Neap

Trial Examination 2021

VCE Mathematical Methods Units 3&4

Written Examination 2

Suggested Solutions

SECTION A – MULTIPLE-CHOICE QUESTIONS

Question	Answer	Question	Answer
1	A	11	C
2	D	12	B
3	A	13	B
4	B	14	D
5	C	15	E
6	C	16	B
7	D	17	C
8	E	18	D
9	E	19	B
10	B	20	A

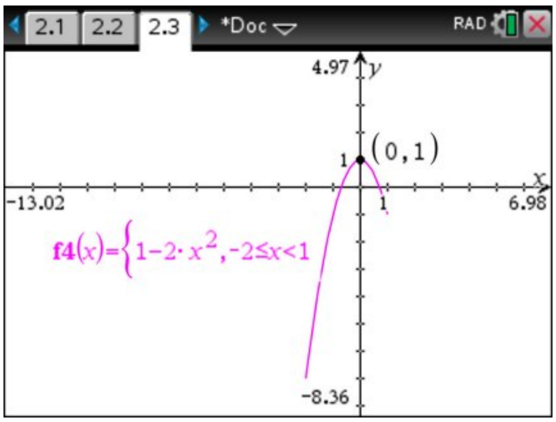
Question 1 A

$\text{tangentLine}\left(x^2+2\cdot x-1,x,1\right)$	$4\cdot x-2$

Question 2 D

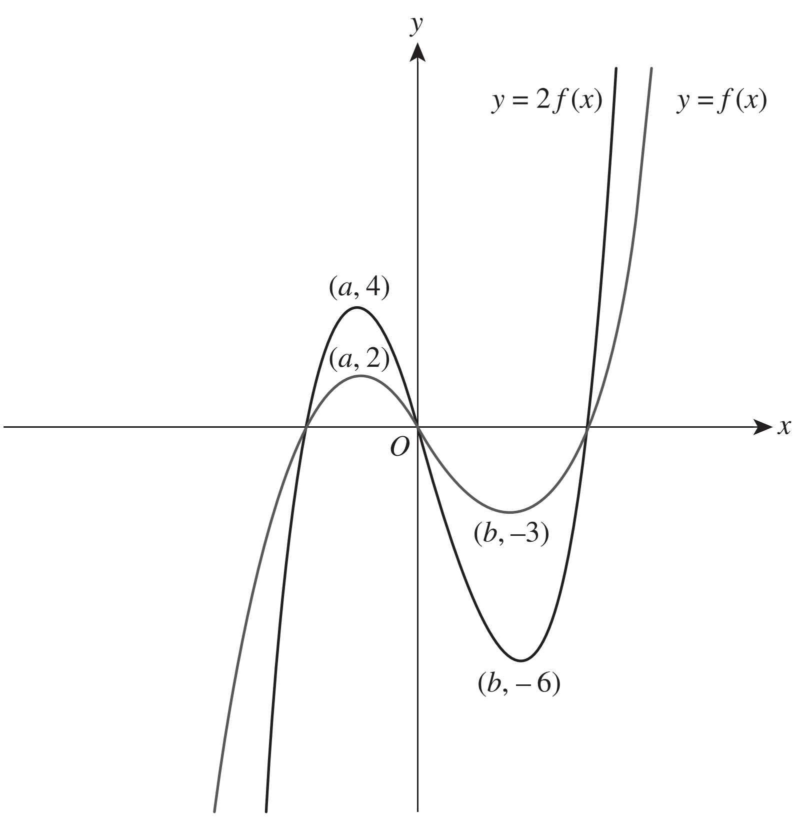
$f(-2)=-7$

$\therefore$ range $=[-7,1]$



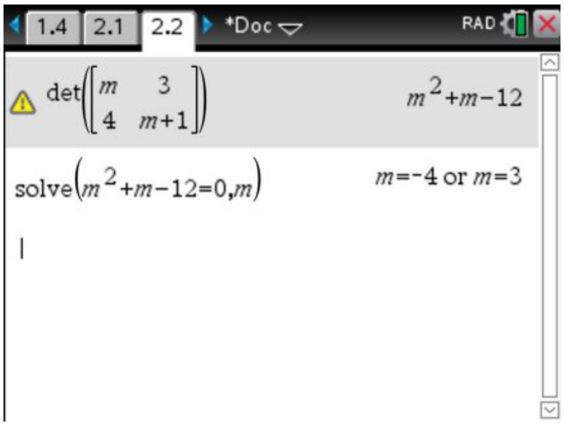
Question 3 A

Graphing $y=2f(x)$ gives turning points that provide the restrictions for c. The graph of $y=c$ is a horizontal line. The solution of $2f(x)=c$ is the intersection point of $y=2f(x)$ and $y=c$. Therefore, $c<-6$ or $c>4$ gives only one point of intersection and one solution.

