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YEAR 12 *Trial Exam Paper*

2021

MATHEMATICAL METHODS

Written examination 2

Reading time: 15 minutes
Writing time: 2 hours

STUDENT NAME:

QUESTION AND ANSWER BOOK

Structure of book

Section	*Number of questions*	*Number of questions to be answered*	*Number of marks*
A	20	20	20
B	5	5	60
			Total 80

- Students are permitted to bring into the examination room: pens, pencils, highlighters, erasers, sharpeners, rulers, a protractor, set squares, aids for curve sketching, one bound reference, one approved technology (calculator or software) and, if desired, one scientific calculator. Calculator memory DOES NOT need to be cleared. For approved computer-based CAS, full functionality may be used.
- Students are NOT permitted to bring into the examination room: blank sheets of paper and/or correction fluid/tape.

Materials supplied

- Question and answer book of 31 pages
- Formula sheet
- Answer sheet for multiple-choice questions

Instructions

- Write your **name** in the box provided above on this page, and on your answer sheet for multiple-choice questions.
- Unless otherwise indicated, the diagrams in this book are **not** drawn to scale.
- All written responses must be in English.

At the end of the examination

- Place the answer sheet for multiple-choice questions inside the front cover of this book.
- You may keep the formula sheet.

Students are NOT permitted to bring mobile phones and/or any other unauthorised electronic devices into the examination room.

SECTION A – Multiple-choice questions

Instructions for Section A

Answer **all** questions in pencil on the answer sheet provided for multiple-choice questions.

Choose the response that is **correct** for the question.

A correct answer scores 1; an incorrect answer scores 0.

Marks will **not** be deducted for incorrect answers.

No marks will be given if more than one answer is completed for any question.

Unless otherwise indicated, the diagrams in this book are **not** drawn to scale.

Question 1

The range of the function $f:(-3,2]\rightarrow R,\ f(x)=-x^2+2x+2$ is

A. $(-13, 3]$

B. $(-13, 2]$

C. $(-13, 3)$

D. $(-13, 1]$

E. $(-13, 1)$

Question 2

The graph of the function $f:D\rightarrow R,\ f(x)=\dfrac{2x+1}{3-x}$, where D is the maximal domain, has asymptotes

A. $x=-3,\ y=\dfrac{2}{3}$

B. $x=3,\ y=2$

C. $x=-\dfrac{1}{2},\ y=-2$

D. $x=3,\ y=-2$

E. $x=-3,\ y=2$

Question 3

The solutions of the equation $2\sin\left(3x-\frac{\pi}{3}\right)-\sqrt{3}=0$ are

A. $x=\frac{\pi(-1+12k)}{18}$, for $k \in Z$

B. $x=\frac{2\pi(1+3k)}{9}$ or $x=\frac{\pi(1+2k)}{3}$, for $k \in Z$

C. $x=\frac{\pi(-5+12k)}{18}$ or $x=\frac{\pi(1+4k)}{6}$, for $k \in Z$

D. $x=\frac{\pi(-1+4k)}{6}$ or $x=\frac{\pi(1+12k)}{18}$, for $k \in Z$

E. $x=\frac{\pi(-1+6k)}{9}$ or $x=\frac{2\pi k}{3}$, for $k \in Z$

Question 4

Let $p(x)=2x^3+x^2-x+1$.

The remainder when p is divided by $2x + 3$ is

A. $\frac{41}{27}$

B. 2

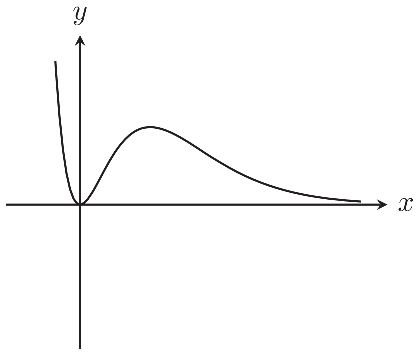
C. 1

D. $\frac{17}{2}$

E. –2

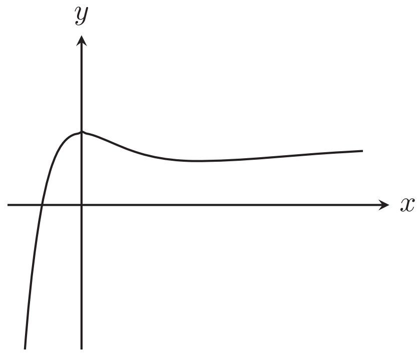
Question 5

Part of the graph of $y = f(x)$ is shown below.



An antiderivative of f is best represented by

A.



