MATHEMATICAL METHODS UNIT 4 SAC 3 SUGGESTED SOLUTIONS

As a rule of thumb, if an answer is correct (or an alternative valid form of the correct answer is given) and there is evidence of sufficient valid working then the student should get full marks. Consequential answer marks to be given *provided the mistake does not trivialise the calculation.*

NB: Marking scheme and distribution of outcomes are suggestive only- they are NOT intended to be prescriptive. Teacher discretion should be used.

a. Require $\frac{1}{12} + \frac{1}{2-k} + \frac{5}{21} + \frac{1}{4} + k = 1$. **O1**

Solve using a CAS calculator:

$$k = \frac{17+\sqrt{317}}{14}, \quad \frac{17-\sqrt{317}}{14}.$$ **O3**

Reject $k = \frac{17+\sqrt{317}}{14} > 2$ because by inspection some of the resulting probabilities are either greater than 1 or less than 0:

$$\frac{1}{2-k} < 0 \text{ and } \frac{1}{4} + k > 1.$$

Answer: $k = \frac{17-\sqrt{317}}{14}$. **O2**

b. i. $\Pr(Y = 14) \times \Pr(Y = 14) = (0.21)^2$.

Answer: 0.0441. **O3**

ii. Use a lattice diagram:

	14	15	16	17
14				
15				▨
16			▨	
17		▨		

The given condition of 32 strawberries (restricted sample space) corresponds to the shaded region.

$$\Pr(Y_1 = 15) \times \Pr(Y_2 = 17)$$
$$+ \Pr(Y_1 = 16) \times \Pr(Y_2 = 16)$$
$$+ \Pr(Y_1 = 17) \times \Pr(Y_2 = 15) \quad *$$

$$= 2(0.38)(0.18) + (0.23)^2. \quad *$$

Either of the lines marked *: **O1**

Answer: 0.1897. **O3**

iii. Use a lattice diagram:

	14	15	16	17
14				
15				■
16			▨	
17		■		

The favourable events in the restricted sample space are dark shaded.

From **part ii.**:

Pr(two packets have 32 strawberries) $= 0.1897$.

Pr(one of the two packets has 15 strawberries) $= 2(0.38)(0.18) = 0.1368$. **O3**

Pr(one packet has 15 strawberries | 32 slices)

$$= \frac{0.1368}{0.1897}.$$ **O2**

Answer: 0.7211. **O3**

Give full marks consequential on answer to part ii.

c. **i.** • *Define the random variable*:

Let A denote the random variable

Number of packets with 16 strawberries.

• *Define the distribution that the random variable follows*:

$A \sim \text{Binomial}\,(n = 7,\ p = 0.23)$. **O1**

Note: $p = \Pr(Y = 16)$.

• *Define the problem in terms of a probability statement*:

$\Pr(A \geq 3) = ?$

Answer: 0.2033. **O3**

ii. • *Define the random variable*:

Let B denote the random variable

Number of packets with 15 strawberries.

• *Define the distribution that the random variable follows*:

$B \sim \text{Binomial}\,(n = 7,\ p = 0.38)$. **O1**

Note: $p = \Pr(Y = 15)$.

• *Define the problem in terms of a probability statement*:

$\Pr(B = 4 \mid B \geq 2) = ?$

$$\Pr(B = 4 \mid B \geq 2) = \frac{\Pr(B = 4)}{\Pr(B \geq 2)} \quad *$$

$$= \frac{0.17393}{0.813695} \quad *$$

Either of the lines marked * or equivalent (both numerator and denominator must be correct to at least 4 decimal places): **O2**

Note: Using only 4 decimal places in * will get penalised via the answer mark if the answer is wrong in the fourth decimal place.

Answer: 0.2138. **O3**

Note: greater than 4 decimal place accuracy must be used during the calculation to avoid rounding error in accuracy (4 decimal places) of the final answer.

d. **i.** • *Define the random variable*:

Let L denote the random variable

Number of packets with less than 16 strawberries.

