2021 Mathematical Methods Trial Exam 2 Solutions

Use CAS whenever practical

SECTION A – Multiple-choice questions

1	2	3	4	5	6	7	8	9	10
E	A	C	A	C	A	B	D	D	C

11	12	13	14	15	16	17	18	19	20
A	B	D	E	B	E	D	C	D	B

Q1 Horizontal dilation factor $\sqrt{2}$, vertical dilation factor 2

$(a, b) \to \left(\sqrt{2}a, 2b\right)$ E

Q2 $g'(x) = 2(f(x))f'(x)$, $f(x)$ is an odd function $\therefore f(0) = 0$

$\therefore g'(0) = 2(f(0))f'(0) = 0$ A

Q3 $\alpha = ba$, $\beta = b^2$ $\therefore \beta - \alpha = b^2 - ab$ C

Q4 $y = a^{\log_b x}$, $\log_a y = \log_b x$, $\dfrac{\log_b y}{\log_b a} = \log_b x$,

$\log_b y = \log_b x^{\log_b a}$, $y = x^{\log_b a}$ is a power function

$\log_a (y - x) = \log_b (b - a)$, $y - x = a^{\log_b (b-a)}$ is a linear function if $b > a$, and is undefined if $b \le a$. A

Q5 The two tangents are inverse of each other. They intersect on the line $y = x$, $\therefore \beta = \alpha$ C

Q6 The average value of $f(x)$ equals the average rate of change of $f(x)$ with respect to x over the interval $[1, 3]$.

$\therefore \dfrac{a+b}{2} = \dfrac{b-a}{3-1}$ $\therefore a = 0$ A

Q7 $b > a$, $-2b < -2a$

$\dfrac{1}{2} f(-x)$ is the horizontal dilation by factor 2 and vertical dilation by factor $\dfrac{1}{2}$ of $f(2x)$.

$\therefore \int_{-2b}^{-2a} \dfrac{1}{2} f(-x)\, dx = c$ $\therefore \int_{-2a}^{-2b} \dfrac{1}{2} f(-x)\, dx = -c$ B

Q8 Range of $g \subseteq$ domain of f $\therefore$ range of $g \subseteq \left(\dfrac{\pi}{4}, \dfrac{\pi}{2}\right)$

$\therefore$ range of g is $\left(\dfrac{\pi}{4}, \dfrac{\pi}{2}\right)$ $\therefore$ domain of g is $\left(\dfrac{1}{2}, \dfrac{1}{\sqrt{2}}\right)$

$\therefore$ domain of $f(g(x))$ is $\left(\dfrac{1}{2}, \dfrac{1}{\sqrt{2}}\right)$ and range of $f(g(x))$ is $(-1, 0)$ D

Q9 $f(x) = a(x-1)(x+1) + 3$, $\dfrac{f(x)}{x^2 - 1} = a + \dfrac{3}{x^2 - 1}$, the remainder is 3 D

Q10 $\sqrt{(a\sin(nx))^2 + (b\cos(nx))^2} = \sqrt{a^2 - a^2\cos^2(nx) + b^2\cos^2(nx)}$

$= \sqrt{a^2 + \left(b^2 - a^2\right)\cos^2(nx)}$

Min is a when $\cos^2(nx) = 0$, max is b when $\cos^2(nx) = 1$ C

Q11 $f'(x) = 3ax^2 + 2bx + c \ne 0$, $(2b)^2 - 4(3a)c < 0$, $b^2 < 3ac$ A

Q12 $T_3T_2T_1\begin{bmatrix} x \\ y \end{bmatrix} = T_3T_2\begin{bmatrix} qx \\ by \end{bmatrix} = T_3\begin{bmatrix} ax-c \\ by-d \end{bmatrix} = \begin{bmatrix} ax-c \\ -\frac{1}{2}(by-d) \end{bmatrix} = \begin{bmatrix} ax-c \\ \frac{1}{2}(d-by) \end{bmatrix}$ B

Q13 $y = \log_e ax$ and $y = e^{x-b}$ intersect at $y = x$, and $y = x$ is a common tangent to the curves $\therefore \dfrac{dy}{dx} = 1$ at the intersection.

