convert one of the functions into the other so that both functions are in terms of the same circular function. This can be done with either of the identities:

$$\cos(x) = \sin\left(\frac{\pi}{2} - x\right) \text{ or } \sin(x) = \cos\left(\frac{\pi}{2} - x\right) = \cos\left(x - \frac{\pi}{2}\right).$$

So $g(x) = 2\sin(x) + 1 = 2\cos\left(x - \frac{\pi}{2}\right) + 1.$

Original

$$y = 2\cos\left(x - \frac{\pi}{2}\right) + 1$$

$$\frac{y-1}{2} = \cos\left(x - \frac{\pi}{2}\right)$$

Image

$y_T = \cos(2x_T)$

From the rules for the original and image,

$$2x_T = x - \frac{\pi}{2}$$

$$x_T = \frac{1}{2}x - \frac{\pi}{4}$$

$$y_T = \frac{y-1}{2} = \frac{1}{2}y - \frac{1}{2}$$

Hence, $a = \frac{1}{2}$, $b = \frac{1}{2}$, $c = -\frac{\pi}{4}$ and $d = -\frac{1}{2}$.

Alternative approach

The original $y = 2\cos\left(x - \frac{\pi}{2}\right) + 1$ is the rule $y = \cos(x)$ after:

- a vertical dilation by a factor of 2
- a vertical translation by 1
- a horizontal translation by $\frac{\pi}{2}$.

To reverse these requires (in order):

- a horizontal translation by $-\frac{\pi}{2}$
- a vertical translation by -1
- a vertical dilation by a factor of $\frac{1}{2}$.

The image $y_T = \cos(2x_T)$ is the rule $y = \cos(x)$ after a horizontal dilation by a factor of $\frac{1}{2}$.

Putting these together gives

$x_T = \frac{1}{2}\left(x - \frac{\pi}{2}\right) = \frac{1}{2}x - \frac{\pi}{4}$

$y_T = \frac{1}{2}(y-1) = \frac{1}{2}y - \frac{1}{2}$

Hence, $a = \frac{1}{2}$, $b = \frac{1}{2}$, $c = -\frac{\pi}{4}$ and $d = -\frac{1}{2}$.

Mark allocation: 3 marks

- 1 method mark for identifying how the original is mapped to the image, either as rules such as $2x_T = x - \frac{\pi}{2}$ and $y_T = \frac{y-1}{2}$, or in words or equivalent
- 1 answer mark for finding the correct value for at least two of a, b, c and d) **OR**
  2 answer marks for finding the correct value of a, b, c and d

Question 6a.

Worked solution

The mean of X is 10, hence, $\Pr(X<10)=0.5$

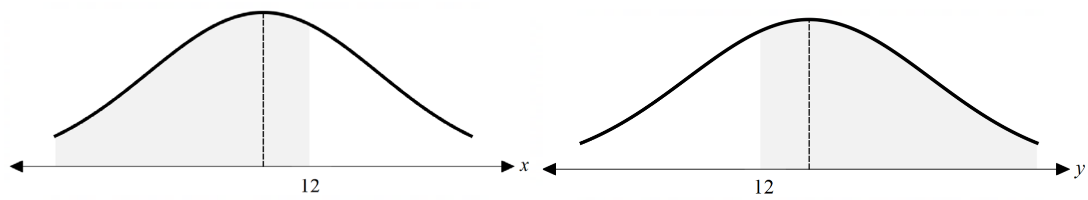
Mark allocation: 1 mark

- 1 answer mark for finding 0.5

Question 6b.