• *Define the distribution that the random variable follows*:

$L \sim \text{Binomial}\,(n = ?,\ p = 0.59)$. **O2**

Note: $p = \Pr(Y = 14) + \Pr(Y = 15)$.

• *Define the problem in terms of a probability statement*:

The **smallest** value of n such that

$$\Pr\left(L \geq \frac{n}{2}\right) > 0.86 \quad \Rightarrow \Pr\left(L < \frac{n}{2}\right) \leq 0.14$$

is required.

Either one of the above inequalities: **O2**

Use the CAS calculator to solve either of the above inequalities:

• To find $\Pr\left(L \geq \frac{n}{2}\right) > 0.86$, define

f1(x)=binomcdf(x, 0.59, x/2, x).

$x = 24 \Rightarrow n = 24$.

• To find $\Pr\left(L < \frac{n}{2}\right) \leq 0.14$ is more difficult because the function to be defined depends on whether n is even or odd:

If n is even then $n = 2m$ and so

$$\Pr\left(L < \frac{n}{2}\right) = \Pr(L \leq m - 1) \leq 0.14.$$

Define f1(x)=binomcdf(2x, 0.59, 0, x – 1).

$x = 12 \Rightarrow n = 24$.

If n is odd then $n = 2m + 1$ and so

$\Pr\left(L < \frac{n}{2}\right) = \Pr(L \le m) \le 0.14$.

Define f1(x)=binomcdf(2x+1, 0.59, 0, x).

$x = 17 \Rightarrow n = 35$.

But the **smallest** value of n is required and so $n = 35$ is rejected.

Answer: $n = 24$. O3

Check: $\Pr(L \ge 11 \mid n = 22) = 0.8585$.

$\Pr(L \ge 12 \mid n = 23) = 0.8105$.

$\Pr(L \ge 12 \mid n = 24) = \underline{0.8648}$.

$\Pr(L \ge 13 \mid n = 25) = 0.8203$.

$\Pr(L \ge 13 \mid n = 26) = 0.8708$.

If $n = 35$ then

$\Pr(L \ge 18 \mid n = 35) = 0.86024 > 0.86$.

Note: The probabilities for both odd n and even n increase as n increases because $p > \frac{1}{2}$ and so $E(L) > \frac{n}{2}$.

ii. • *Define the random variable*:

Let D denote the random variable

Number of regular strawberries in the sample.

80% or more of a sample size of 11
$\Rightarrow 9 \le D \le 11$.

• *Define the problem in terms of a probability statement*:

$\Pr(D = 9) + \Pr(D = 10) + \Pr(D = 11)$ O1

$= \frac{\binom{12}{9}\binom{3}{2}}{\binom{15}{11}} + \frac{\binom{12}{10}\binom{3}{1}}{\binom{15}{11}} + \frac{\binom{12}{11}\binom{3}{0}}{\binom{15}{11}}$. O1

Answer: $\frac{58}{91}$. O3

e. **i.** Population is large.

$p = 0.9,\ n = 51$.

$E\left(\hat{P}\right) = p$:

Answer: $E\left(\hat{P}\right) = 0.9$. O3

$sd\left(\hat{P}\right) = \sqrt{\frac{p(1-p)}{n}}$:

Answer: $sd\left(\hat{P}\right) = 0.042$. O3

ii. • *Define the random variable*:

Let U denote the random variable

Number of regular strawberries.