$\dfrac{dy}{dx} = \dfrac{1}{x} = 1$ and $\dfrac{dy}{dx} = e^{x-b} = 1$ $\therefore x = 1$ and $1 - b = 0$, $b = 1$

Intersection is $(1, 1)$ $\therefore \log_e a = 1$ $\therefore a = e$ D

Q14 Find the point where $\dfrac{dy}{dx} = m$ for each curve.

For $y = \dfrac{3}{4}x^2$, $\dfrac{dy}{dx} = \dfrac{3}{2}x = m$; for $y = \log_e x$, $\dfrac{dy}{dx} = \dfrac{1}{x} = m$

$\left(\dfrac{2m}{3}, \dfrac{m^2}{3}\right)$, $\left(\dfrac{1}{m}, \log_e \dfrac{1}{m}\right)$

Common normal at both points: $\dfrac{\frac{m^2}{3} - \log_e \frac{1}{m}}{\frac{2m}{3} - \frac{1}{m}} = -\dfrac{1}{m}$, $m = 1$

$\left(\dfrac{2}{3}, \dfrac{1}{3}\right)$, $(1, 0)$ $\therefore$ distance$^2 = \left(1 - \dfrac{2}{3}\right)^2 + \left(0 - \dfrac{1}{3}\right)^2 = \dfrac{2}{9}$ E

Q15 $\int_0^{\pi} (a\cos 2x + b)\,dx = 1$ $\therefore b = \dfrac{1}{\pi}$

$a\cos 2x + \dfrac{1}{\pi} \ge 0$ $\therefore -\dfrac{1}{\pi} \le a \le \dfrac{1}{\pi}$ B

Q16 Sum of the number of dots on opposite faces = 7

$\Pr(33) = \Pr(5 \text{ dots on the uppermost face}) = \dfrac{1}{6}$ E

Q17 A, B, C and E are events.
The difference is a numerical variable and its value depends on the outcome of tossing 8 coins. D

Q18 $\mu = np = a$, $\sigma = \sqrt{np(1-p)} = \sqrt{b}$, $np(1-p) = b$, $a - ap = b$

$\therefore a > b$ C

Q19 $p + 1.5p^2 + p^3 + 0.25p^4 = 1$ $\therefore 4p + 6p^2 + 4p^3 + p^4 = 4$

$1 + 4p + 6p^2 + 4p^3 + p^4 = 5$, $(1+p)^4 = 5$, $p = \sqrt[4]{5} - 1 \approx 0.49535$

$\text{E}(X) = p(1) + 1.5p^2(2) + p^3(3) + 0.25p^4(4) \approx 1.656$ D

Q20 $n = 1000$, $\hat{p} = 0.60$, $\text{sd}(\hat{p}) = \sqrt{\dfrac{0.60 \times 0.40}{1000}} \approx 0.0155$

invnorm$(0.8) \approx 0.842$

$0.60 \pm 0.842 \times 0.0155 \approx 0.587$ or 0.613 B

SECTION B

Q1a $OK=OP+r=1$, $\sqrt{1-\left(\frac{a}{2}\right)^2}+r+r=1$ ∴ $r=\frac{1}{2}\left(1-\frac{\sqrt{4-a^2}}{2}\right)$

Q1b $OJ=OG+R=1$, $\sqrt{x^2+R^2}+R=1$ ∴ $x=\sqrt{1-2R}$

Q1c ΔEGF and ΔACB are similar triangles ∴ $\frac{y}{R}=\frac{2}{\sqrt{4-a^2}}$

∴ $y=\frac{2R}{\sqrt{4-a^2}}$

Q1d ΔADE and ΔACB are similar triangles ∴ $\frac{z}{R}=\frac{a}{\sqrt{4-a^2}}$

∴ $z=\frac{aR}{\sqrt{4-a^2}}$

Q1e $1+x=y+z$ ∴ $x=y+z-1$

Q1f $\sqrt{1-2R}=\frac{aR}{\sqrt{4-a^2}}+\frac{2R}{\sqrt{4-a^2}}-1$, $\sqrt{1-2R}=\frac{(2+a)R}{\sqrt{4-a^2}}-1$,

$\sqrt{1-2R}=R\sqrt{\frac{2+a}{2-a}}-1$.

