# 2021 VCE Mathematical Methods Trial Examination 2

# Detailed Answers

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## SECTION A

## ANSWERS

| | | | | | |
|---|---|---|---|---|---|
| 1 | **A** | B | C | D | E |
| 2 | A | **B** | C | D | E |
| 3 | A | B | **C** | D | E |
| 4 | A | B | C | **D** | E |
| 5 | A | B | C | D | **E** |
| 6 | **A** | B | C | D | E |
| 7 | A | **B** | C | D | E |
| 8 | A | B | **C** | D | E |
| 9 | A | B | C | **D** | E |
| 10 | A | B | C | D | **E** |
| 11 | A | B | **C** | D | E |
| 12 | A | B | **C** | D | E |
| 13 | A | B | C | D | **E** |
| 14 | A | B | C | **D** | E |
| 15 | A | B | C | D | **E** |
| 16 | **A** | B | C | D | E |
| 17 | **A** | B | C | D | E |
| 18 | A | B | C | **D** | E |
| 19 | A | **B** | C | D | E |
| 20 | A | **B** | C | D | E |

## SECTION A

**Question 1** **Answer A**

The period is $T = \dfrac{2\pi}{\frac{\pi}{b}} = 2b$, the range of $\cos(x)$ is $[-1,1]$, the range of $\cos\left(\dfrac{\pi x}{b}\right)-1$ is $[-2,0]$ but since $a<0$, reflecting in the $x$-axis, the range is $[0,2]$ dilating by a factor of $a$, the range of $a\left(\cos\left(\dfrac{\pi x}{b}\right)-1\right)$ becomes $[0,-2a]$.

**Question 2** **Answer B**

Let $f:[0,3\pi]\to R$, $f(x)=3\sin\left(\dfrac{x}{3}\right)-3$. The period is $T=\dfrac{2\pi}{\frac{1}{3}}=6\pi$

The graph of $f$ is transformed by a reflection in the $x$-axis, the rule is
$g(x)=3-3\sin\left(\dfrac{x}{3}\right)$, we only have one-half of a cycle.

Now a dilation of factor 3 from the $y$-axis, replace $x$ with $\dfrac{x}{3}$.

$g(x)=3-3\sin\left(\dfrac{x}{9}\right)$, the period is $T=\dfrac{2\pi}{\frac{1}{9}}=18\pi$, since we must have one-half of a cycle, the new domain is $[0,9\pi]$, then a dilation by a factor of 3 from the $x$-axis, multiply $y$ by 3

the equation becomes $g:[0,9\pi]\to R$, $g(x)=9-9\sin\left(\dfrac{x}{9}\right)$

**Question 3** **Answer C**

$\dfrac{dy}{dx}=3\sin\left(\dfrac{x}{3}\right)$

$y=\displaystyle\int 3\sin\left(\dfrac{x}{3}\right)dx=-9\cos\left(\dfrac{x}{3}\right)+c$

to find $c$ use $y\left(\dfrac{\pi}{2}\right)=0$

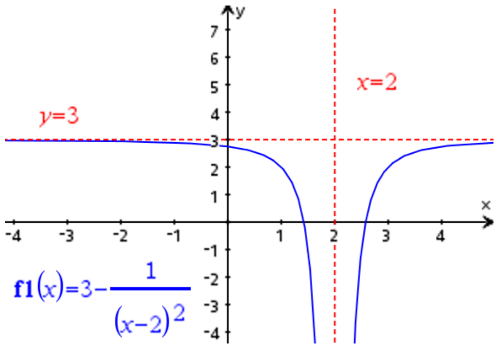
$0=-9\cos\left(\dfrac{\pi}{6}\right)+c$ , $c=9\cos\left(\dfrac{\pi}{6}\right)=\dfrac{9\sqrt{3}}{2}$

$y(x)=9\left(\dfrac{\sqrt{3}}{2}-\cos\left(\dfrac{x}{3}\right)\right)$

when $x=0$ , $y(0)=9\left(\dfrac{\sqrt{3}}{2}-1\right)=\dfrac{9}{2}\left(\sqrt{3}-2\right)$

| | |
|---|---|
| Define $y(x)=\int\left(3\cdot\sin\left(\frac{x}{3}\right)\right)dx+c$ | *Done* |
| solve$\left(y\left(\frac{\pi}{2}\right)=0,c\right)$ | $c=\frac{9\cdot\sqrt{3}}{2}$ |
| $y(x)$ | $\frac{9\cdot\sqrt{3}}{2}-9\cdot\cos\left(\frac{x}{3}\right)$ |
| factor$(y(0))$ | $\frac{9\cdot\left(\sqrt{3}-2\right)}{2}$ |

**Question 4** **Answer D**

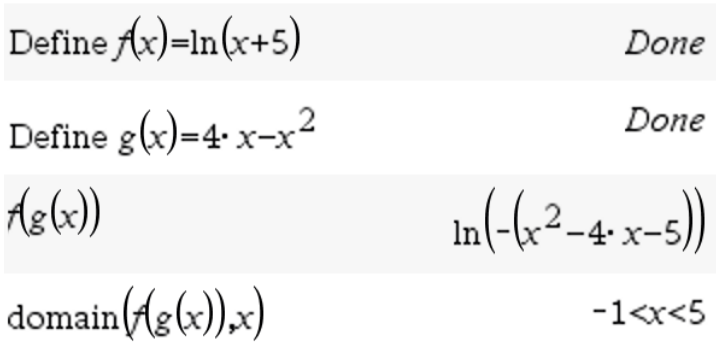



$y = b - \frac{1}{(x-a)^2}$ is the only curve which

has a vertical asymptote at $x = a$ and

a horizontal asymptote at $y = b$.

**Question 5** **Answer E**



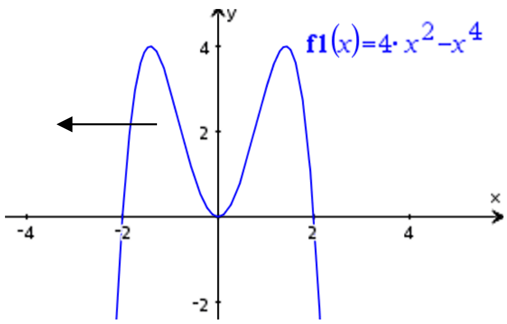

$f(x) = \log_e(x+5)$ and $g(x) = 4x - x^2$

$$\begin{aligned} f(g(x)) &= \log_e(g(x)+5) \\ &= \log_e(4x - x^2 + 5) \\ &= \log_e(-(x^2 - 4x - 5)) \\ &= \log_e(-(x-5)(x+1)) \end{aligned}$$

we require $-(x-5)(x+1) > 0$ or

$(x-5)(x+1) < 0$ that is $(-1,5)$, since $\text{dom } f(g(x)) = \text{dom } g(x) = D_g$

**Question 6** **Answer A**




$f(x) = kx^2 - x^4 = x^2(k - x^2)$

crosses the $x$-axis at $x = 0$ , $x = \pm\sqrt{k}$

$f'(x) = 2kx - 4x^3 = 2x(k - 2x^2)$

has stationary points at $x = 0$ , $x = \pm\sqrt{\frac{k}{2}}$

since $k > 0$. The function will have an inverse provided that the function is one-one, increasing function that is $b < -\sqrt{\frac{k}{2}}$.