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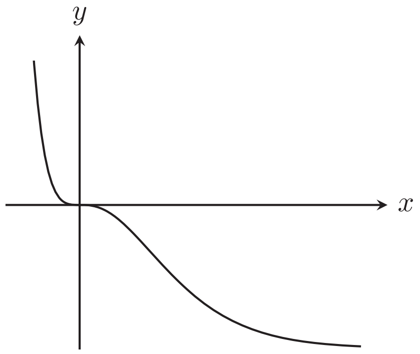
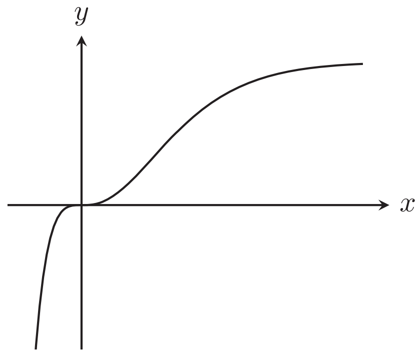
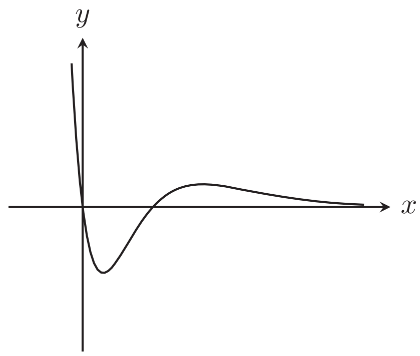


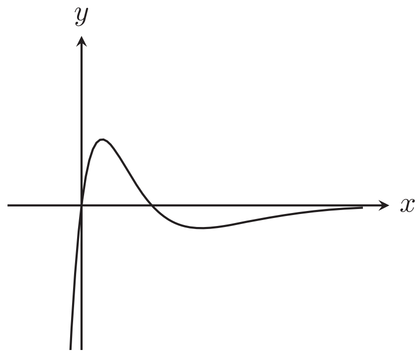
C.



D.



E.



Question 6

Consider the transformation T defined as

$$T: R^2 \to R^2, T\left(\begin{bmatrix} x \\ y \end{bmatrix}\right) = \begin{bmatrix} -2 & 0 \\ 0 & 3 \end{bmatrix}\begin{bmatrix} x \\ y \end{bmatrix} + \begin{bmatrix} 0 \\ 1 \end{bmatrix}$$

T maps the graph of $y = f(x)$ onto the graph of $y = g(x)$.

If $\int_1^3 f(x)dx = 6$, then $\int_{-6}^{-2} g(x)dx$ is equal to

A. 21

B. 30

C. 32

D. 40

E. 45

Question 7

A binomial random variable X has $E(X) = 15$ and $sd(X) = \sqrt{6}$.

$\Pr(X \geq 15)$ is closest to

A. 0.5858

B. 0.6101

C. 0.6343

D. 0.6721

E. 0.7895

Question 8

The random variable X is normally distributed with a mean of 75 and a standard deviation of 4.

Z is the standard normal random variable.

If $\Pr(X > c) = \Pr(Z < -1.5)$, then the value of c is equal to

A. 81

B. 77

C. 85

D. 73

E. 89

Question 9

The discrete random variable X has the following probability distribution.

x	0	1	2	3	4
$\Pr(X = x)$	a	$2a$	$\frac{2}{5}a$	$\frac{3}{5} - a$	$4a$

The value of $E(2X + 1)$ is

A. $\frac{263}{20}$

B. $\frac{263}{40}$

C. $\frac{263}{80}$

D. $\frac{223}{40}$

E. $\frac{223}{80}$

Question 10

The function f is a probability density function with rule

$$f(x) = \begin{cases} \frac{ax(5-x)}{50} e^{-\frac{x}{10}} & 0 \le x \le 5 \\ 0 & \text{elsewhere} \end{cases}$$

The value of a is

A. $\frac{50}{\sqrt{e}} - 30$

B. $\frac{\sqrt{e}}{30\sqrt{e} - 50}$

C. $\frac{\sqrt{e}}{50\sqrt{e} - 30}$

D. $\frac{\sqrt{e}}{50} - \frac{1}{30}$

E. $\frac{\sqrt{e}}{50 - 30\sqrt{e}}$

Question 11

$e^{2x} + ae^{x} = \frac{a}{2}$, where $a \in R$, has two real solutions if

A. $a \in (-\infty, 0)$

B. $a \in (-\infty, 0) \cup (2, \infty)$

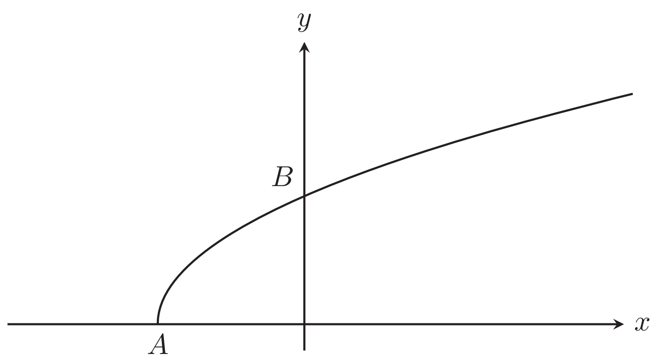
C. $a \in (-\infty, -2)$

D. $a \in (-\infty, -2) \cup (0, \infty)$

E. $a \in R$

Question 12

Part of the graph of $y = f(x) = \sqrt{x+b}$, $b > 0$, is shown below. The x- and y-intercepts are denoted by A and B respectively.