• *Define the distribution that the random variable follows*:

Sample drawn from a large population therefore binomial approximation is valid:

$U \sim \text{Binomial}\,(n = 51,\ p = 0.9)$. O1

• *Define the problem in terms of a probability statement*:

85% or more of a sample size of 51
$\Rightarrow 44 \le U \le 51$. O1

$\Pr(U \ge 44) = ?$

Answer: 0.8671. O3

iii. • *Define the distribution that the random variable* $\hat{P}$ *follows*:

Large sample size therefore

$\hat{P} \sim \text{Normal}\,(\mu = 0.9,\ \sigma = 0.042)$. O1

• *Define the problem in terms of a probability statement*:

$\Pr\left(\hat{P} > 0.80\right) = ?$

Answer: 0.991. O3

iv. Sample proportion: $\hat{p} = \frac{47}{51}$. **O1**

98% confidence interval therefore $z_{0.01}$ is required:

$\Pr(Z > z_{0.01}) = 0.01 \Rightarrow z_{0.01} = 2.33$. **O3**

Substitute $\hat{p} = \frac{47}{51}$, $n = 51$ and $z_{0.01} = 2.33$ into

$$\left(\hat{p} - z_{0.01}\sqrt{\frac{\hat{p}(1-\hat{p})}{n}},\ \hat{p} + z_{0.01}\sqrt{\frac{\hat{p}(1-\hat{p})}{n}} \right):$$

Answer: $(0.834,\ 1)$. **O3**

f. Recognition of conditional probability. **O2**

Recognition includes but is not limited to any of the following:

- A statement of the form Pr(4 tweets in total | tweets on both Thursday and Friday).
- Statement of a formula such as $\frac{\Pr(\text{4 tweets in total} \cap \text{tweets on both Thursday and Friday})}{\Pr(\text{tweets on Thursday and Friday})}$.
- Indication of a restricted sample space on a table or tree diagram (by shading etc.).

$\Pr(\text{tweets on both Thursday and Friday}) = 0.81$.

$\Pr(\text{4 tweets in total}) = 0.28$. Both probabilities: **O2**

Answer: $\frac{28}{81}$. Numerator and denominator must both be integers. **O2**

One possible approach is to construct a lattice diagram to visualise the situation:

Number of tweets on Friday

Number of tweets on Thursday

	0	1	2	3	
0					0.1
1				0.06	0.3
2			0.16		0.4
3		0.06			0.2
	0.1	0.3	0.4	0.2	1

- The given condition that there were tweets on both Thursday and Friday restricts the sample space to the shaded region.

- The diagonal cells in the shaded region are the favourable events (total number of tweets is equal to 4) within the restricted sample space.

- The probability in the shaded region (restricted sample space) is equal to $(0.9)(0.9) = 0.81$.

- The probability in the diagonal cells of the shaded region is equal to 0.28.

- Therefore $\Pr(\text{4 tweets in total} \mid \text{tweets on both Thursday and Friday}) = \frac{0.28}{0.81} = \frac{28}{81}$.

g. **i.** If A and B are independent then $\Pr(A' \mid B) = \Pr(A')$.

$$\Pr(A') = 1 - \Pr(A) = 1 - \frac{1}{5} = \frac{4}{5}.$$

Answer: $\frac{4}{5}$. **O1**

ii. If A and B are mutually exclusive then A cannot happen if B happens:

$$\Pr(A \cap B) = 0.$$

$$\Pr(A \cap B) = 0 \Rightarrow \Pr(A' \mid B) = 1.$$

Answer: 1. **O1**

iii. $\Pr(A \cup B) = \frac{3}{4}$

$$\Rightarrow \Pr(A' \cap B') = 1 - \frac{3}{4} = \frac{1}{4}.$$ **O1**

$$\Pr(A' \mid B) = \frac{\Pr(A' \cap B)}{\Pr(B)} = \frac{\frac{33}{60}}{\frac{2}{3}}.$$ **O1**

Answer: $\frac{33}{40}$. **O3**

One possible approach is to use a Karnaugh table. Construct the table as follows:

Insert the given information:

	A	A'	
B			$\frac{2}{3}$
B'		$\frac{1}{4}$	
	$\frac{1}{5}$		1

Fill the empty cells:

	A	A'	
B	$\frac{7}{60}$	$\frac{33}{60}$	$\frac{2}{3}$
B'	$\frac{1}{12}$	$\frac{1}{4}$	$\frac{1}{3}$
	$\frac{1}{5}$	$\frac{4}{5}$	1

The given condition (restricted sample space) corresponds to the shaded region.