**Question 7** **Answer B**

original curve $\frac{y}{2} = \cos(x)$ image curve $y' = \sin(2x') = \cos\left(\frac{\pi}{2} - 2x'\right)$

$x = \frac{\pi}{2} - 2x'$ , $x' = -\frac{x}{2} + \frac{\pi}{4}$ , $\frac{y}{2} = y'$ in matrix form $T\left(\begin{bmatrix} x \\ y \end{bmatrix}\right) = \begin{bmatrix} -\frac{1}{2} & 0 \\ 0 & \frac{1}{2} \end{bmatrix}\begin{bmatrix} x \\ y \end{bmatrix} + \begin{bmatrix} \frac{\pi}{4} \\ 0 \end{bmatrix}$

**Question 8** **Answer C**

$\int_a^b \left( f(x)+k \right) dx = \int_a^b f(x)\,dx + \left[ kx \right]_a^b = A + k(b-a)$ **A.** is true

$\frac{1}{k}\int_{ak}^{bk} f\left(\frac{x}{k}\right) dx = \int_a^b f(x)\,dx = A$

the area gets dilation by a factor of $k$ parallel to the $x$-axis. **B.** is true

$\int_{ak}^{bk} \frac{f(kx)}{k}\,dx = \frac{1}{k}\int_{ak}^{bk} f(kx)\,dx \neq A$ **C.** is false

$A = \int_{a-k}^{b-k} f(x+k)\,dx = \int_{a+k}^{b+k} f(x-k)\,dx = \int_a^b f(x)\,dx$ the area remains the same under a transformation $k$ units to the left or right parallel to the $x$-axis. **D**. and **E**. are both true.

**Question 9** **Answer D**

Total $b+r$ marbles, let $A$ be the event two marbles of the same color, drawn without replacement, that is 1 red and 2 blue or 1 blue and 2 red, but there are 3 ways of doing each of these,

*BBR*, *BRB*, *RBB*, *RRB*, *RBR*, *BRR*,

$$\Pr(A) = \frac{3b(b-1)r}{(b+r)(b+r-1)(b+r-2)} + \frac{3r(r-1)b}{(b+r)(b+r-1)(b+r-2)}$$

$$= \frac{3br(b-1)+3br(r-1)}{(b+r)(b+r-1)(b+r-2)} = \frac{3br(b-1+r-1)}{(b+r)(b+r-1)(b+r-2)} = \frac{3br(b+r-2)}{(b+r)(b+r-1)(b+r-2)}$$

$$= \frac{3br}{(b+r)(b+r-1)}$$

**Question 10** **Answer E**

(1) $4y - nx = n^2 \quad \Rightarrow \quad y = \frac{nx}{4} + \frac{n^2}{4}$

(2) $ny - mx = n \quad \Rightarrow \quad y = \frac{mx}{n} + 1$

equal gradients when $\frac{n}{4} = \frac{m}{n}$ , $n^2 = 4m$

equal $y$-intercepts when $\frac{n^2}{4} = 1$ , $n = \pm 2$,

so that when. $n = \pm 2$ and $m = 1$,

there is an infinite number of solutions,

and when $n = \pm 2$ and $m = -1$ there is a unique solution.

$eq1 := 4\cdot y - n\cdot x = n^2$ $\quad 4\cdot y - n\cdot x = n^2$

$eq2 := n\cdot y - m\cdot x = n$ $\quad n\cdot y - m\cdot x = n$

solve($eq1$ and $eq2$, $\{x,y\}$)|$n=-2$ and $m=1$
$x=-2\cdot(c12-1)$ and $y=c12$

solve($eq1$ and $eq2$, $\{x,y\}$)|$n=-2$ and $m=-1$
$x=0$ and $y=1$

solve($eq1$ and $eq2$, $\{x,y\}$)|$n=2$ and $m=1$
$x=2\cdot(c13-1)$ and $y=c13$

solve($eq1$ and $eq2$, $\{x,y\}$)|$n=2$ and $m=-1$
$x=0$ and $y=1$

**Question 11** **Answer C**

$$\frac{d}{dx}\left(x^2e^{-kx}\right)=xe^{-kx}\left(2-kx\right)=2xe^{-kx}-kx^2e^{-kx}$$

$$\int\left(2xe^{-kx}-kx^2e^{-kx}\right)dx=x^2e^{-kx}$$

$$2\int xe^{-kx}\,dx-k\int x^2e^{-kx}\,dx=x^2e^{-kx}$$

$$k\int x^2e^{-kx}\,dx=2\int xe^{-kx}\,dx-x^2e^{-kx}$$

$$\int x^2e^{-kx}\,dx=\frac{2}{k}\int xe^{-kx}\,dx-\frac{x^2}{k}e^{-kx}+c$$

**Question 12** **Answer C**

$$\Pr\left(X=k\right)=\frac{e^{-\lambda}\lambda^k}{k!}\ ,\quad \Pr\left(X=0\right)=e^{-\lambda}\ ,\quad \Pr\left(X=1\right)=\lambda e^{-\lambda}\ ,\quad \Pr\left(X=2\right)=\frac{\lambda^2e^{-\lambda}}{2}$$

$$\Pr\left(X<3\mid X\ge 1\right)=\Pr\left(X\le 2\mid X\ge 1\right)=\frac{\Pr\left(1\le X\le 2\right)}{\Pr\left(X\ge 1\right)}=\frac{\Pr\left(X=1\right)+\Pr\left(X=2\right)}{1-\Pr\left(X=0\right)}$$

$$\Pr\left(X<3\mid X\ge 1\right)=\frac{\lambda e^{-\lambda}+\dfrac{\lambda^2e^{-\lambda}}{2}}{1-e^{-\lambda}}=\frac{\lambda e^{-\lambda}\left(1+\dfrac{\lambda}{2}\right)}{1-e^{-\lambda}}=\frac{\lambda e^{-\lambda}\left(\lambda+2\right)}{2\left(1-e^{-\lambda}\right)}$$

**Question 13** **Answer E**

| | $A$ | $A'$ | |
|---|---|---|---|
| $B$ | $p^2$ | $b^2-p^2$ | $b^2$ |
| $B'$ | $a^2-p^2$ | $\Pr\left(A'\cap B'\right)$ | $1-b^2$ |
| | $a^2$ | $1-a^2$ | |

$$\Pr\left(A'\cap B'\right)=1-a^2-\left(b^2-p^2\right)=1-b^2-\left(a^2-p^2\right)$$

$$\Pr\left(A'\cap B'\right)=\left(1-a^2\right)+\left(p^2-b^2\right)=\left(1-b^2\right)+\left(p^2-a^2\right)$$

Given that $(1-a)(1+a)+(p-b)(p+b)=\left(1-a^2\right)\left(1-b^2\right) \iff p^2=a^2b^2$

$$\left(1-a^2\right)+a^2b^2-b^2=\left(1-a^2\right)-b^2\left(1-a^2\right)=\left(1-a^2\right)\left(1-b^2\right)$$

Since $\Pr\left(A\cap B'\right)\ne 0$ , $\Pr\left(A'\cap B'\right)\ne 0$ **A**. and **B**. are false.