The graph of $y = f(x)$ has a tangent parallel to the line joining A and B when

A. $x = \frac{5b}{4}$

B. $x = \frac{1}{4b} - b$

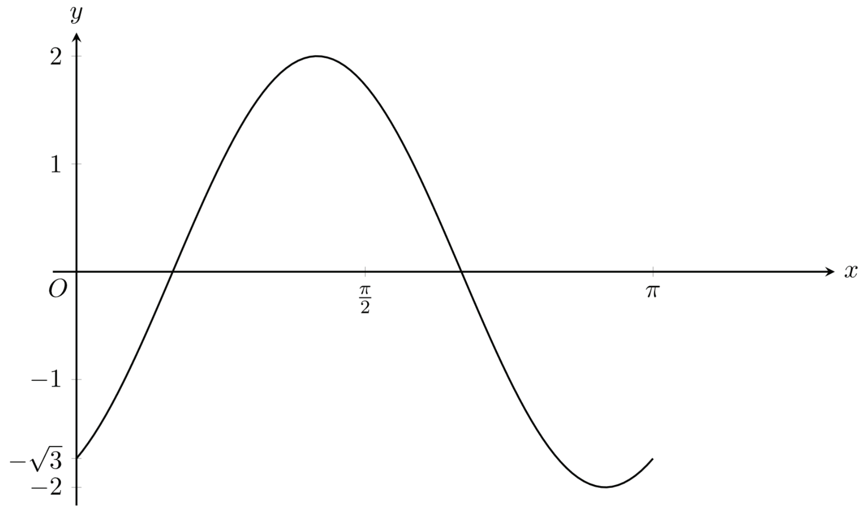
C. $x = -\frac{b}{2}$

D. $x = -\frac{3b}{4}$

E. $x = -\frac{b}{4}$

Question 13

One complete cycle of the graph of $y = f(x) = a\cos(bx + c)$ is shown below.



The values of a, b and c could be

A. $a = 2,\ b = \frac{1}{2},\ c = -\frac{5\pi}{6}$

B. $a = 2,\ b = 2,\ c = -\frac{5\pi}{6}$

C. $a = -2,\ b = 2,\ c = -\frac{\pi}{6}$

D. $a = -2,\ b = \frac{1}{2},\ c = \frac{\pi}{6}$

E. $a = -2,\ b = \frac{1}{2},\ c = \frac{5\pi}{6}$

Question 14

Let $g(3) = 2$, $g'(3) = -1$, $h(3) = 5$ and $h'(3) = 2$.

If $f(x) = \frac{xg(x)}{h(x)}$, then the value of $f'(3)$ is

A. $-\frac{9}{25}$

B. $-\frac{17}{25}$

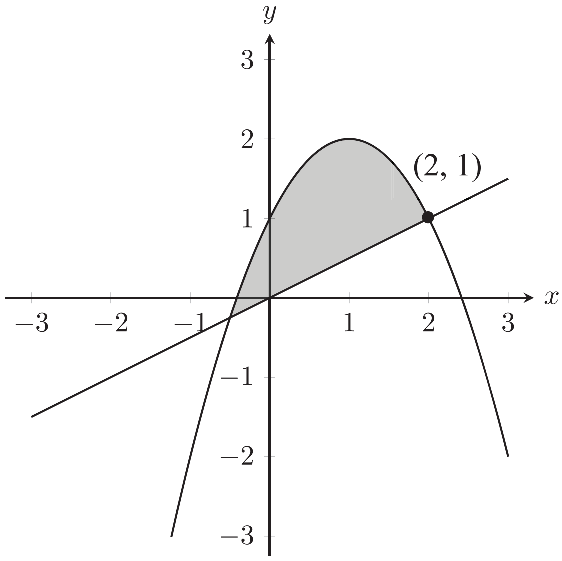
C. $\frac{21}{25}$

D. $-\frac{9}{5}$

E. $\frac{9}{5}$

Question 15

The area enclosed by the graph of $y = f(x) = 2 - (x-1)^2$ and the straight line perpendicular to the tangent to the graph of $y = f(x)$ at point (2, 1) is shaded below.



The shaded area is closest to

A. $\frac{215}{48}$

B. $\frac{125}{48}$

C. $\frac{62}{12}$

D. $\frac{5}{3}$

E. $\frac{145}{48}$

Question 16

The function $f : D \to R,\ f(x) = 2x^3 + 3x^2 - 36x + 4$ will have an inverse function for

A. $D = [0, 3]$

B. $D = [0, 2]$

C. $D = R$

D. $D = (-\infty, 2]$

E. $D = [-3, \infty)$

Question 17

A bag contains six red and four blue marbles. Three marbles are randomly drawn from the bag without replacement.

Given that at least two of the marbles drawn from the bag are red, the probability that all three marbles drawn from the bag are red is equal to

A. $\frac{1}{8}$

B. $\frac{1}{4}$

C. $\frac{2}{3}$

D. $\frac{1}{6}$

E. $\frac{1}{24}$

Question 18

Events A and B are independent events from a sample space with $\Pr(A) = a$ and $\Pr(B) = b$.