By squaring both sides: $1-2R=R^2\frac{2+a}{2-a}-2R\sqrt{\frac{2+a}{2-a}}+1$

∴ $R^2\frac{2+a}{2-a}+2R\left(1-\sqrt{\frac{2+a}{2-a}}\right)=0$, $R\frac{2+a}{2-a}+2\left(1-\sqrt{\frac{2+a}{2-a}}\right)=0$

∴ $R=2\left(\frac{2-a}{2+a}\right)\left(\sqrt{\frac{2+a}{2-a}}-1\right)$ ∴ $R=2\left(\sqrt{\frac{2-a}{2+a}}-\frac{2-a}{2+a}\right)$

Q1g R is a maximum when $a=1.2$ ∴ $r=\frac{1}{2}\left(1-\frac{\sqrt{4-a^2}}{2}\right)=\frac{1}{10}$

Q1h $A=\pi\left(r^2+R^2\right)=\frac{\pi}{4}\left(\left(1-\frac{\sqrt{4-a^2}}{2}\right)^2+16\left(\sqrt{\frac{2-a}{2+a}}-\frac{2-a}{2+a}\right)^2\right)$

Q1i $A_{\max}\approx 0.821$ when $a\approx 1.277$

Q1j $A_{\min}\approx 0.384$ when $a=\frac{1}{2}$

Q2a Period $T=\frac{2\pi}{\frac{\pi}{2}}=4$

Q2b When $A(x)$ changes from $A(x)=1$ to $A(x)=1-\frac{x}{2}$, $f(x)$ changes from $f(x)=\cos\left(\frac{\pi x}{2}\right)$ to $f(x)=\left(1-\frac{x}{2}\right)\cos\left(\frac{\pi x}{2}\right)$, i.e. from an even function to neither even nor odd.

The amplitude increases as the value of x increases in both directions. The change has no effects on the period.

Q2c $g(x)=f(x-b)=\left(1-\frac{x-b}{2}\right)\cos\left(\frac{\pi(x-b)}{2}\right)$ is an odd function,

i.e. $g(-x)=-g(x)$.

∴ $\left(1-\frac{-x-b}{2}\right)\cos\left(\frac{\pi(-x-b)}{2}\right)=-\left(1-\frac{x-b}{2}\right)\cos\left(\frac{\pi(x-b)}{2}\right)$

$\left(1+\frac{x+b}{2}\right)\cos\left(\frac{\pi(x+b)}{2}\right)=-\left(1-\frac{x-b}{2}\right)\cos\left(\frac{\pi(x-b)}{2}\right)$

∴ At $x=0$, $g(0)=0$, ∴ $b=-2$

∴ $g(x)=-\frac{x}{2}\cos\left(\frac{\pi x}{2}+\pi\right)=\frac{x}{2}\cos\left(\frac{\pi x}{2}\right)$

Q2d $A(x)=\frac{x}{2}$, $\frac{d}{dx}A(x)=\frac{1}{2}$

Q2ei Let $\frac{x}{2}\cos\left(\frac{\pi x}{2}\right)=\frac{x}{2}$, $\frac{x}{2}\cos\left(\frac{\pi x}{2}\right)-\frac{x}{2}=0$, $\frac{x}{2}\left(\cos\left(\frac{\pi x}{2}\right)-1\right)=0$

∴ $x=0$ or $\cos\left(\frac{\pi x}{2}\right)-1=0$, $\frac{\pi x}{2}=2n\pi$, $x=4n$ and $y=2n$.

General solution $(4n, 2n)$ where $n\in$ set of integers.

Q2eii $g'(x(n))=\frac{1}{2}$

$g(x)=\frac{x}{2}\cos\left(\frac{\pi x}{2}\right)$, $g'(x)=\frac{1}{2}\cos\left(\frac{\pi x}{2}\right)-\frac{x}{2}.\frac{\pi}{2}\sin\left(\frac{\pi x}{2}\right)$ ∴ $g'(4n)=\frac{1}{2}$

Q2f $\int_0^{4n}\frac{x}{2}\cos\left(\frac{\pi x}{2}\right)dx=0$ ∴ the regions above and below the *x*-axis have the same area.