In fact the all the events, $A$ and $B$, $A$ and $B'$, $A'$ and $B'$, $A'$ and $B'$ are all independent.

$$\Pr\left(A'\cap B'\right)=\left(1-a^2\right)\left(1-b^2\right)=\Pr\left(A'\right)\Pr\left(B'\right)$$

$$\Pr\left(A\cap B\right)=p^2=a^2b^2=\Pr\left(A\right)\Pr\left(B\right)$$

$$\Pr\left(A\cap B'\right)=a^2-p^2=a^2-a^2b^2=a^2\left(1-b^2\right)=\Pr\left(A\right)\Pr\left(B'\right)$$

$$\Pr\left(A'\cap B\right)=b^2-p^2=b^2-a^2b^2=\left(1-a^2\right)b^2=\Pr\left(A'\right)\Pr\left(B\right)$$

**Question 14** **Answer D**

$f: \quad y = \dfrac{x-a}{x-1} \quad , \quad a \in R \setminus \{1\}$ swap $x$ and $y$

$f^{-1}: \quad x = \dfrac{y-a}{y-1} \quad , \quad x(y-1) = y-a$

$xy - x = y - a \quad , \quad xy - y = x - a \quad , \quad y(x-1) = x - a$

Since $f^{-1}(x) = \dfrac{x-a}{x-1} = f(x)$ they are the same graph and therefore have an infinite number of points of intersection.

**Question 15** **Answer E**

Given $f(a) = b \Rightarrow f^{-1}(b) = a \; , \; f'(a) = c$

let $y = g(x) = f^{-1}(x)$

$x = f(y)$ differentiate wrt $y$ $\dfrac{dx}{dy} = f'(y)$

inverting $\dfrac{dy}{dx} = \dfrac{d}{dx}\left[f^{-1}(x)\right] = g'(x) = \dfrac{1}{f'(y)} = \dfrac{1}{f'\left(f^{-1}(x)\right)}$

$g'(b) = \dfrac{1}{f'\left(f^{-1}(b)\right)} = \dfrac{1}{f'(a)} = \dfrac{1}{c}$

**Question 16** **Answer A**

$n = 25 \; , \; p = 0.1 \; , \; \hat{P} = \dfrac{X}{n} = \dfrac{X}{25} \quad X \overset{d}{=} Bi(n = 25, p = 0.1)$

binomCdf(25,0.1,0,4) 0.902006

$\Pr\left(\hat{P} < 0.2\right) = \Pr\left(\hat{P} < \dfrac{1}{5}\right) = \Pr(X < 5) = \Pr(X \le 4) = 0.902$

**Question 17** **Answer A**

$f(x) = kx$, since it is a probability density function $\int_0^a kx\,dx = 1$

the mean value $\dfrac{1}{a-0}\int_0^a kx\,dx = \dfrac{1}{a} \times 1 = 2 \Rightarrow a = \dfrac{1}{2}$

$\int_0^a kx\,dx = \int_0^{\frac{1}{2}} kx\,dx = k\left[\dfrac{x^2}{2}\right]_0^{\frac{1}{2}} = k\left(\dfrac{1}{8} - 0\right) = 1, \quad k = 8$

**Question 18** **Answer D**

The confidence interval is

$$\left(\hat{p}-1.96\sqrt{\frac{\hat{p}(1-\hat{p})}{n}}, \hat{p}+1.96\sqrt{\frac{\hat{p}(1-\hat{p})}{n}}\right)$$

$\hat{p}=\frac{x}{n}$, $n=36$

$$f(x)=\frac{x}{36}+1.96\sqrt{\frac{\frac{x}{36}\left(1-\frac{x}{36}\right)}{36}}$$

$f'(x)=0 \quad \Rightarrow \quad x=35$

Define $f(x)=\frac{x}{n}+1.96\cdot\sqrt{\frac{\frac{x}{n}\cdot\left(1-\frac{x}{n}\right)}{n}}\,|n=36$ — *Done*

$\text{solve}\left(\frac{d}{dx}(f(x))=0,x\right)$ — $x=35.1102$

$f(34)$ — 1.01927

$f(35)$ — 1.02591

**Question 19** **Answer B**

$\sum \Pr(X=x)=\log_{81}(a)+\log_{81}(b)=\log_{81}(ab)=1$

$(1) \quad ab=81$

$E(X)=\sum x\Pr(X=x)=2\log_{81}(b)-2\log_{81}(a)=1$

$E(X)=2\log_{81}\left(\frac{b}{a}\right)=1 \quad \Rightarrow \quad \log_{81}\left(\frac{b}{a}\right)=\frac{1}{2}$

$(2)\ \frac{b}{a}=9 \quad b=9a$ solving $(1)$ and $(2)$

$9a^2=81$ , $a^2=9$ , $a=3$ , since $a>0$, $b>0$

$a=3$ , $b=27$

$eq1:=\log_{81}(a)+\log_{81}(b)=1$ — $\log_{81}(a)+\log_{81}(b)=1$

$eq2:=2\cdot\left(\log_{81}(b)-\log_{81}(a)\right)=1$ — $-2\cdot\left(\log_{81}(a)-\log_{81}(b)\right)=1$

$\text{solve}(eq1 \text{ and } eq2,\{a,b\})$ — $a=3$ and $b=27$

**Question 20** **Answer B**

$\Pr(X>b)=0.03 \qquad \Pr(X<c)=0.1$

$\Pr(X<b)=0.97$

invNorm(0.97,0,1) — 1.88079

invNorm(0.1,0,1) — -1.28155

$\frac{b-\mu}{\sigma}=1.88$ and $\frac{c-\mu}{\sigma}=-1.28 \qquad \frac{b-\mu}{\sigma}=1.88$ and $\frac{\mu-c}{\sigma}=1.28$.

**END OF SECTION A SUGGESTED ANSWERS**

## SECTION B

### Question 1

**a.** $f(x)=x^4-6x^2+bx+c$

$f(3)=0 \Rightarrow 81-54+3b+c=0$ , $(1)$ $3b+c=-27$

$f'(x)=4x^3-12x+b$ M1

$f'(-2)=0 \Rightarrow -32+24+b=0$ , $b=8$

$b=8$ , $c=-27-24=-51$ A1

**b.** $f(x)=x^4-6x^2+c$

$f(x)=x^4-6x^2+c=0$

$\Rightarrow x^2=\dfrac{6\pm\sqrt{36-4\times c}}{2}=3\pm\sqrt{9-c}$

$f'(x)=4x^3-12x=4x(x^2-3)=0$ for stationary points M1

$f'(x)=0 \Rightarrow x=0$ , $\pm\sqrt{3}$

$f(0)=c$ , $f(\sqrt{3})=c-9$ , $f(-\sqrt{3})=c-9$

A2

| $y=f(x)=x^4-6x^2+c$ | values of $c$ |
|---|---|
| crosses the $x$-axis four times, that is, there are four solutions for $f(x)=0$. | $0<c<9$ |
| crosses the $x$-axis three times, that is, there are three solutions for $f(x)=0$. | $c=0$ |
| crosses the $x$-axis twice, that is, there are two solutions for $f(x)=0$. | $c=9$ or $c<0$ |
| does not cross the $x$-axis, that is, there are no solutions for $f(x)=0$. | $c>9$ |

**c.i.** $f(x)=x^4-6x^2-4x$

$f'(x)=4x^3-12x-4=0$

$x=-1.53,-0,35,1.88$

$(-1.53,-2.45)$, $(-0.35,0.68)$, $(1.88,-16.23)$ A1

**c.ii.** $m(x) = f'(x) = 4x^3 - 12x - 4$

$m'(x) = 12x^2 - 12 = 12(x^2 - 1) = 0$ for maximum and minimum gradient

$x = \pm 1 \quad m(-1) = f'(-1) = -4 + 12 - 4 = 4$ , $m(1) = f'(1) = 4 - 12 - 4 = -12$ A1