$\Pr(A \cup B')$ is equal to

A. $a + 1 - b$

B. $1 + b - ab$

C. $1 - b + ab$

D. $1 - b - ab$

E. $b + 1 - a$

Question 19

A random sample of students was surveyed about whether they are studying chemistry. From this random sample, an approximate 95% confidence interval for the proportion of students who are studying chemistry was calculated to be $(0.3828, 0.5283)$.

The number of students studying chemistry is closest to

A. 82

B. 90

C. 112

D. 180

E. 192

Question 20

If $f(x)=\int_1^x \frac{1}{\sqrt{t^2+1}}dt$, then $f'\left(\frac{1}{\sqrt{3}}\right)$ equals

A. 1

B. $\frac{1}{2}$

C. $\frac{1}{\sqrt{3}}$

D. $\frac{\sqrt{3}}{2}-\frac{1}{\sqrt{2}}$

E. $\frac{\sqrt{3}}{2}$

END OF SECTION A
TURN OVER

SECTION B

Instructions for Section B

Answer **all** questions in the spaces provided.

In all questions where a numerical answer is required, an exact value must be given unless otherwise specified.

In questions where more than one mark is available, appropriate working **must** be shown.

Unless otherwise indicated, the diagrams in this book are **not** drawn to scale.

Question 1 (10 marks)

Let $f : R \to R,\ f(x) = -\frac{1}{18}\left(7x^4 + 18x^3 - 21x^2 - 58x\right)$.

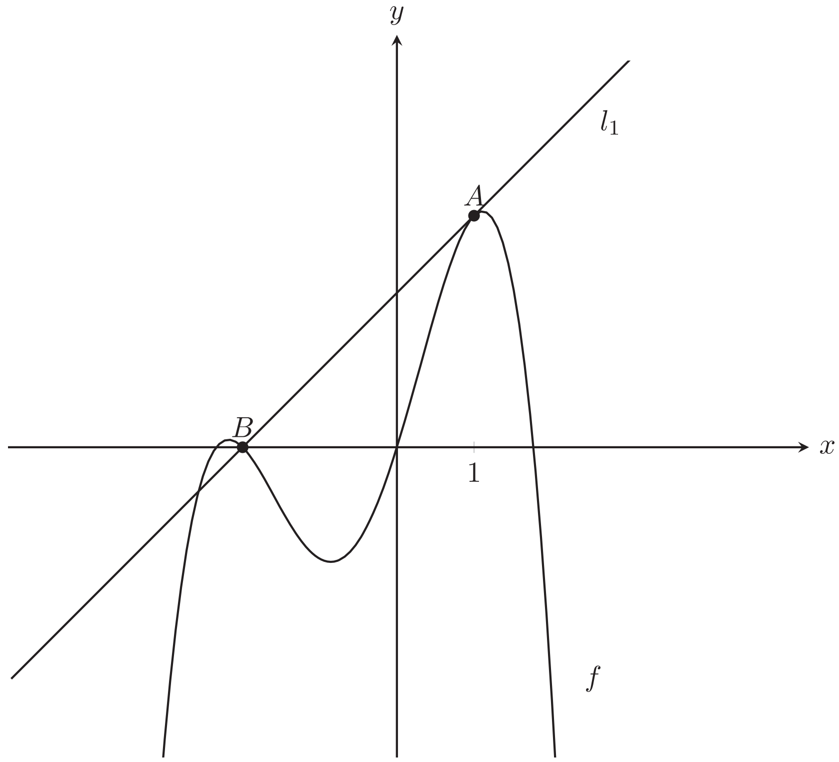
a. Express $f(x)$ in the form $-\frac{1}{18}x(x+2)\left(ax^2 + bx + c\right)$, where a, b and c are integers.

1 mark

b. Find all solutions of the equation $f(x) = 0$.

1 mark

Part of the graph of *f* and the tangent to the curve at *A* when $x = 1$ is shown below.



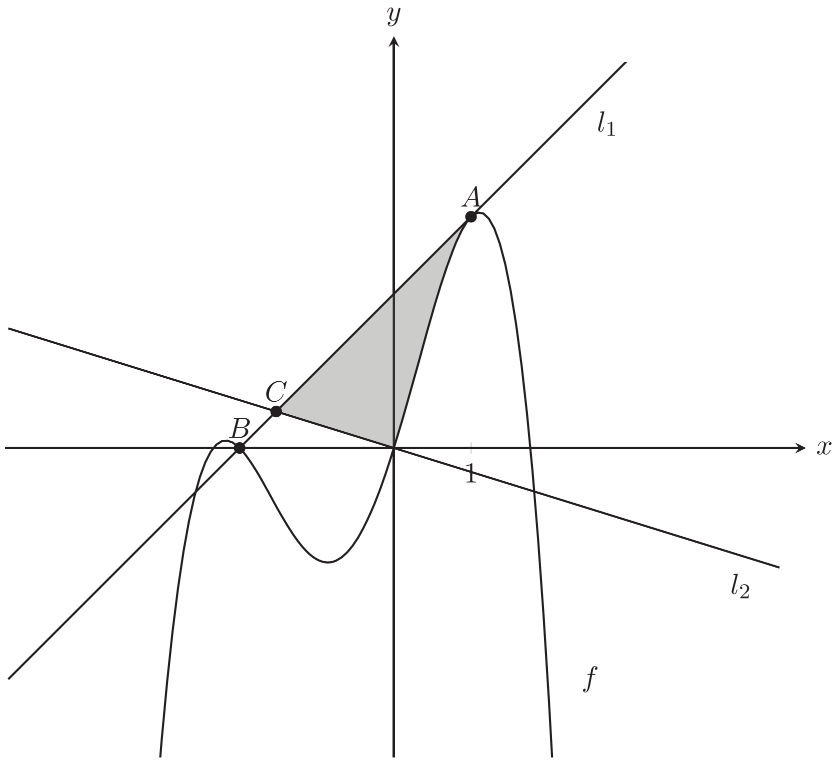
c. Find the equation of the tangent $l_1(x)$ to the graph of *f* at $x = 1$.