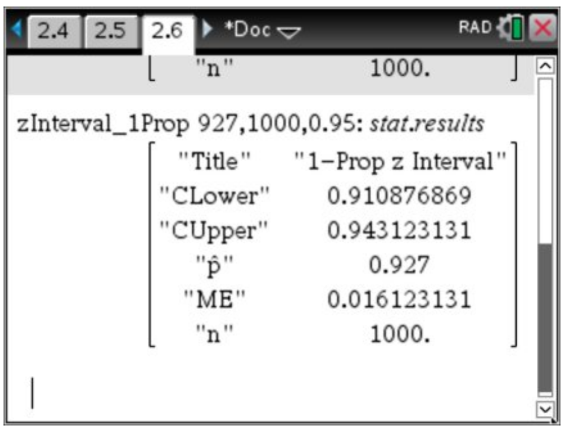


Question 4 B

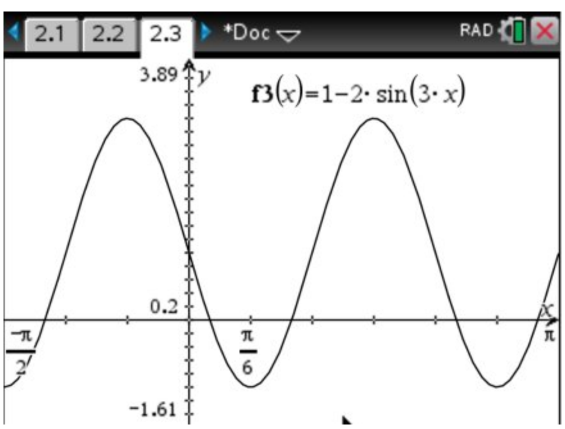
$$\begin{bmatrix} m & 3 \\ 4 & m+1 \end{bmatrix}\begin{bmatrix} x \\ y \end{bmatrix}=\begin{bmatrix} 0 \\ m \end{bmatrix}$$



Question 5 C



Question 6 C



The local minimum is $\left(\dfrac{\pi}{6},-1\right)$ and the local maximum is $\left(\dfrac{\pi}{2},3\right)$. Therefore, the range is [–1, 3].

$$\begin{aligned}\text{period} &= \frac{2\pi}{n} \\ &= \frac{2\pi}{3}\end{aligned}$$

Question 7 D

$g(x) = 2f(x-5)+1$

Transformations:

- dilation factor of 2 from x-axis $(-1,2) \to (-1,4)$
- translation of 5 units right $(-1,4) \to (4,4)$
- translation of 1 unit up $(4,4) \to (4,5)$

$\to g(4) = 5$

Question 8 E

$$\begin{aligned}\int_5^3 1-2f(x)\,dx &= \int_3^5 2f(x)-1\,dx \\ &= 2\int_3^5 f(x)\,dx - \int_3^5 1\,dx \\ &= 2\times 10 - 2 \\ &= 18\end{aligned}$$

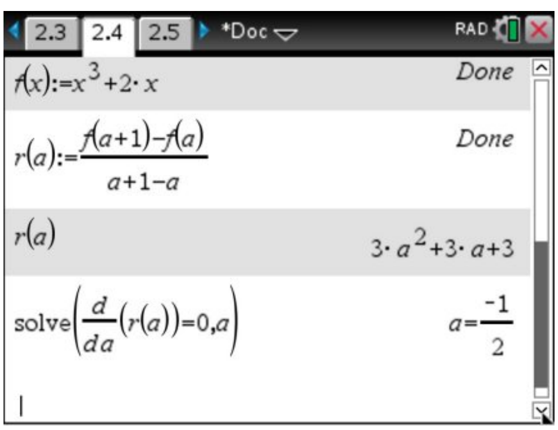
Question 9 E

As $0 \le \Pr(x=2) \le 1$, $0 \le k - \frac{1}{4} \le 1$, therefore the minimum value of k is equal to $\frac{1}{4}$.