**c.iii.** $f(-1) = -1$ , $f(1) = 9$ , $P(-1,-1)$ , $Q(1,-9)$

gradient joining $m(PQ) = \dfrac{-9+1}{1-(-1)} = -\dfrac{8}{2} = -4$ A1

equation of the line joining $PQ$: $y + 1 = -4(x+1)$

$y = -4x - 5$, and drawing the line on the graph below A1

**c.iv.** solving $x^4 - 6x^2 - 4x = -4x - 5$

$x^4 - 6x^2 + 5 = 0$

$(x^2 - 5)(x^2 - 1) = 0$ M1

$x = \pm\sqrt{5}$ , $\pm 1$

$x = \pm 1$ are the points $P$ and $Q$

$x_S = -\sqrt{5}$ , $x_R = \sqrt{5}$ A1

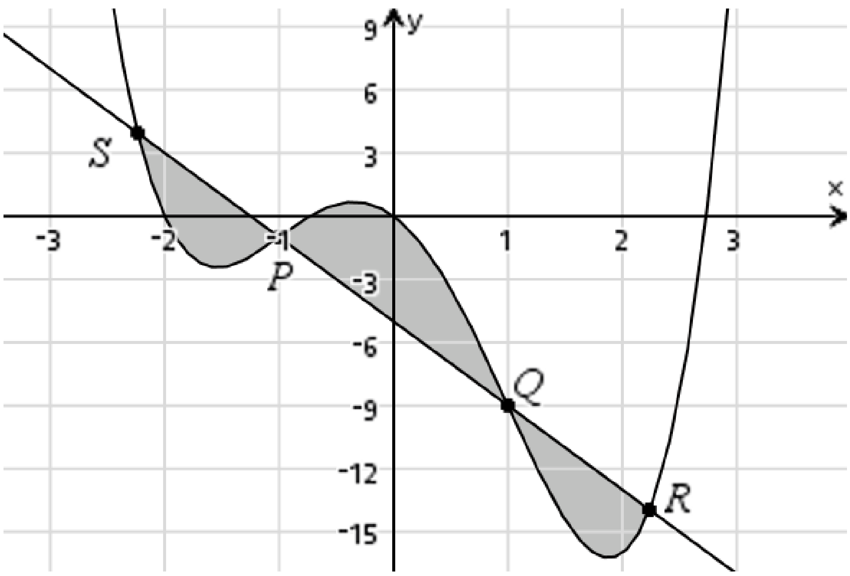
**c.v.** $f(-\sqrt{5}) = 4\sqrt{5} - 5$ , $f(\sqrt{5}) = -4\sqrt{5} - 5$

$R(\sqrt{5}, -4\sqrt{5} - 5)$ , $S(-\sqrt{5}, 4\sqrt{5} - 5)$, $P(-1,-1)$ , $Q(1,-9)$

$d(SP) = \sqrt{(-1+\sqrt{5})^2 + (-4\sqrt{5}+4)^2} = \sqrt{17}(\sqrt{5} - 1)$ M1

$d(QR) = \sqrt{(\sqrt{5}-1)^2 + (4 - 4\sqrt{5})^2} = \sqrt{17}(\sqrt{5} - 1)$

so $d(SP) = d(QR) = \sqrt{17}(\sqrt{5} - 1)$ A1



**c.vi.** Let $t(x) = -4x - 5$

$$A_1 = \int_{x_S}^{x_P} \left(t(x) - f(x)\right) dx = \int_{-\sqrt{5}}^{-1} \left(-x^4 + 6x^2 - 5\right) dx = \frac{16}{5} = 3\frac{1}{5} = 3.2$$

M1

$$A_2 = \int_{x_P}^{x_Q} \left(f(x) - t(x)\right) dx = \int_{-1}^{1} \left(x^4 - 6x^2 + 5\right) dx = \frac{32}{5} = 6\frac{2}{5} = 6.4$$

$$A_3 = \int_{x_Q}^{x_R} \left(t(x) - f(x)\right) dx = \int_{1}^{\sqrt{5}} \left(-x^4 + 6x^2 - 5\right) dx = \frac{16}{5} = 3\frac{1}{5} = 3.2$$

A1

$$A_1 = A_3 = \frac{1}{2} A_2 \quad , \quad A_1 + A_3 = A_2$$

| | |
|---|---|
| Define $f1(x) = x^4 - 6 \cdot x^2 + b \cdot x + c$ | *Done* |
| $f1(3) = 0$ | $3 \cdot b + c + 27 = 0$ |
| Define $df1(x) = \frac{d}{dx}(f1(x))$ | *Done* |
| $df1(-2) = 0$ | $b - 8 = 0$ |
| solve$(3 \cdot b + c + 27 = 0, c) \mid b = 8$ | $c = -51$ |
| Define $f1(x) = x^4 - 6 \cdot x^2 + c$ | *Done* |
| $u^2 - 6 \cdot u + c = 0$ | $u^2 - 6 \cdot u + c = 0$ |
| solve$(u^2 - 6 \cdot u + c = 0, u)$ | $u = -(\sqrt{9-c} - 3)$ or $u = \sqrt{9-c} + 3$ |
| Define $f2(x) = x^4 - 6 \cdot x^2 - 4 \cdot x$ | *Done* |
| solve$\left(\frac{d}{dx}(f2(x)) = 0, x\right)$ | $x = -1.5321$ or $x = -0.3473$ or $x = 1.8794$ |
| zeros$\left(\frac{d}{dx}(f1(x)), x\right) \rightarrow xtps$ | $\{-1.5321, -0.3473, 1.8794\}$ |
| $f1(xtps)$ | $\{-2.4456, 0.6800, -16.2344\}$ |
| $f1(\sqrt{5})$ | $-4 \cdot \sqrt{5} - 5$ |
| $f1(-\sqrt{5})$ | $4 \cdot \sqrt{5} - 5$ |
| $\sqrt{(\sqrt{5} - 1)^2 + (4 - 4 \cdot \sqrt{5})^2}$ | $(\sqrt{5} - 1) \cdot \sqrt{17}$ |
| $f2(x) - (-4 \cdot x - 5)$ | $x^4 - 6 \cdot x^2 + 5$ |
| $a1 := \int_{-\sqrt{5}}^{-1} (-x^4 + 6 \cdot x^2 - 5) dx$ | $\frac{16}{5}$ |
| $a2 := \int_{-1}^{1} (x^4 - 6 \cdot x^2 + 5) dx$ | $\frac{32}{5}$ |
| $a3 := \int_{1}^{\sqrt{5}} (-x^4 + 6 \cdot x^2 - 5) dx$ | $\frac{16}{5}$ |

**Question 2**

**a.** $L = h\left(y_0 + y_1 + y_2 + y_3 + y_4\right)$

$L = 0.2\left(2 + 2.18 + 2.33 + h + 1.24\right) = 1.93$

$h = 1.90 = y_3 \quad , \quad b = y_5 = ?$

$R = h\left(y_1 + y_2 + y_3 + y_4 + y_5\right)$

$R = 0.2\left(2.18 + 2.33 + 1.90 + 1.24 + b\right) = 1.79$

$b = 1.30$ A1

**b.** $f\left(1\right) = 1\sin\left(\frac{7\pi}{4}\right) + 2 = 2 - \frac{\sqrt{2}}{2} = \frac{1}{2}\left(4 - \sqrt{2}\right)$ A1

**c.** $f\left(x\right) = x\sin\left(\frac{7\pi x}{4}\right) + 2$

$f'\left(x\right) = \sin\left(\frac{7\pi x}{4}\right) + \frac{7\pi x}{4}\cos\left(\frac{7\pi x}{4}\right)$ A1

solving $f'\left(x\right) = 0 \quad x \in \left(0.1\right)$

gives $x = 0.369$ , $0.894$ , $f\left(0.369\right) = 2.331$ , $f\left(0.894\right) = 1.124$

closest point $\left(0.894, 1.124\right)$ furthest point $\left(0.369, 2.331\right)$ A1