1 mark

d. Explain why the tangent to the graph of *f* at *A* is the perpendicular to the tangent to the graph of *f* at *B*.

1 mark

The line $l_2(x)$ is perpendicular to the graph of f at the origin and intersects the line through A and B at point C as shown in the diagram below.



e. Find the coordinates of point C.

2 marks

f. Find the acute angle between lines l_1 and l_2. Give your answer in degrees, correct to two decimal places.

2 marks

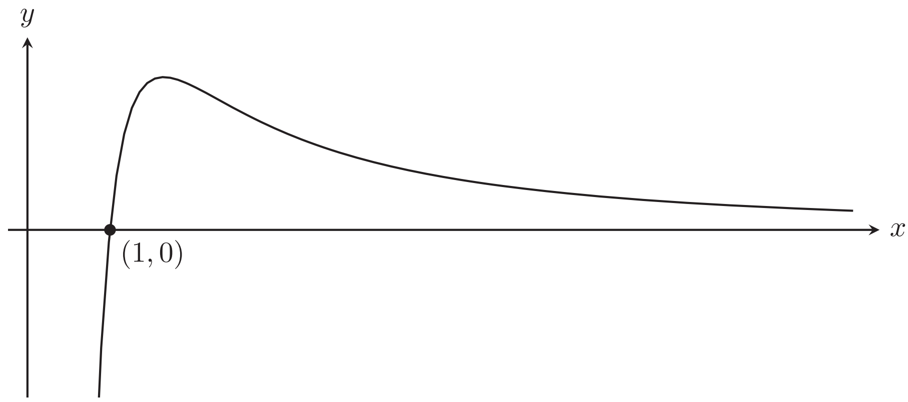
g. Find the total area of the shaded region in the graph on page 14. Give your answer correct to two decimal places.

2 marks

Question 2 (14 marks)

Let $f:(0,\infty)\to R,\ f(x)=\dfrac{\log_e(x)}{x^2}$.

Part of the graph of f is shown below.



a. Find the coordinates of the turning point of f.

1 mark

b. **i**. Calculate $\dfrac{d}{dx}\left(\dfrac{\log_e(x)}{x}\right)$.

1 mark

ii. Hence, determine an antiderivative of $f(x)$.

2 marks

iii. Use your result from **part b.ii.** to determine $\int_1^{e^2} f(x)dx$.

2 marks

iv. Hence, find the average value of f on the interval $\left[1, e^2\right]$.

1 mark

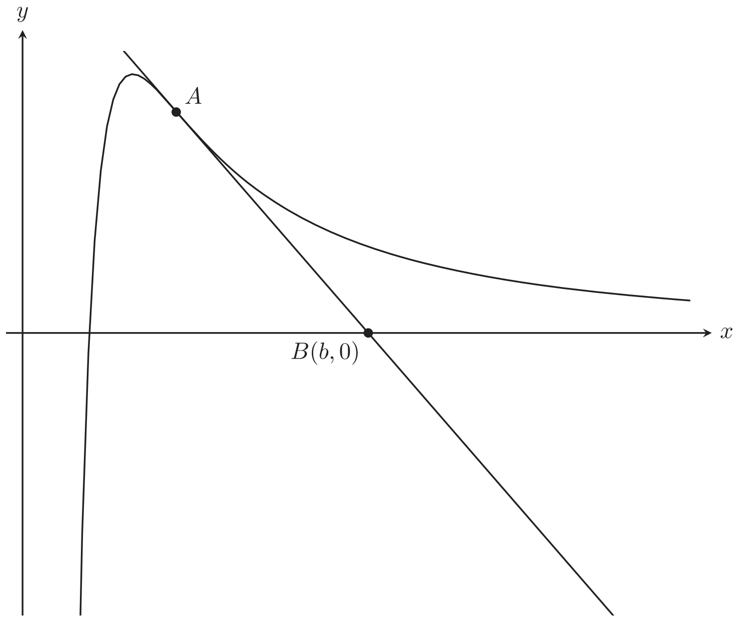
c. **i**. Find the rule for f' in terms of x.

1 mark

ii. Find the minimum value of the graph of f'.

1 mark

Shown on the axes below are the graph of *f* and the tangent to *f* where the gradient to *f* is a minimum.