Q2gi $\int_0^{4n}\left(\frac{x}{2}-\frac{x}{2}\cos\left(\frac{\pi x}{2}\right)\right)dx$

Q2gii The definite integral has a value equal to the area of the triangle bounded by $y=\frac{x}{2}$, $x=4n$ and the *x*-axis, i.e. $\frac{1}{2}\times 4n\times 2n=4n^2$

Q3a Radius r of water surface: $\frac{r}{1-h}=\frac{1}{1}$, $r=1-h$

$V(h)=\frac{1}{3}\pi 1^2 1-\frac{1}{3}\pi(1-h)^2(1-h)=\frac{\pi}{3}\left(1-(1-h)^3\right)=\pi h\left(1-h+\frac{h^2}{3}\right)$

Q3b $h_{\text{full}}=1-0.25=\frac{3}{4}$, $V_{av}=\frac{\int_0^{\frac{3}{4}}\pi h\left(1-h+\frac{h^2}{3}\right)dh}{\frac{3}{4}-0}=\frac{171\pi}{1024}$

Q3c $\left(\frac{dV}{dh}\right)_{av}=\frac{V\left(\frac{3}{4}\right)-V(0)}{\frac{3}{4}-0}=\frac{7\pi}{16}$

Q3d $\frac{dV}{dh}=\pi(1-h)^2$, $\frac{dV}{dt}=\frac{dV}{dh}\times\frac{dh}{dt}$, $\frac{\pi}{3600}=\pi(1-h)^2\times\frac{dh}{dt}$,

$\frac{dh}{dt}=\frac{1}{3600(1-h)^2}$

itute.com

Q3e When $h=\frac{1}{2}\times\frac{3}{4}=\frac{3}{8}$, $V\left(\frac{3}{8}\right)=\pi\frac{3}{8}\left(1-\frac{3}{8}+\frac{\left(\frac{3}{8}\right)^2}{3}\right)\approx 0.2520\pi$,

time required $\approx\frac{0.2520\pi}{\frac{\pi}{3600}}\approx 907$ s

Q3f Time required to fill the tank $=\frac{V\left(\frac{3}{4}\right)}{\frac{\pi}{3600}}=\frac{\frac{21\pi}{64}}{\frac{\pi}{3600}}=\frac{4725}{4}$,

$\left(\frac{dh}{dt}\right)_{av}=\frac{h_{\text{full}}-0}{t_{\text{full}}-0}=\frac{\frac{3}{4}}{\frac{4725}{4}}=\frac{1}{1575}$ m/s

Q3g $\frac{dV}{dt}=\frac{dV}{dh}\times\frac{dh}{dt}$, $\frac{dV}{dt}=\pi(1-h)^2\times\frac{1}{1000\pi}=\frac{(1-h)^2}{1000}$

Q3h Time required $=\frac{\frac{3}{4}}{\frac{1}{1000\pi}}=750\pi$ s

Q3i $V=\pi h\left(1-h+\frac{h^2}{3}\right)=\frac{1}{2}\times\frac{21\pi}{64}$, $h\approx 0.2022$,

time required $\approx\frac{0.2022}{\frac{1}{1000\pi}}\approx 635$ s

Q3j $\left(\frac{dV}{dt}\right)_{av}=\frac{V\left(\frac{3}{4}\right)}{750\pi}=\frac{\frac{21\pi}{64}}{750\pi}=\frac{7}{16000}$ m^3/s

Q4a $g'(3)=f'(3)$ $\therefore\ \frac{a}{2\sqrt{3-b}}=1$ $\therefore\ \sqrt{3-b}=\frac{a}{2}$

$g(3)=a\sqrt{3-b}+c=1$ $\therefore\ c<1$ and $a\times\frac{a}{2}+c=1$ $\therefore\ a=\sqrt{2(1-c)}$