**d.** $A\left(0.3, 0.4\right) \quad R\left(u, f\left(u\right)\right)$

$d\left(AR\right) = s\left(u\right) = \sqrt{\left(u - 0.3\right)^2 + \left(f\left(u\right) - 0.4\right)^2}$ M1

solving $\frac{ds}{du} = 0$ gives $u = 0.0162$ , $0.3715$ , $0.8659$ M1

the minimum distance is $s_{\min} = s\left(0.8659\right) = 0.9277\,\text{km}$ A1

**e.** using average of the left and right rectangles $A = \frac{1}{2}\left(L + R\right) = 0.5\left(1.93 + 1.79\right) = 1.86\,\text{km}^2$

using the river equation $A = \int_0^1 \left(x\sin\left(\frac{7\pi x}{4}\right) + 2\right)dx = 1.85\ \text{km}^2$

the average of the left and right rectangles over-estimate the modelled area. A1

**f.** solving $\int_0^m \left(x\sin\left(\frac{7\pi x}{4}\right) + 2\right)dx = \frac{1.848}{2} = \int_m^1 \left(x\sin\left(\frac{7\pi x}{4}\right) + 2\right)dx$

or solving $\int_0^m \left(x\sin\left(\frac{7\pi x}{4}\right) + 2\right)dx = \int_m^1 \left(x\sin\left(\frac{7\pi x}{4}\right) + 2\right)dx$ M1

gives solving $m = 0.4235$

$f\left(m\right) = 2.3077$ , the length of the fence is 2.3077 km A1

| | |
|---|---|
| $\text{solve}(0.2\cdot(2+2.18+2.33+h+1.24)=1.93,h)$ | $h=1.90$ |
| $\text{solve}(0.2\cdot(2.18+2.33+1.9+1.24+b)=1.79,b)$ | $b=1.30$ |
| Define $f1(x)=x\cdot\sin\left(\frac{7\cdot\pi\cdot x}{4}\right)+2$ | *Done* |
| $f1(1)$ | $2-\frac{\sqrt{2}}{2}$ |
| $\frac{d}{dx}(f1(x))$ | $\frac{7\cdot\pi\cdot x\cdot\cos\left(\frac{7\cdot\pi\cdot x}{4}\right)}{4}+\sin\left(\frac{7\cdot\pi\cdot x}{4}\right)$ |
| ⚠ $\text{solve}\left(\frac{d}{dx}(f1(x))=0,x\right)\|0<x<1$ | $x=0.369$ or $x=0.894$ |
| $f1(0.369)$ | 2.331 |
| $f1(0.894)$ | 1.124 |
| Define $s(u)=\sqrt{(u-0.3)^2+(f1(u)-0.4)^2}$ | *Done* |
| ⚠ $\text{solve}\left(\frac{d}{du}(s(u))=0,u\right)\|0<u<1$ | $u=0.0162$ or $u=0.3715$ or $u=0.8659$ |
| $s(0.0162)$ | 1.6264 |
| $s(0.3715)$ | 1.9323 |
| $s(0.8659)$ | 0.9277 |
| $0.5\cdot(1.93+1.79)$ | 1.8600 |
| $\int_0^1 f1(x)\,dx$ | 1.8480 |
| ⚠ $\text{solve}\left(\int_0^m f1(x)\,dx=\int_m^1 f1(x)\,dx,m\right)\|0<m<1$ | $m=0.4235$ |
| $f1(0.42351)$ | 2.3077 |

**Question 3**

tennis balls $T \stackrel{d}{=} N\left(57, 0.937^2\right)$

**a.i.** $\Pr(T < 56) = 0.1429$ A1

**ii.** containers $C \stackrel{d}{=} Bi(n = 3, p = 0.1429)$

$\Pr(C \geq 1) = 1 - \Pr(C = 0) = 1 - (1 - 0.1429)^3 = 0.3704$ A1

| | |
|---|---|
| normCdf(-∞,56,57,0.937) | 0.142933 |
| p:=0.1429327019929 | 0.142933 |
| $1-(1-p)^3$ | 0.370429 |
| binomCdf(3,p,1,3) | 0.370429 |

**b.** containers $Cn \stackrel{d}{=} Bi(n = ?, p = 0.1429)$

$\Pr(Cn = 2) + \Pr(Cn = 3) = 0.5$

$\binom{n}{2} 0.1429^2 \times (1 - 0.1429)^{n-2} + \binom{n}{3} 0.1429^3 \times (1 - 0.1429)^{n-3} = 0.5$ M1

solving gives $n = 15.34$ or $18.12$

so $n = 16, 17, 18$ A1

$\text{nCr}(n,2)\cdot p^2\cdot(1-p)^{n-2}+\text{nCr}(n,3)\cdot p^3\cdot(1-p)^{n-3}=0.5$
$0.000773\cdot n\cdot(n-1)\cdot(n+15.9889)\cdot(0.857067)^n=0.5$

solve$(7.730343401055\text{E-}4\cdot n\cdot(n-1)\cdot(n+15.98889$
$n=15.3405$ or $n=18.1177$

$\text{nCr}(n,2)\cdot p^2\cdot(1-p)^{n-2}+\text{nCr}(n,3)\cdot p^3\cdot(1-p)^{n-3}|n=15$
0.497563

$\text{nCr}(n,2)\cdot p^2\cdot(1-p)^{n-2}+\text{nCr}(n,3)\cdot p^3\cdot(1-p)^{n-3}|n=16$
0.503093

$\text{nCr}(n,2)\cdot p^2\cdot(1-p)^{n-2}+\text{nCr}(n,3)\cdot p^3\cdot(1-p)^{n-3}|n=17$
0.503952

$\text{nCr}(n,2)\cdot p^2\cdot(1-p)^{n-2}+\text{nCr}(n,3)\cdot p^3\cdot(1-p)^{n-3}|n=18$
0.50064

$\text{nCr}(n,2)\cdot p^2\cdot(1-p)^{n-2}+\text{nCr}(n,3)\cdot p^3\cdot(1-p)^{n-3}|n=19$
0.493672

**c.i.** $E\left(\hat{P}\right) = 0.143$

$sd\left(\hat{P}\right) = \sqrt{\dfrac{0.1429 \times (1 - 0.1429)}{49}} = 0.050$ A1

**ii.** $\Pr(0.1429-2\times 0.050 \le T \le 0.1429+2\times 0.050)$

$=\Pr(0.0429 \le T \le 0.2429)\ \times 49$

$=\Pr(2.1 \le T_b \le 11.9)$ M1

$=\Pr(3 \le T_b \le 11)$ A1

$=0.938$

| | |
|---|---|
| $49\cdot p$ | 7.0037 |
| $\sqrt{\frac{p\cdot(1-p)}{49}}$ | 0.050001 |
| $sd:=0.05000060095433$ | 0.050001 |
| $p+2\cdot sd$ | 0.242934 |
| $p-2\cdot sd$ | 0.042932 |
| $49\cdot 0.04293150008424$ | 2.10364 |
| $49\cdot 0.24293390390156$ | 11.9038 |
| binomCdf(49,p,2.1,11.9) | 0.938049 |
| binomCdf(49,p,3,11) | 0.938049 |