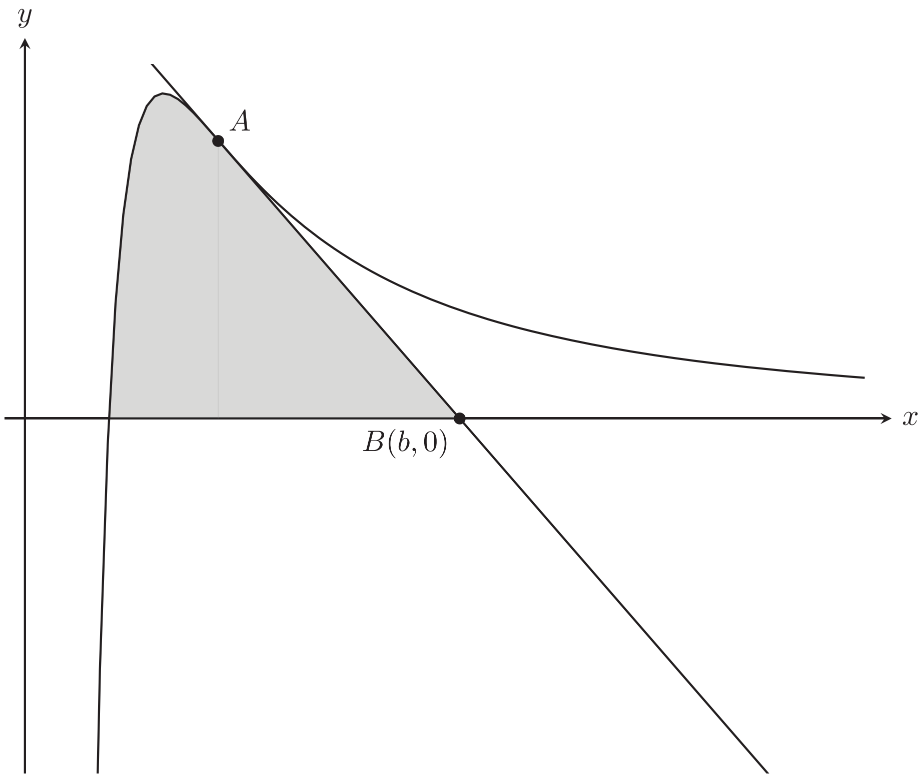
d. **i.** Write down the coordinates of *A*.

1 mark

ii. Determine the value of *b*.

1 mark

The area bounded by the graph of f, the x-axis and the tangent to f at the point A is shaded below.



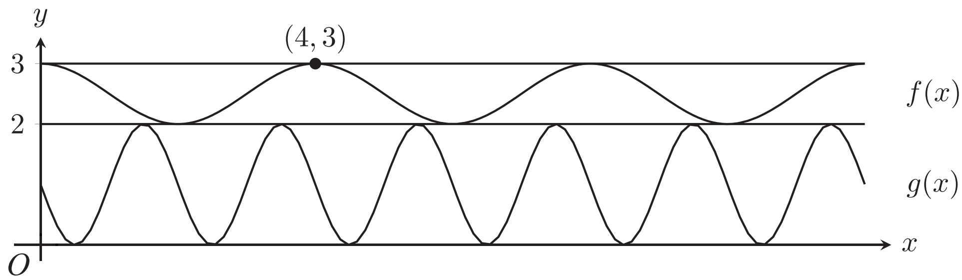
e. Determine the area of the shaded region, giving your answer in the form $a + be^c$ where a, b and c are rational numbers.

3 marks

Question 3 (12 marks)

Corrugated cardboard is produced by gluing corrugated paper between flat sheets of paper.

A particular type of corrugated cardboard is made up of two corrugated sheets of paper glued between three flat sheets of paper as shown below.



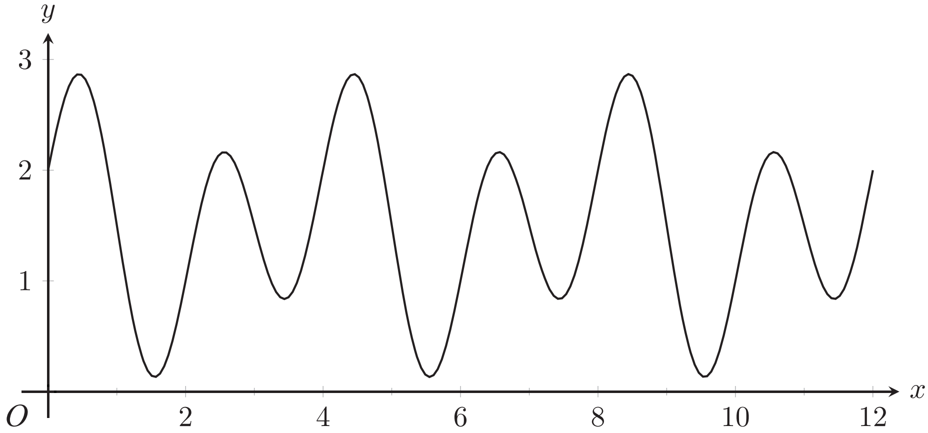
The flat sheets of paper are described by the horizontal lines $y = 0$, $y = 2$ and $y = 3$. The corrugations are described by two functions $f(x)$ and $g(x) = -\sin(\pi x) + 1$.

a. Given that $f(x) = a\cos(bx) + c$, write down the values of a, b and c.