The favourable event in the restricted sample space is dark shaded.

From the Karnaugh table:

$$\Pr(A' \mid B) = \frac{\frac{33}{60}}{\frac{2}{3}} = \frac{99}{120} = \frac{33}{40}.$$

h. **i.** $X = a \Rightarrow Z = \dfrac{a - 1.5}{0.4}$.

Therefore:

$$\Pr(X < a) = \Pr\left(Z > \frac{a}{3}\right)$$

$$\Rightarrow \Pr\left(Z < \frac{a-1.5}{0.4}\right) = \Pr\left(Z > \frac{a}{3}\right).$$ **O1**

By symmetry:

$$\Pr\left(Z < -\frac{a}{3}\right) = \Pr\left(Z > \frac{a}{3}\right).$$

Therefore:

$$\Pr\left(Z < \frac{a-1.5}{0.4}\right) = \Pr\left(Z < -\frac{a}{3}\right)$$

$$\Rightarrow \frac{a-1.5}{0.4} = -\frac{a}{3}.$$ **O2**

Solve for a.

Answer: $a = \frac{45}{34}$. **O3**

ii.
- $X = 2 \Rightarrow Z = \dfrac{2-1.5}{0.4} = \dfrac{5}{4}$.
- $X = k \Rightarrow Z = \dfrac{k-1.5}{0.4} = z_k$.

Both 'values' of *Z*: **O1**

Therefore:

$$\Pr(k < X < 2) = 0.846$$

$$\Rightarrow \Pr\left(z_k < Z < \frac{5}{4}\right) = 0.846$$

$$\Rightarrow \Pr\left(Z < \frac{5}{4}\right) - \Pr(Z < z_k) = 0.846$$

$$\Rightarrow \left(1 - \Pr\left(Z > \frac{5}{4}\right)\right) - \Pr(Z < z_k) = 0.846$$

$$\Rightarrow (1 - 0.106) - \Pr(Z < z_k) = 0.846$$

$$\Rightarrow \Pr(Z < z_k) = 0.048. \qquad \ldots.\ (1)$$ **O1**

$$\Pr\left(Z > \frac{5}{3}\right) = 0.048$$

$$\Rightarrow \Pr\left(Z < -\frac{5}{3}\right) = 0.048. \qquad \ldots.\ (2)$$

by symmetry.

Compare (1) and (2): $z_k = -\dfrac{5}{3}$.

Therefore:

$$\frac{k-1.5}{0.4} = -\frac{5}{3}.$$ **O2**

Solve for *k*:

Answer: $k = \dfrac{5}{6}$. **O2**

iii.
- By definition:

$$\int_0^1 4bx^3 + 3x^2 - b^2 dx = 1$$ **O1**

$$\Rightarrow b = 0, 1.$$ **O3**

- By definition:

$f(x) \geq 0$ for $0 < x \leq 1$.

$b = 0$:

$$f(x) = \begin{cases} 3x^2 & \text{if } 0 < x \leq 1 \\ 0 & \text{otherwise} \end{cases}$$

$3x^2 > 0$ over $0 < x < 1$
therefore $b = 0$ is a valid solution.

$b = 1$:

$$f(x) = \begin{cases} 4x^3 + 3x^2 - 1 & \text{if } 0 < x \leq 1 \\ 0 & \text{otherwise} \end{cases}$$

$-1 < 4x^3 + 3x^2 - 1 < 6$ over $0 < x < 1$.
Therefore $f(x)$ is not always greater than or equal to zero over $0 < x < 1$.

Therefore $b = 1$ is NOT a valid solution.

Answer: $b = 0$. **O3**

Justification for the rejection of $b = 1$ is required.