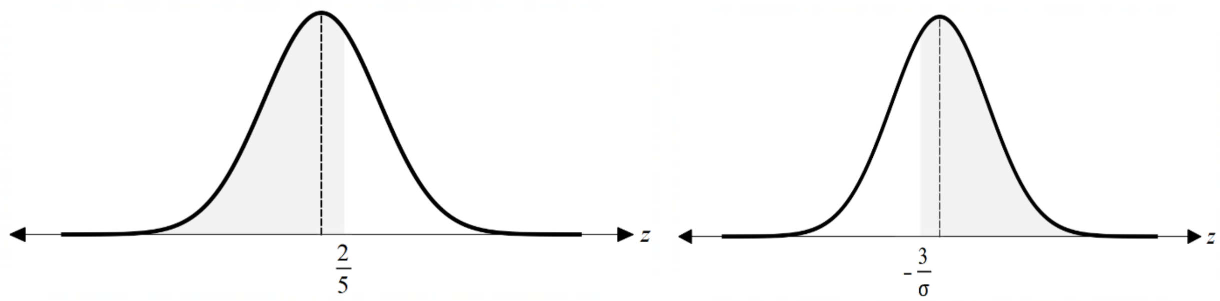
Worked solution

$\Pr\left(X<12\right)=\Pr\left(Y>12\right)$



Convert this to the standard normal distribution.

$$\Pr\left(Z<\frac{12-10}{5}\right)=\Pr\left(Y>\frac{12-15}{\sigma}\right)=\Pr\left(Y<-\frac{12-15}{\sigma}\right)$$



$$\Pr\left(Z<\frac{2}{5}\right)=\Pr\left(Y<\frac{3}{\sigma}\right)$$

Equate the two z values.

$$\frac{2}{5}=\frac{3}{\sigma}$$

$$\sigma=\frac{15}{2}$$

Mark allocation: 2 marks

- 1 method mark for deriving $\dfrac{12-10}{5}=-\dfrac{12-15}{\sigma}$, or equivalent
- 1 answer mark for calculating $\sigma=\dfrac{15}{2}$

Tip

- *You may find it helpful to draw diagrams for normal distribution questions, as diagrams can help you to visualise the inequalities.*

Question 7

Worked solution

$$1=\int_0^a \frac{k}{x+1}dx$$
$$=\left[k\cdot\log_e(x+1)\right]_0^a$$
$$=k\cdot\log_e(a+1)-k\cdot\log_e(0+1)$$
$$=k\cdot\log_e(a+1)$$
$$k=\frac{1}{\log_e(a+1)}$$

Mark allocation: 3 marks

- 1 method mark for stating that $1=\int_0^a \frac{k}{x+1}dx$
- 1 method mark for evaluating the antiderivative of $\frac{k}{x+1}$ as $k\cdot\log_e(x+1)$ for $x>-1$
- 1 answer mark for deriving $k=\frac{1}{\log_e(a+1)}$

Question 8a.

Worked solution

successes	0	1	2	3
probability	$(1-p)^3$	$3p(1-p)^2$	$3p^2(1-p)$	p^3
points gained	0	-2	1	3

$$\text{points gained}=-2\times 3p(1-p)^2+1\times 3p^2(1-p)+3\times p^3$$
$$=-6p\left(1-2p+p^2\right)+3p^2(1-p)+3p^3$$
$$=-6p+12p^2-6p^3+3p^2-3p^3+3p^3$$
$$=-6p^3+15p^2-6p$$

Mark allocation: 2 marks

- 1 answer mark for stating the sum of the pairwise products of probability and points
- 1 answer mark for correctly expanding and simplifying the expression

Question 8b.

Worked solution

Solve $-6p^3+15p^2-6p>0$ for $0\le p\le 1$.