1 mark

Let $d(x) = f(x) - g(x)$ be the vertical distance between the corrugated sheets of paper.

The graph of $d(x)$ is plotted below.



b. **i.** Show that $d(x) = \frac{3}{2} + \frac{1}{2}\cos\left(\frac{\pi x}{2}\right) + \sin(\pi x)$.

1 mark

ii. Write down the period of $d(x)$.

1 mark

iii. Write down the range of $d(x)$, giving your answers correct to three decimal places.

2 marks

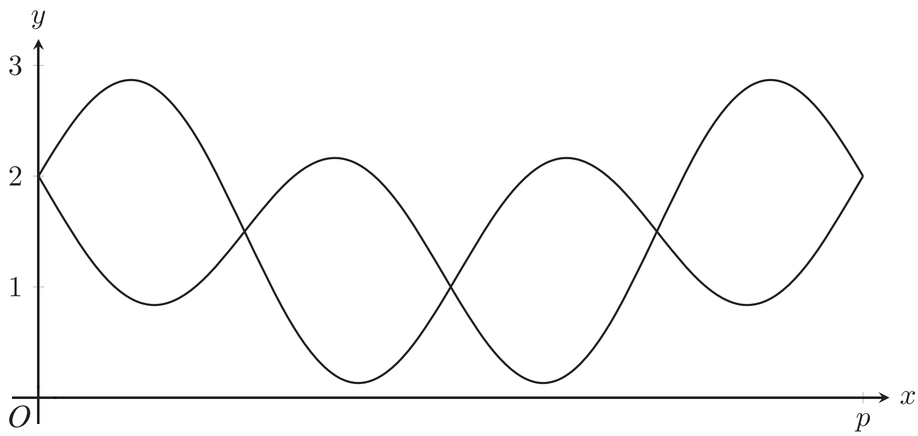
c. Find the average value of $d(x)$ over one period.

1 mark

d. Find the values of x for which $d(x) = \frac{3}{2}$, $1 \le x \le 5$. Give non-integer solutions correct to three decimal places.

2 marks

The graphs of $d(x)$ and $d(-x)$ for $0 \le x \le p$, where p is the length of one period of $d(x)$, are shown below.



e. **i.** Find the coordinates of the points of intersection of the graphs of $d(x)$ and $d(-x)$ for $0 \le x \le p$.

2 marks

ii. Determine the area bounded by the graphs of $d(x)$ and $d(-x)$ for $0 \le x \le p$.

2 marks

Question 4 (12 marks)

A fleet of cars uses two types of tyres, Type A and Type B.

The depth of the tread on a Type A tyre is normally distributed with a mean of 5.2 mm and a standard deviation of 1.1 mm.

a. Find the probability that a randomly selected Type A tyre has a tread depth greater than 5.7 mm given that it has a tread depth greater than 5.0 mm. Give your answer correct to four decimal places.

2 marks

Type B tyres have a tread depth described by the probability density function

$$f(x) = \begin{cases} \dfrac{3}{512}(9-x)(x-1)\left(e^{\frac{9-x}{4}}-1\right) & 1 \le x \le 9 \\ 0 & \text{elsewhere} \end{cases}$$

b. Determine the expected value of the depth of the tread on a randomly selected Type B tyre. Give your answer in millimetres, correct to four decimal places.

2 marks

c. Determine the standard deviation of the depth of the tread of a randomly selected Type B tyre. Give your answer in millimetres, correct to four decimal places.

2 marks

d. Find the probability that a randomly selected Type B tyre will have a tread depth greater than 7 mm given that it has a tread depth greater than 5 mm. Give your answer correct to four decimal places.

1 mark

A Type B tyre is deemed to be unsafe if it has a tread depth of less than 2 mm.

The probability that a randomly selected Type B tyre is unsafe is 0.1134, correct to four decimal places.

e. Find the probability that in a random sample of 30 Type B tyres, five or more tyres are unsafe. Give your answer correct to two decimal places.

1 mark

f. Let $\hat{P}$ be the random variable that represents the proportion of Type B tyres that are unsafe in a random sample of 30 tyres.

Find $\Pr\left(\hat{P} > 0.1 | \hat{P} < 0.2\right)$. Give your answer correct to two decimal places.

2 marks

g. In a particular sample of 30 Type B tyres, an approximate c% confidence interval for the proportion of tyres that were unsafe was calculated to be (0.10641, 0.29395).

Determine the value of c correct to the nearest integer.