Maximum value of m:

$$\begin{aligned}m &= 1-\left(\frac{1}{4}+0+\left(\frac{1}{4}\right)^2\right) \\ &= \frac{11}{16}\end{aligned}$$

Question 10 B



Question 11 C

Degree must be even as range $\neq R$.

There is a point of inflection at $x=-a$ and a local maximum at $x=a$.

Therefore, f is at minimum a degree-4 polynomial.

Question 12 B

Let S = success, M = miss, A = Any (success or miss)

Pr(exactly 3) = Pr(SSSM) + Pr(MSSS)

$$\Pr(\text{exactly } 3) = \left(\frac{4}{5}\right)^3 \times \left(\frac{1}{5}\right) + \left(\frac{4}{5}\right)^3 \times \left(\frac{1}{5}\right)$$

$$= \frac{27}{128}$$

Question 13 B

$$\text{solve}\left(\int_{\pi}^{a} \sin(2\cdot x)\ dx = \frac{1}{4}, a\right) \Big| \pi < a < \frac{3\cdot\pi}{2}$$

$$a = \frac{7\cdot\pi}{6}$$

Question 14 D

$$\text{solve}(1 - m\cdot x > 0, x) | m > 0 \qquad x < \frac{1}{m} \text{ and } m > 0$$

Question 15 E

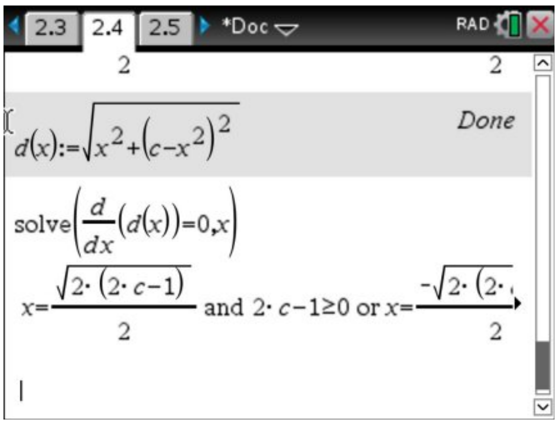
E is correct. This graph has two x-intercepts and has matching gradients with the derivative function f'.

Question 16   B

Differentiating the distance formula from the origin to a point $P\left(x, c-x^2\right)$ reveals that the magnitude of the minimum (or maximum) distance occurs at an x-value of $x=\frac{\pm\sqrt{2(2c-1)}}{2}$ if $2c-1 \geq 0$ or $c \geq \frac{1}{2}$.

If $c \leq \frac{1}{2}$, however, the magnitude of the minimum distance will always be found at the y-intercept, which has a value of c.

Note: A sketch graph could also be used.



Note: $x=0$ gives the magnitude of the minimum distance when $c \geq \frac{1}{2}$, but the derivative indicates a local maximum when $c > \frac{1}{2}$.

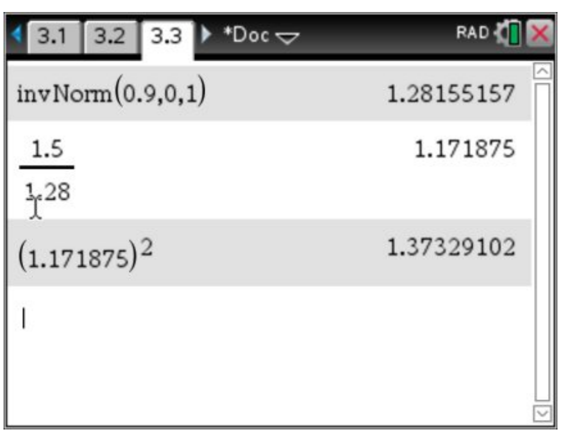
Question 17   C

$$z=\frac{x-\mu}{\sigma}$$

$$\mu=0$$

$$\sigma=\frac{x}{z}$$

$$Var(X)=\sigma^2$$



Question 18 D

$$y=\sqrt{\frac{1}{f(x)}}$$

$$=\left[f(x)\right]^{-\frac{1}{2}}$$

$$\frac{dy}{dx}=-\frac{1}{2}\times\left[f(x)\right]^{-\frac{3}{2}}\times f'(x)$$

$$\frac{dy}{dx}=\frac{-f'(x)}{2\left[\sqrt{f(x)}\right]^{3}}$$

Question 19 B

$\sin(\theta)$ is a one-to-one function for $0\le\theta\le\frac{\pi}{2}$.