$-6p^3+15p^2-6p=0$

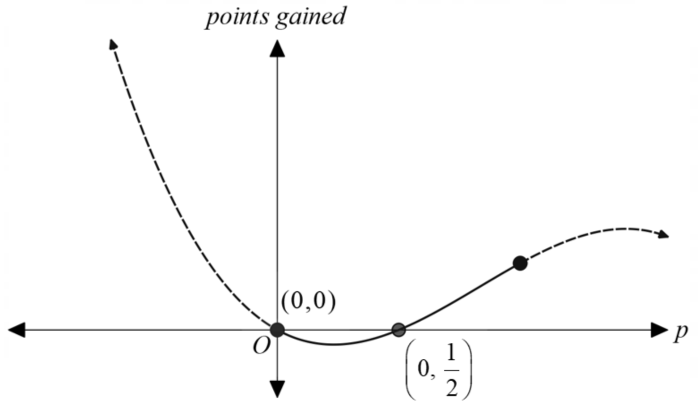
$p=0$, or

$-6p^2+15p-6=0$

$2p^2-5p+2=0$

$(2p-1)(p-2)=0$

$p=\frac{1}{2},\ 2$



A player should expect to gain points for $p\in\left(\frac{1}{2},1\right]$.

Mark allocation: 3 marks

- 1 answer mark for deriving $-6p^3+15p^2-6p>0$ or $-6p^3+15p^2-6p=0$
- 1 answer mark for solving the equation to give $p=0,\ \frac{1}{2},\ 2$
- 1 answer mark for identifying the correct interval: $p\in\left(\frac{1}{2},1\right]$

Question 8c.

Worked solution

$$\frac{d(\text{points})}{dp} = -18p^2 + 30p - 6$$

The minimum points gained (that is, the most points lost) will occur when $\frac{d(\text{points})}{dp} = 0$.

$$0 = -18p^2 + 30p - 6$$
$$= 3p^2 - 5p + 1$$
$$p = \frac{5 \pm \sqrt{5^2 - 4(3)(1)}}{2(3)}$$
$$= \frac{5 \pm \sqrt{13}}{6}$$
$$0 \le p \le 1$$
$$p = \frac{5 - \sqrt{13}}{6}$$

Mark allocation: 3 marks

- 1 answer mark for calculating the derivative: $\frac{d(\text{points})}{dp} = -18p^2 + 30p - 6$
- 1 answer mark for equating the derivative to zero: $0 = -18p^2 + 30p - 6$
- 1 answer mark for deriving the solution: $p = \frac{5 - \sqrt{13}}{6}$

Question 9a.

Worked solution

$$f'(x) = \frac{1}{2\sqrt{x}}$$

The gradient of line perpendicular to f at P is

$$-\frac{1}{f'(p)} = -2\sqrt{p}$$

The line passes through $\left(p, \sqrt{p}\right)$:

$$y - \sqrt{p} = -2\sqrt{p}(x - p)$$

x-intercept occurs when $y = 0$:

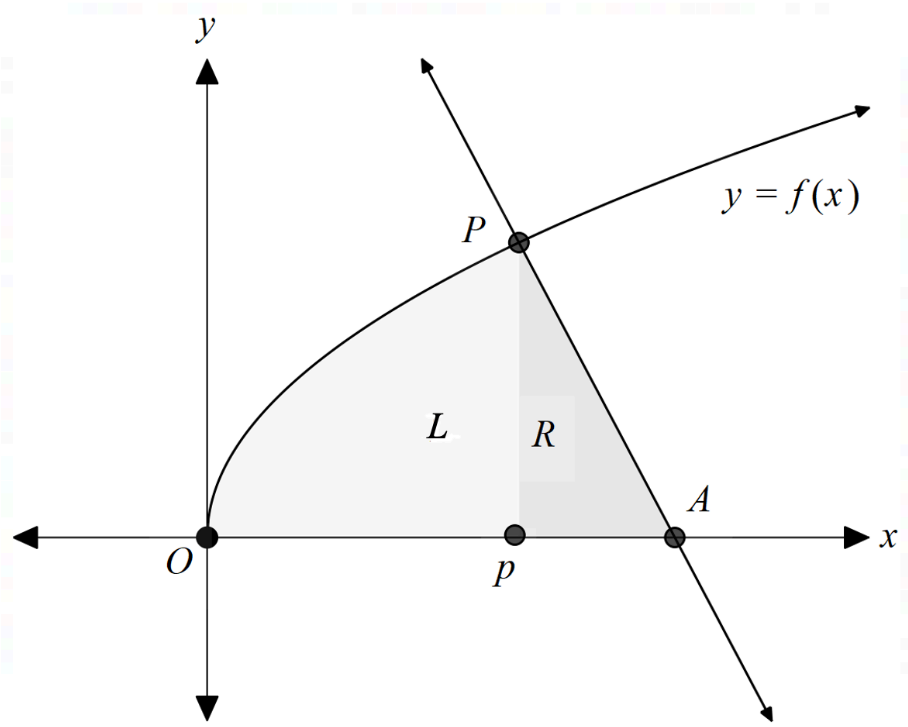
$$-\sqrt{p} = -2\sqrt{p}(x - p)$$
$$\frac{1}{2} = x - p$$
$$x = p + \frac{1}{2}$$