2 marks

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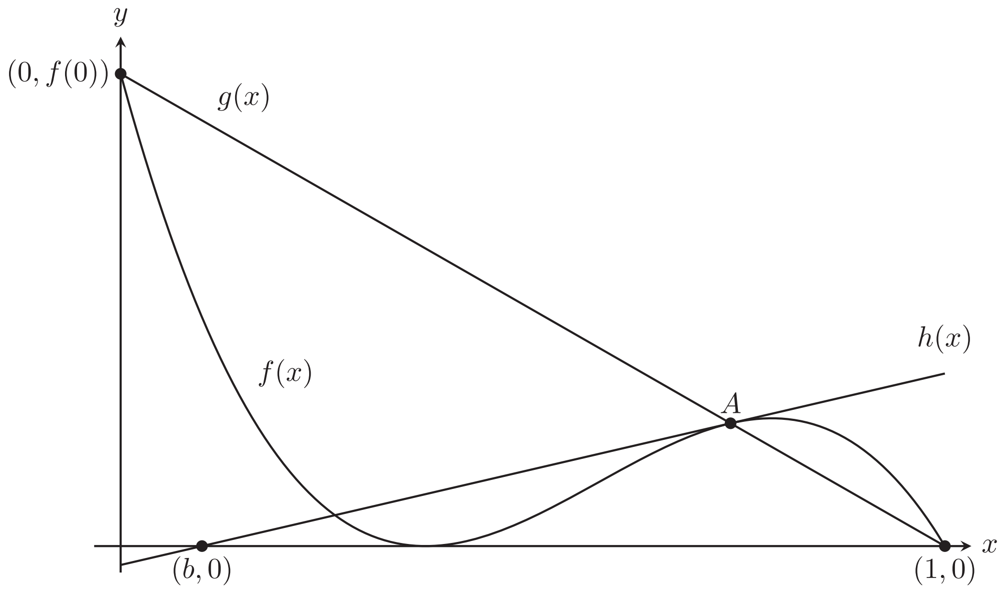
Question 5 (12 marks)

Let $f:[0,1]\rightarrow R$, $f(x)=(x-a)^2(1-x)$, $0<a<\frac{2}{5}$.

Let $g(x)$ be the line joining the y-intercept of $f(x)$ and the point (1, 0) as shown on the diagram below.

Further, let $h(x)$ be tangent to $f(x)$ at the point where $g(x)$ and $f(x)$ meet and $(b, 0)$ be the x-intercept of $h(x)$.



a. Write down the equation of $g(x)$ in terms of a.

1 mark

b. Find the coordinates of the point A in terms of a.

1 mark

c. **i.** Find the equation of $h(x)$ in terms of a.

1 mark

ii. Hence, show that $b = \dfrac{a(8a-3)}{5a-2}$.

1 mark

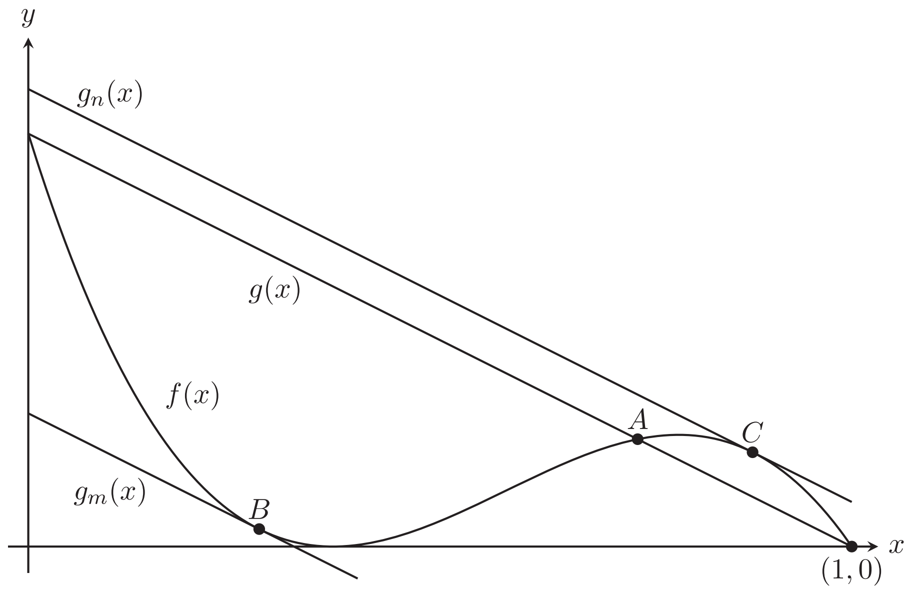
iii. Find the maximum value of b and the value of a at which the maximum occurs.

2 marks

iv. Find the values of a such that $0 \le a \le b$.

1 mark

Let g_m and g_n be the two tangents to the graph of $y = f(x)$ at the points where their graphs are parallel to $g(x)$.



Let d_m and d_n be the vertical distance from $g(x)$ to $g_m(x)$ and $g(x)$ to $g_n(x)$ respectively.

d. **i.** Determine the x-coordinates of the points B and C. Give your answers in terms of a.

2 marks

d. **ii.** Find $\frac{g_n(0)+g_m(0)}{2}$. Express your answer as a cubic polynomial in terms of a.

1 mark

iii. Hence, determine the value of a for which $d_m = d_n$ and find the value of d_m.

2 marks

END OF QUESTION AND ANSWER BOOK