**d.** $\left(\hat{p}-z\sqrt{\frac{\hat{p}(1-\hat{p})}{n}}\ ,\ \hat{p}+z\sqrt{\frac{\hat{p}(1-\hat{p})}{n}}\right)=(\ 0.0523, 0.2334)$ M1

since $\hat{p}=0.1429$ , $n=49$ , $sd=0.05$ , $z=?$

$z=\frac{0.2334-0.0523}{2\times 0.050}=1.811$ , $Z\overset{d}{=}N(0,1)$

$\Pr(-1.811 \le Z \le 1.811)=0.93$ A1

93%

| | |
|---|---|
| $\frac{0.2334-0.0523}{2\cdot sd}$ | 1.81098 |
| normCdf(-1.811,1.811,0,1) | 0.929859 |
| zInterval_1Prop 7,49,0.93: *stat.results* | "Title" "1−Prop z Interval"; "CLower" 0.05228; "CUpper" 0.233434; "p̂" 0.142857; "ME" 0.090577; "n" 49. |

**e.** $a\left[\int_0^2 \sin\left(\frac{\pi t}{4}\right)dt + \int_2^4 \frac{4-t}{2}\,dt\right]=1$ since the total area is equal to one

$$a\left(\left[-\frac{4}{\pi}\cos\left(\frac{\pi t}{4}\right)\right]_0^2 + \left[\frac{1}{2}\left(4t-\frac{1}{2}t^2\right)\right]_2^4\right)=1 \qquad \text{A1}$$

$$a\left(-\frac{4}{\pi}\cos\left(\frac{\pi}{2}\right)+\frac{4}{\pi}\cos(0)+\frac{1}{2}(16-8-8+2)\right)=1$$

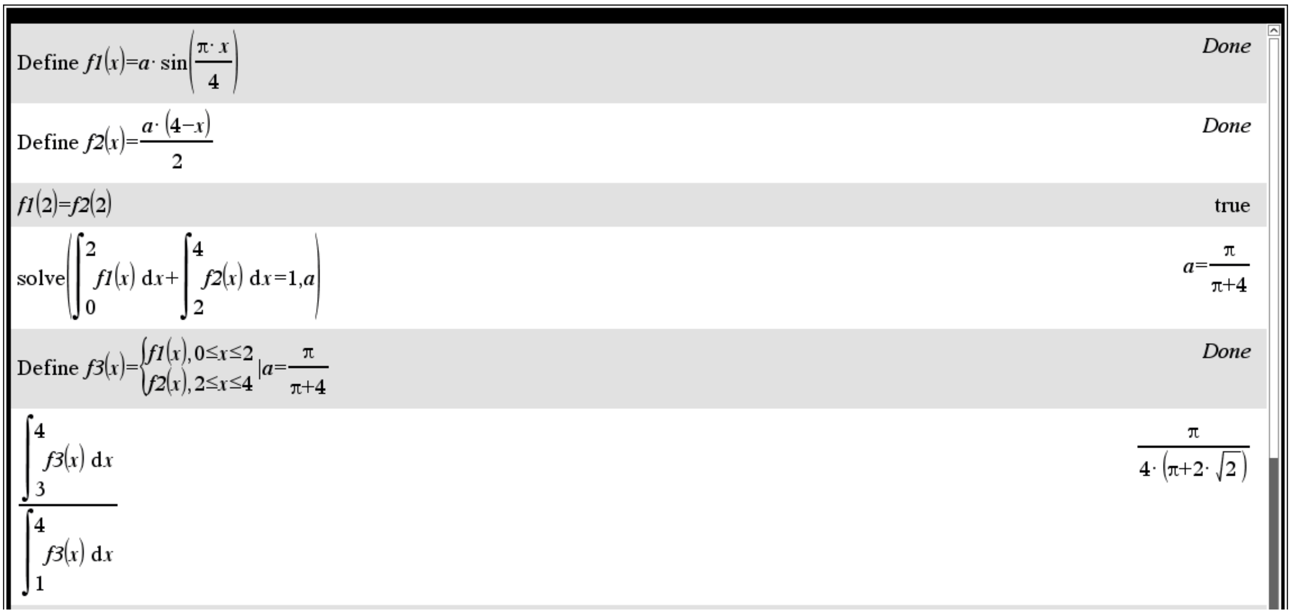
$$a\left(\frac{4}{\pi}+1\right)=a\left(\frac{4+\pi}{\pi}\right)=1 \qquad \text{M1}$$

$$a=\frac{\pi}{4+\pi}$$

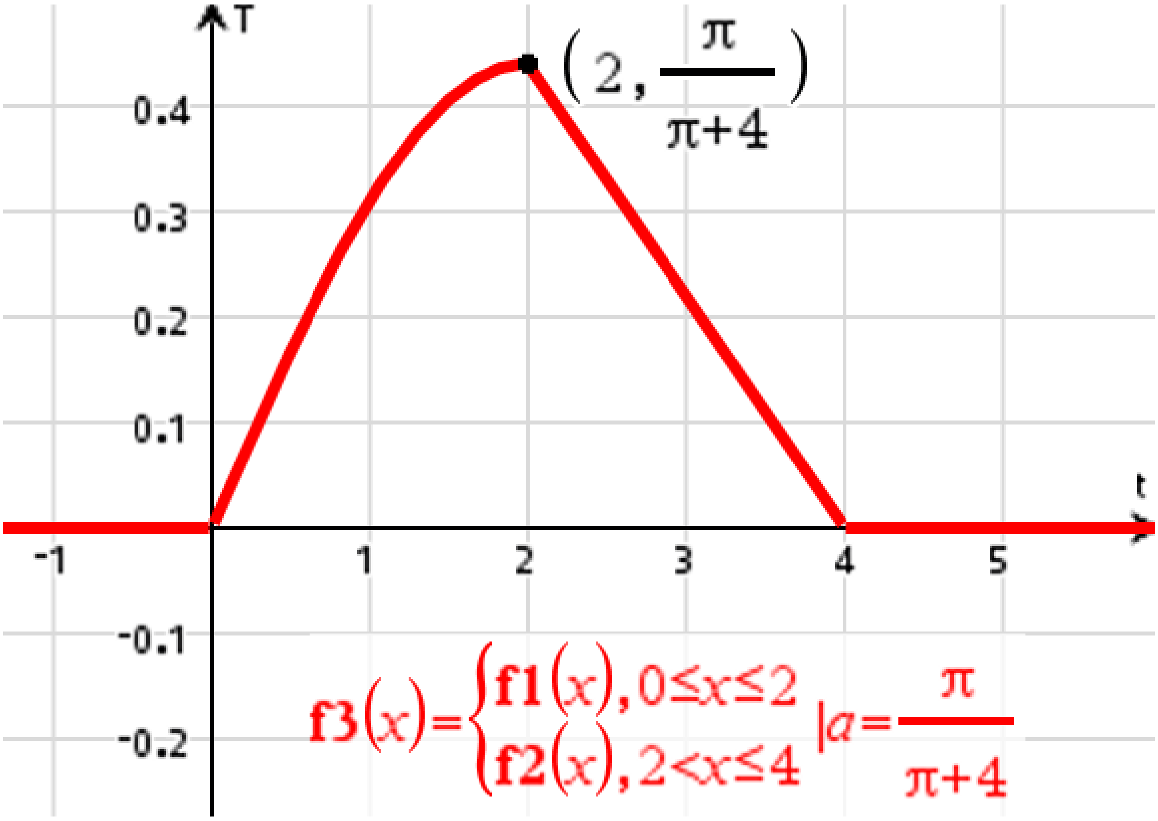
**f.** $\Pr(T>3)\,|\,\Pr(T>1)=\dfrac{\Pr(T>3)}{\Pr(T>1)}$

$$=\frac{\int_3^4 \frac{1}{2}(4-t)\,dt}{\int_1^2 \sin\left(\frac{\pi t}{4}\right)dt + \int_2^4 \frac{1}{2}(4-t)\,dt} \qquad \text{M1}$$

$$=\frac{\pi}{4\left(\pi+2\sqrt{2}\right)} \qquad \text{A1}$$



**g.** G2




**h.** $E(T)=\frac{\pi}{4+\pi}\left[\int_0^2 t\sin\left(\frac{\pi t}{4}\right)dt+\int_2^4 \frac{t(4-t)}{2}dt\right]=1.8862\,\text{hr}$ A1