$$0\le a^{x}\le\frac{\pi}{2}$$

$$-\infty<x\le\log_{a}\left(\frac{\pi}{2}\right)$$

Question 20 A

Maximum area occurs when the parabola is tangential to the line at $x=a$.

$$g(x)=kx(x-a)$$

$$g'(x)=k(2x-a)$$

When $x=a,\ g'(a)=ak$.

$$m_T=-\frac{b}{a}$$

Let $g'(a)=m_T$.

$$ak=-\frac{b}{a}$$

$$k=-\frac{b}{a^2}$$

$$g(x)=-\frac{b}{a^2}x(x-a)$$

$$\text{maximum area}=\int_0^a-\frac{b}{a^2}x(x-a)dx$$

$$=\frac{ab}{6}$$

SECTION B

Question 1 (10 marks)

a. $y = 3x - 2$ A1

$f(x) := x^3$ *Done*

$\text{tangentLine}(f(x), x, 1)$ $3 \cdot x - 2$

b. $A = \int_0^{\frac{2}{3}} x^3 \, dx + \int_{\frac{2}{3}}^{1} x^3 - (3x - 2) dx$ M2

$= \frac{1}{12}$ A1

$\int_0^{\frac{2}{3}} f(x) \, dx + \int_{\frac{2}{3}}^{1} (f(x) - (3 \cdot x - 2)) dx$ $\frac{1}{12}$

c. $y = 3x - 2 + k$ M1

Let $y = 0$.

$3x - 2 + k = 0$

$x = \frac{2}{3} - \frac{k}{3}$

$\left(\frac{2}{3} - \frac{k}{3}, 0\right)$ A1

d. $\text{area } 1 = \int_0^{\frac{2}{3}-\frac{k}{3}} x^3 + k\,dx + \int_{\frac{2}{3}-\frac{k}{3}}^{1} x^3 + k - (3x - 2 + k)\,dx$ M1

$= -\frac{k^2}{6} + \frac{2k}{3} + \frac{1}{12}$ M1

$\text{area } 2 = -\frac{1}{2} \times (k-2) \times \left(\frac{2}{3} - \frac{k}{3}\right)$

$= \frac{k^2}{6} - \frac{2k}{3} + \frac{2}{3}$

$\text{area } 1 = \text{area } 2$

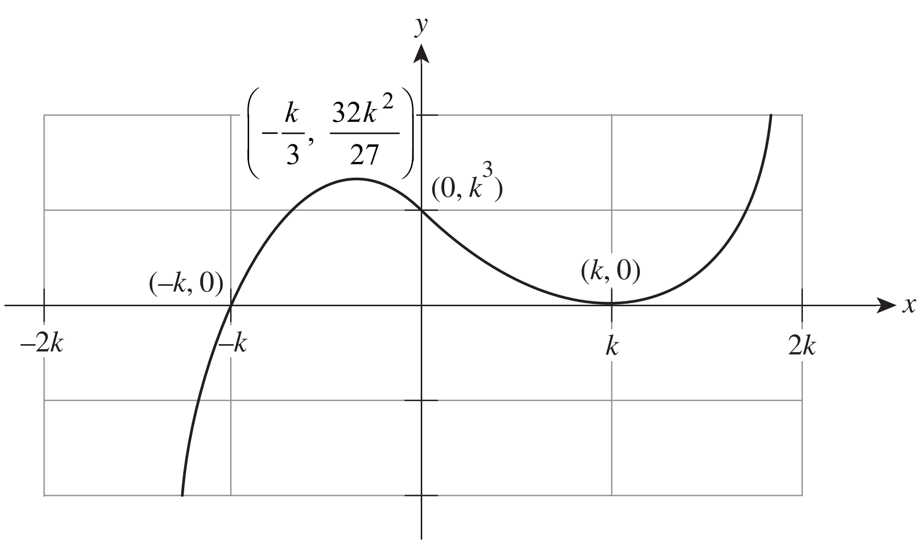
$-\frac{k^2}{6} + \frac{2k}{3} + \frac{1}{12} = \frac{k^2}{6} - \frac{2k}{3} + \frac{2}{3}$ M1