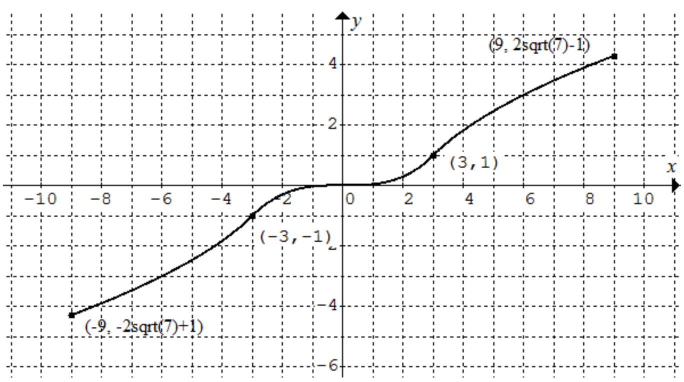
Q4b $\sqrt{3-b}=\frac{1-c}{a}$, $3-b=\frac{(1-c)^2}{a^2}$, $b=3-\frac{(1-c)^2}{a^2}$

Q4c $c=-1$, $a=\sqrt{2(1-c)}=2$, $b=3-\frac{(1-c)^2}{a^2}=2$

$\therefore\ g(x)=2\sqrt{x-2}-1$

Q4d $h(x)=\begin{cases}-2\sqrt{-x-2}+1 & -9\le x<-3\\ \frac{x^3}{27} & -3\le x\le 3\\ 2\sqrt{x-2}-1 & 3<x\le 9\end{cases}$

Q4e



Q4f $f(x)=\frac{x^3}{27}$, $f'(x)=\frac{x^2}{9}=m$, $x=-3\sqrt{m}$, $y=-m\sqrt{m}$,

$g(x)=2\sqrt{x-2}-1$, $g'(x)=\frac{1}{\sqrt{x-2}}=m$, $x=\frac{1}{m^2}+2$, $y=\frac{2}{m}-1$,

common tangent: $\frac{\frac{2}{m}-1+m\sqrt{m}}{\frac{1}{m^2}+2+3\sqrt{m}}=m$

$\therefore\ \frac{2}{m}-1+m\sqrt{m}=\frac{1}{m}+2m+3m\sqrt{m}$

$\therefore\ 2m\sqrt{m}+2m+1-\frac{1}{m}=0$, $m\approx 0.4196$

Q5a $n=2\,000\,000$, $p=0.75$, $\mu=np=1500000$,

$\sigma=\sqrt{np(1-p)}\approx 612.3724$

Normal approximation : $\Pr(N>1501000)\approx 0.05$

Q5b $X\sim Bin(100, 0.75)$, $\Pr(78\le X\le 86)\approx 0.2839$

Number of samples $\approx 25\times 0.2839\approx 7$

Q5c $\mathrm{E}(\hat{P})=p=0.75$, $\mathrm{sd}(\hat{P})=\sqrt{\frac{p(1-p)}{n}}=\sqrt{\frac{0.75\times 0.25}{100}}\approx 0.0433$,

$\Pr(0.78\le\hat{P}\le 0.86)\approx 0.2387$

Number of samples $\approx 25\times 0.2387\approx 6$

The difference arises due to small sample size and skewed distribution of $\hat{P}$ (p not close to 0.5).

Q5d

	w	w'	
c	0.75	$1.125x$	$0.75+1.125x$
c'	x	0.05	$x+0.05$
	$0.75+x$	$1.125x+0.05$	1

$0.75+1.125x+x+0.05=1$, $x=\frac{8}{85}$, $\Pr(c\cap w')=1.125\times\frac{8}{85}=\frac{9}{85}$

Q5e $\Pr(c'\cup w')=1-\Pr(c\cap w)=1-0.75=0.25$

Q5fi $\Pr(c\mid w)=\frac{\Pr(c\cap w)}{\Pr(w)}=\frac{\frac{3}{4}}{\frac{3}{4}+\frac{8}{85}}=\frac{255}{287}$

Q5fii $\Pr(c)=0.75+1.125\times\frac{8}{85}=\frac{291}{340}\ne\Pr(c\mid w)$ $\therefore$ not independent

Please inform admin@itute.com re conceptual and/or mathematical errors