$E(T^2)=\frac{\pi}{4+\pi}\left[\int_0^2 t^2\sin\left(\frac{\pi t}{4}\right)dt+\int_2^4 \frac{t^2(4-t)}{2}dt\right]=4.2625$

$\text{var}(T)=E(T^2)-(E(T))^2=4.2625-1.8862^2=0.7047\,\text{hr}^2$ A1

**i.** $\int_0^m f(t)\,dt=\frac{1}{2}$

$m=1.86$ hours or $m=111.88$ minutes

$m=112$ A1

| | |
|---|---|
| $\int_0^4 f3(x)\, dx$ | 1 |
| $\int_0^4 (x \cdot f3(x))\, dx$ | 1.88621 |
| $\int_0^4 (x^2 \cdot f3(x))\, dx$ | 4.2625 |
| $4.262504-(1.8862093)^2$ | 0.704718 |
| $\text{solve}\left(\int_0^m f3(x)\, dx=0.5, m\right)$ | $m=1.86312$ |
| $1.8631166 \cdot 60$ | 111.787 |

**Question 4**

**a.** original $y=\log_e(x)$ new $y'=\log_3(2x'-5)-5$

$y=\log_e(x)=\dfrac{\log_3(x)}{\log_3(e)}$ , $\log_3(e)y=\log_3(x)$

$y'+5=\log_3(2x'-5)$ , $y'+5=\log_3(e)y$ , $y'=\log_3(e)y-5$

$x=2x'-5$ , $x'=\dfrac{x}{2}+\dfrac{5}{2}$ M1

$\begin{bmatrix} x' \\ y' \end{bmatrix}=\begin{bmatrix} \frac{1}{2} & 0 \\ 0 & \log_3(e) \end{bmatrix}\begin{bmatrix} x \\ y \end{bmatrix}+\begin{bmatrix} \frac{5}{2} \\ -5 \end{bmatrix}$ , $a=\dfrac{1}{2}$ , $b=\log_3(e)$ , $p=\dfrac{5}{2}$ , $q=-5$ A1

**b.** At the point $(u,\log_3(2u-5)-5)$

$y=\log_3(2x-5)-5=\dfrac{\log_e(2x-5)}{\log_e(3)}-5$

$\dfrac{dy}{dx}=\dfrac{2}{(2x-5)\log_e(3)}$ at $x=u$ $m_T=\dfrac{2}{(2u-5)\log_e(3)}$

$T:\quad y-(\log_3(2u-5)-5)=\dfrac{2}{(2u-5)\log_e(3)}(x-u)$

$y=\dfrac{2x}{(2u-5)\log_e(3)}+\log_3(2u-5)-5-\dfrac{2u}{(2u-5)\log_e(3)}$ M1

$y=\dfrac{nx}{(2u-5)}+\log_3(2u-5)-5-\dfrac{nu}{2u-5}$

$n=\dfrac{2}{\log_e(3)}=2\log_3(e)=\log_3(e^2)$ , $m=e^2$ A1

**c.** $y=\log_3(2x-k)-k=\dfrac{\log_e(2x-k)}{\log_e(3)}-k$

$\dfrac{dy}{dx}=\dfrac{2}{(2x-k)\log_e(3)}$ at $x=v$

$m_T=\dfrac{2}{(2v-k)\log_e(3)}=\tan(45^0)=1$

$2v-k=\dfrac{2}{\log_e(3)}=n$

$v=\dfrac{k}{2}+\dfrac{1}{\log_e(3)}=\dfrac{k}{2}+\log_3(e)=\dfrac{1}{2}\left(k+\log_3(e^2)\right)$ A1

**d.** The tangent at $x=3$ is $y=\dfrac{2x}{\log_e(3)(6-k)}+\log_3(6-k)-k-\dfrac{6}{\log_e(3)(6-k)}$

if this passes through the origin then $\log_3(6-k)-k-\dfrac{6}{\log_e(3)(6-k)}=0$

solving this for $k$ gives $k=0.5436$, A1

the gradient of the line is then $\dfrac{2}{\log_e(3)(6-0.5436)}=0.3336=\tan(\theta)$

$\theta=\tan^{-1}(0.3336)$

$\theta=18.5^0$ A1

| | |
|---|---|
| Define $g(x)=\log_3(2\cdot x-k)-k$ | *Done* |
| tangentLine$(g(x),x,3)$ | $\dfrac{(k-6)\cdot\ln(-(k-6))-k^2\cdot\ln(3)+6\cdot k\cdot\ln(3)+6}{(k-6)\cdot\ln(3)}-\dfrac{2\cdot x}{(k-6)\cdot\ln(3)}$ |
| solve$\left(\dfrac{(k-6)\cdot\ln(-(k-6))-k^2\cdot\ln(3)+6\cdot k\cdot\ln(3)+6}{(k-6)\cdot\ln(3)}=0,k\right)$ | $k=0.5436$ |
| tangentLine$(g(x),x,3)\mid k=0.5435713$ | $0.3336\cdot x+3.4868\text{E-}8$ |
| $\tan^{-1}(0.33363919027)$ | 18.5 |

**e.** $g:\quad y=\log_3(2x-k)-k$ swap $x$ and $y$

$g^{-1}:\ x=\log_3(2y-k)-k$

$x+k=\log_3(2y-k)$

$2y-k=3^{x+k}$ M1

$y=\dfrac{1}{2}\left(3^{x+k}+k\right)$

dom $g=\left(\dfrac{k}{2},b\right)=$ ran $g^{-1}$

ran $g=\left(-\infty,\log_3(2b-k)-k\right)=$ dom $g^{-1}$

$g^{-1}:\left(-\infty,\log_3(2b-k)-k\right)\to R\ ,\ g^{-1}(x)=\dfrac{1}{2}\left(3^{x+k}+k\right)$ A1

**f.** $g^{-1}\left(\log_3(2b-k)-k\right)=\frac{1}{2}\left(3^{\log_3(2b-k)-k+k}+k\right)$

$g^{-1}\left(\log_3(2b-k)-k\right)=\frac{1}{2}\left(3^{\log_3(2b-k)}+k\right)$

$g^{-1}\left(\log_3(2b-k)-k\right)=\frac{1}{2}\left(2b-k+k\right)$

$g^{-1}\left(\log_3(2b-k)-k\right)=b$

alternatively, at the endpoint of the function

$g(b)=\log_3(2b-k)-k \quad \Leftrightarrow \quad g^{-1}\left(\log_3(2b-k)-k\right)=b$ A1

**g.** $y=\frac{1}{2}\left(3^{x+k}+k\right)=\frac{1}{2}\left(3^k\times 3^x+k\right)$ at $x=1$ $\left(1,\frac{1}{2}\left(3^{k+1}+k\right)\right)$

Now $\frac{d}{dx}\left(3^x\right)=\log_e(3)\times 3^x$

$\frac{dy}{dx}=\frac{3^k}{2}\log_e(3)\times 3^x$ at $x=1$ $m_T=\frac{3^{k+1}}{2}\log_e(3)$