$k = \frac{1}{2} \text{ or } k = \frac{7}{2}$

Since $0 < k < 2$, $k = \frac{1}{2}$ A1

Question 2 (6 marks)

a.



correct intercepts A1
correct turning point A1
correct shape A1

b. i. $x^2 - k^2 = \frac{1}{x-k}$

$(x-k)(x+k) = \frac{1}{x-k}$

$(x-k)^2(x+k) = 1$ A1

ii. Let $f(x) = (x-k)^2(x+k)$.

Since $(x-k)^2(x+k) = 1,\ f(x) = 1$.

Two solutions occur when $\dfrac{32k^3}{27} = 1$, as seen on the graph in **part a.** M1

$k = \dfrac{3 \times 2^{\frac{1}{3}}}{4}$ A1

$\text{solve}\left(\dfrac{32 \cdot k^3}{27} = 1, k\right)$ $\quad k = \dfrac{3 \cdot 2^{\frac{1}{3}}}{4}$

Question 3 (17 marks)

a. Let $B \sim N(280, 24^2)$

$\Pr(250 < B < 300) = 0.692$ A1

1.1 1.2 1.3 *Doc RAD

normCdf(250,300,280,24) 0.692021836

b. $\Pr(\text{elite}) = \Pr(B < 240)$

$= 0.04779...$ M1

$\Pr(\text{purebred} \cap \text{elite}) = 0.35 \times 0.04779...$

$= 0.017$ A1

c. **i.** $X \sim Bi(20, 0.35)$ A1

$\Pr(X = 6) = 0.171$ A1

binomPdf(20,0.35,6) 0.171229673

ii. $\Pr(\hat{P} > 33\%) = \Pr(X \geq 7)$ M1

$= 0.583$ A1

binomCdf(20,0.35,7,20) 0.583374582

iii. $\Pr\left(\hat{P} \geq \dfrac{2}{n}\right) = 1 - \Pr(X = 0) - \Pr(X = 1)$ M1

$0.65^n + n \times 0.35^1 \times 0.65^{n-1} < 0.04$

Let $0.65^n + n \times 0.35^1 \times 0.65^{n-1} = 0.04$ to solve on CAS.

$\text{solve}\left((0.65)^n + n \cdot 0.35 \cdot (0.65)^{n-1} = 0.04, n\right)$

$n = -1.82327443$ or $n = 12.1650004$

As $n > 12.165\ldots$, $n = 13$. A1

d. **i.** $\hat{p} = \dfrac{0.1184 + 0.3047}{2}$

$= 0.21155$ A1

ii. Confidence interval formula: $\left(\hat{p} - z\sqrt{\dfrac{\hat{p}(1-\hat{p})}{n}}, \hat{p} + z\sqrt{\dfrac{\hat{p}(1-\hat{p})}{n}}\right)$

$$0.21155 - z\sqrt{\frac{0.21155(1-0.21155)}{52}} = 0.1184$$

$$z = 1.6438...$$ A1

$\Pr(-1.64 < Z < 1.64) = 0.9$

$= 90\%$ confidence A1

e. **i.** Let $y = \left(t^2 + 2t + 2\right)e^{-t}$.

$\dfrac{dy}{dt} = -t^2 e^{-t}$ A1

$\dfrac{d}{dx}\left(\left(x^2+2\cdot x+2\right)\cdot \mathbf{e}^{-x}\right) \qquad -x^2\cdot \mathbf{e}^{-x}$

$\displaystyle\int -t^2 e^{-t}\,dt = \left(t^2 + 2t + 2\right)e^{-t}$

$\displaystyle\int \frac{mt^2}{e^t}dt = -m\left(t^2 + 2t + 2\right)e^{-t}$

$\displaystyle\int_0^{\infty} \frac{mt^2}{e^t}\,dt = 1$ for a probability density function.

$$\left[\left(t^2 + 2t + 2\right)e^{-t}\right]_0^{\infty} = \lim_{x\to\infty}\left(\left(t^2 + 2t + 2\right)e^{-t}\right) - \left(\frac{0^2 + 2(0) + 2}{e^0}\right)$$ M1