Mark allocation: 2 marks

- 1 answer mark for calculating that the gradient of the line perpendicular to f at P is $-2\sqrt{p}$
- 1 answer mark for writing the equation of the line perpendicular to f at P and determining its x-intercept

Question 9b.

Worked solution



Total area = $\text{Area}_L + \text{Area}_R$

$$= \int_0^p \sqrt{x}\,dx + \frac{1}{2}\cdot\frac{1}{2}\cdot\sqrt{p}$$

$$= \left[\frac{2}{3}x^{\frac{3}{2}}\right]_0^p + \frac{1}{4}\sqrt{p}$$

$$= \frac{2}{3}p^{\frac{3}{2}} + \frac{1}{4}\sqrt{p}$$

Mark allocation: 2 marks

- 1 answer mark for correctly expressing the area of either the left or right section of the shaded region as part of an expression for the whole area

OR

- 2 answer marks for deriving the correct expression $\frac{2}{3}p\sqrt{p}+\frac{1}{4}\sqrt{p}$, or an equivalent, for the area of the shaded region

Tip

- *You should annotate the diagram when you have to find an area that will need to be divided into parts for calculation. This will help you develop appropriate expressions for each part. When one part is a simple geometric shape, you may wish to use measurement formulas for the area instead of definite integration.*

Question 9c.

Worked solution

The area function is continuous. To show that the area is strictly increasing, we will show that the derivative of the area function is positive.

$$A = \frac{2}{3}p^{\frac{3}{2}} + \frac{1}{4}\sqrt{p}$$

$$\frac{dA}{dp} = \sqrt{p} + \frac{1}{8\sqrt{p}}$$

Identify any possible stationary points of the area function.

$$0 = \sqrt{p} + \frac{1}{8\sqrt{p}}$$

$$\frac{1}{8\sqrt{p}} = -\sqrt{p}$$

$$1 = -\sqrt{p} \cdot 8\sqrt{p} = -8p$$

$$p = -\frac{1}{8}$$

As $p > 0$, the area function has no stationary points.

At $p = 1$

$$\frac{dA}{dp} = \sqrt{1} + \frac{1}{8\sqrt{1}}$$

$$= \frac{9}{8}$$

$\frac{dA}{dp}$ is positive at $p = 1$, $\frac{dA}{dp} \neq 0$ and $\frac{dA}{dp}$ is continuous for $p > 0$. Therefore, $\frac{dA}{dp} > 0$ for $p > 0$. Hence, the area is strictly increasing as p increases.

Mark allocation: 2 marks

- 1 answer mark for providing some elements of an appropriate argument that the area is strictly increasing, such as finding the rule for $\frac{dA}{dp}$

OR

- 2 answer marks for a complete and rigorous argument that the area is strictly increasing, such as showing that $\frac{dA}{dp} \neq 0$ for $p > 0$ and that $\frac{dA}{dp} > 0$ for some choice of $p > 0$

END OF WORKED SOLUTIONS