$T:\ y-\left(\frac{1}{2}\left(3^{k+1}+k\right)\right)=\frac{3^{k+1}}{2}\log_e(3)(x-1)$ M1

$y=\frac{3^{k+1}}{2}\log_e(3)x+\frac{1}{2}\left(3^{k+1}+k\right)-\frac{3^{k+1}}{2}\log_e(3)$

if this tangent passes through the origin then $\frac{1}{2}\left(3^{k+1}+k\right)-\frac{3^{k+1}}{2}\log_e(3)=0$

solving this for $k$ gives $k=0.529$ , $1.4415$ A1

the gradient of the line is $m=\frac{3^{k+1}}{2}\log_e(3)$

when $k=0.529$ , $m=\frac{3^{0.529+1}}{2}\log_e(3)=2.9466$ , $\theta=\tan^{-1}(2.9466)$

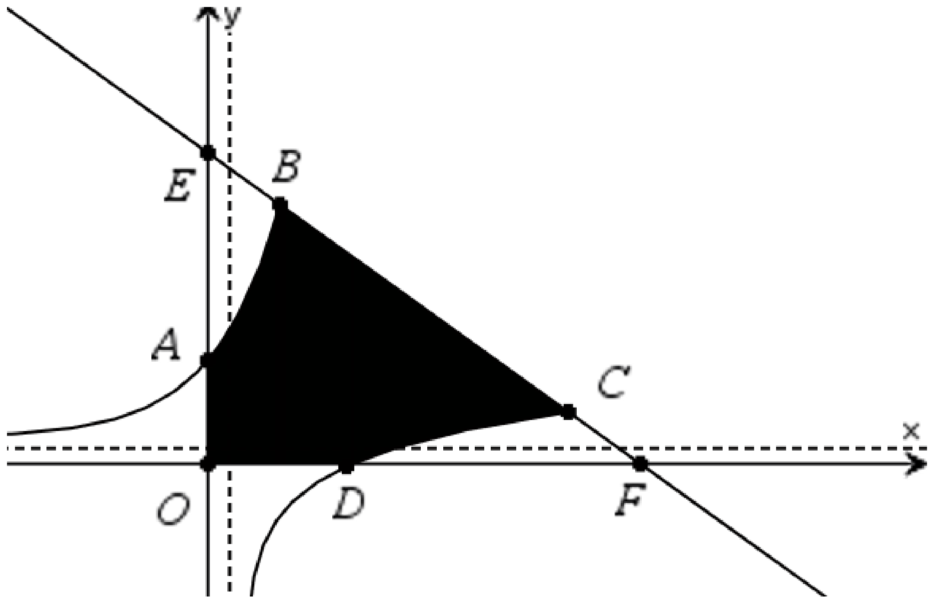
when $k=1.4415$ , $m=\frac{3^{1.4415+1}}{2}\log_e(3)=8.029$ , $\theta=\tan^{-1}(8.029)$

$\theta=71.3^0$ , $82.9^0$ A1

both answers are acceptable.

| | |
|---|---|
| Define $gi(x)=\frac{1}{2}\cdot\left(3^{x+k}+k\right)$ | *Done* |
| tangentLine$(gi(x),x,1)$ | $\frac{3\cdot 3^{k}\cdot \ln(3)\cdot x}{2}-\frac{3\cdot 3^{k}\cdot(\ln(3)-1)-k}{2}$ |
| ⚠ solve$\left(\frac{3\cdot 3^{k}\cdot(\ln(3)-1)-k}{2}=0,k\right)$ | $k=0.5290$ or $k=1.4415$ |
| $\frac{3\cdot 3^{k}\cdot \ln(3)}{2}\mid k=0.528981867$ | 2.9466 |
| $\tan^{-1}(2.94662)$ | 71.3 |
| $\frac{3\cdot 3^{k}\cdot \ln(3)}{2}\mid k=1.441454872$ | 8.0294 |
| $\tan^{-1}(8.02942542)$ | 82.9 |

**h.**




The graph of the function $g(x)=\log_3(2x-k)-k$ has a vertical asymptote at $x=\frac{k}{2}$

it crosses the $x$-axis when $\log_3(2x-k)-k=0$ , $2x-k=3^k$ , $x=\frac{1}{2}\left(3^k+k\right)$

at the point $D\left(\frac{1}{2}\left(3^k+k\right),0\right)$ and has its endpoint at the point, $C\left(b,g(b)\right)$

$C\left(b,\log_e(2b-k)-k\right)$ , so we require $b>\frac{1}{2}\left(3^k+k\right)$

The graph of the function $g^{-1}(x)=\frac{1}{2}\left(3^{x+k}+k\right)$ has a horizontal asymptote at $y=\frac{k}{2}$, it crosses the *y*-axis at the point $A\left(0\ ,\ \frac{1}{2}\left(3^{k}+k\right)\right)$ and has its endpoint at the point $B\left(\log_3(2b-k)-k\ ,b\right)$. A1

Both graphs are symmetrical about the line $y=x$, which has a gradient of 1, the line perpendicular to this line has a gradient of $-1$ and has an equation of the form $y=w-x$, now since it must pass through the endpoints of both functions, that is points *C* and *B*, $b=w-\left(\log_3(2b-k)-k\right)$ so that $w=c=b-k+\log_3(2b-k)$.

The line passing $y=b-k+\log_3(2b-k)-x=c-x$ crosses the *x*-axis at *F* and crosses the *y*-axis at *E*, $E(0,c)$ , $F(c,0)$ , $c>b>\frac{1}{2}\left(3^{k}+k\right)$. A1

**i.** The two non-shaded areas bounded by the coordinates axes are equal, the shaded area is therefore, the area of the triangle *OEF*, minus twice the area bounded by the graph of $y=g(x)$ between the point *D* and the line $x=b$, ( the line through $C$ ) and twice the area bounded by the graph of $y=c-x$ between the line at $x=b$ and the point *F*.

Let $A_1=\int_{\frac{1}{2}\left(3^{k}+k\right)}^{b} g(x)\,dx$ and $A_2=\int_b^c (c-x)\,dx$.

Now $A_2=\int_b^c (c-x)\,dx=\left[cx-\frac{x^2}{2}\right]_b^c=\left(c^2-\frac{c^2}{2}\right)-\left(bc-\frac{b^2}{2}\right)=\frac{1}{2}\left(c^2+b^2\right)-bc$ A1

The area of the triangle *OEF* is $\frac{1}{2}c^2$, so the shaded area is

$$A=\frac{1}{2}c^2-2\left(A_1+A_2\right)=\frac{1}{2}c^2-2A_1-2A_2$$

$$A=\frac{1}{2}c^2-2\int_{\frac{1}{2}\left(3^{k}+k\right)}^{b} g(x)\,dx-2\left(\frac{1}{2}\left(c^2+b^2\right)-bc\right)=\frac{c^2}{2}-\left(c^2+b^2\right)+2bc-2\int_{\frac{1}{2}\left(3^{k}+k\right)}^{b} g(x)\,dx$$

$$A=2bc-b^2-\frac{c^2}{2}-2\int_{\frac{1}{2}\left(3^{k}+k\right)}^{b} g(x)\,dx \quad , \qquad R=2bc-b^2-\frac{c^2}{2}$$ A1

**END OF SECTION B SUGGESTED ANSWERS**

**End of detailed answers for the**
**2021 Kilbaha VCE Mathematical Methods Trial Examination 2**

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