$= 0 - 2$

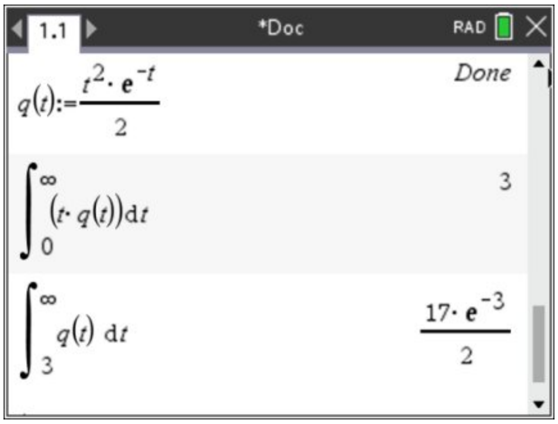
$= -2$

$m \times -2 = -1$

$= \dfrac{1}{2}$ A1

ii. mean age = 3 years A1

$\Pr(Q > 3) = \dfrac{17}{2e^3}$ A1



Question 4 (7 marks)

a. $f(x) = e^{2x} - 2e^x$

$f'(x) = 2e^{2x} - 2e^x$

$= 2e^x\left(e^x - 1\right)$

Let $f'(x) = 0$

$\therefore e^x - 1 = 0$ M1

$x = 0$

$f(0) = -1$

Turning point: $(0, -1)$ A1

b. $a = 0$, as the turning point is at $x = 0$ and g must be a one-to-one function. A1

c. domain g^{-1} = range g

domain $g^{-1} = [-1, \infty)$ A1

Let $y = e^{2x} - 2e^x$.

For inverse, swap x and y.

$x = e^{2y} - 2e^y$

$= (e^y - 1)^2 - 1$ M1

$e^y = \sqrt{x+1} + 1$

$f^{-1}(x) = \log_e\left(\sqrt{x+1} + 1\right)$ A1

d. $(0.86, 0.86)$ A1

solve$\left(\ln\left(\sqrt{x+1}+1\right)=x,x\right)$ $x=0.86032924$

solve$\left(e^{2\cdot x}-2\cdot e^{x}=x,x\right)$

$x=-0.731054944$ or $x=0.86032924$

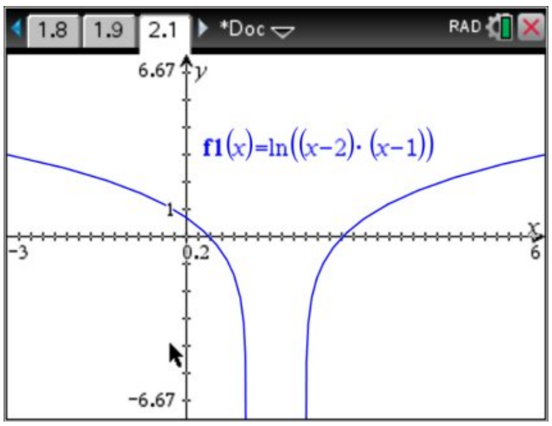
solve$\left(\ln\left(\sqrt{x+1}+1\right)=e^{2\cdot x}-2\cdot e^{x},x\right)$

$x=0.86032924$

Question 5 (20 marks)

a. **i.** domain: $R\,/\,[1,2]$ A1

range: R A1



ii. $$\frac{d\left[g(f(x))\right]}{dx}=\frac{2x-3}{(x-2)(x-1)}$$ A1

Let $\frac{d\left[g(f(x))\right]}{dx}=0.$

$$\frac{2x-3}{(x-2)(x-1)}=0$$

$$x=\frac{3}{2}$$

However, domain is equal to $R\,/\,[1,2]$ and therefore there are no valid solutions and no stationary points. A1

b. i. $h(x) = f(g(x))$

$= \left(\log_e(x) - 1\right)\left(\log_e(x) - 2\right)$ M1

$= \left(\log_e(x)\right)^2 - 3\log_e(x) + 2$

$= \left(\log_e(x) - \frac{3}{2}\right)^2 - \frac{9}{4} + 2$

$= \left(\log_e(x) - \frac{3}{2}\right)^2 - \frac{1}{4}$ A1

ii. $\left(\log_e(x) - \frac{3}{2}\right)^2 \geq 0$

range: $\left[-\frac{1}{4}, \infty\right)$ A1

c. i. $h'(x) = \frac{2\log_e(x) - 3}{x}$ A1

$\frac{d}{dx}(h(x)) \qquad \frac{2\cdot \ln(x) - 3}{x}$

ii. $2\log_e(x) - 3 = 0$

$x_M = e^{\frac{3}{2}}$ A1

d. x_P is the x-coordinate of the intersection of the tangent lines.

$y_1 = 1 - \frac{x}{e}$ and $y_2 = \frac{x}{e^2} - 1$. M1

$1 - \frac{x}{e} = \frac{x}{e^2} - 1$

$2 = \frac{x}{e^2} + \frac{x}{e}$

$2 = x\left(\frac{1}{e^2} + \frac{1}{e}\right)$

$2 = x\left(\frac{1+e}{e^2}\right)$

$x_P = \frac{2e^2}{e+1}$ A1

e. If $x_P < x_M$, then $\dfrac{2e^2}{e+1} < e^{\frac{3}{2}}$.

$$\frac{2e^{\frac{1}{2}}}{e+1} \leq 1$$

$$2e^{\frac{1}{2}} < e+1 \qquad \text{M1}$$

$$4e < (e+1)^2$$

$$4e < e^2 + 2e + 1$$

$$0 < e^2 - 2e + 1$$

$$0 < (e-1)^2$$

As $e > 1$ and $(e-1)^2 > 0$, then $x_P < x_M$ must be true. A1

f. **i.** $h'(e) = -\dfrac{1}{e}$ and $h'(e^2) = \dfrac{1}{e^2}$. M1

$$\angle APB = \pi + \tan^{-1}\left(-\frac{1}{e}\right) - \tan^{-1}\left(\frac{1}{e^2}\right)$$

$$= \tan^{-1}\left(e^2\right) + \tan^{-1}(e) \qquad \text{A1}$$

$\pi + \tan^{-1}\left(-e^{-1}\right) - \tan^{-1}\left(e^{-2}\right) \qquad \tan^{-1}\left(e^{2}\right) + \tan^{-1}(e)$

ii. $\tan^{-1}(1) = \dfrac{\pi}{4}$ and $e > 1$.

$\therefore \tan^{-1}(e) > \dfrac{\pi}{4}$ and $\tan^{-1}\left(e^2\right) > \dfrac{\pi}{4}$.

$$\angle APB = \tan^{-1}\left(e^2\right) + \tan^{-1}(e)$$

$$\angle APB > \frac{\pi}{4} + \frac{\pi}{4}$$

$$> \frac{\pi}{2} \qquad \text{A1}$$

$\therefore \angle APB$ is obtuse

g. area of triangle $ABP = \frac{1}{2} \times \left(e^2 - e\right) \times \left(-y_P\right)$

$x_P = \frac{2e^2}{e+1}$

$y_1 = 1 - \frac{x}{e}$

$y_P = \frac{2}{e+1} - 1$

$A_1 = \frac{1}{2} \times \left(e^2 - e\right) \times \left(1 - \frac{2}{e+1}\right)$ M1

Area bound by $y = h(x)$ and x-axis:

$A_2 = -\int_e^{e^2} h(x)dx$ M1

$= 3e - e^2$

$$-\int_{e}^{e^2} h(x)\,dx \qquad 3\cdot e - e^2$$

total area $= A_1 - A_2$

$= \frac{1}{2} \times \left(e^2 - e\right) \times \left(1 - \frac{2}{e+1}\right) - \left(3e - e^2\right)$ M1

$= \frac{3e^3 - 6e^2 - 5e}{2e+2}$ A1

1.1 1.2 1.3 *Doc RAD

$$\frac{-1}{2}\cdot\left(\frac{2}{e+1}-1\right)\cdot\left(e^2-e\right)-\left(3\cdot e-e^2\right)$$

$$\frac{3\cdot e^2}{2}-\frac{2}{e+1}-\frac{9\cdot e}{2}+2$$

$$\text{comDenom}\left(\frac{3\cdot e^2}{2}-\frac{2}{e+1}-\frac{9\cdot e}{2}+2\right)$$

$$\frac{3\cdot e^3-6\cdot e^2-5\cdot e}{2\cdot e+2}$$