THE SCHOOL FOR EXCELLENCE (TSFX)

UNIT 3 & 4 MATHEMATICAL METHODS 2020

WRITTEN EXAMINATION 2

Reading Time: 15 minutes
Writing Time: 2 hours

QUESTION AND ANSWER BOOKLET

Student Name:

Structure of Booklet

Section	Number of questions	Number of questions to be answered	Number of marks
A	20	20	20
B	5	5	60
			Total 80

- Students are permitted to bring into the examination room: pens, pencils, highlighters, erasers, sharpeners, rulers, a protractor, set-squares, aids for curve sketching, one bound reference, one approved CAS calculator or CAS software and, if desired, one scientific calculator. Calculator memory DOES NOT need to be cleared.
- Students are **NOT** permitted to bring into the examination room: blank sheets of paper and/or white out liquid/tape.

Materials Supplied

- Question and answer book of 26 pages.
- Working space is provided throughout the book.

Instructions

- Write your **name** in the space provided above on this page.
- All written responses must be in English.

Students are **NOT** permitted to bring mobile phones and/or any other unauthorised electronic devices into the examination room.

Note: This examination was written for the revised 2020 VCE Mathematics Study Design in which significant deletions of content to Mathematical Methods Area of Study 4 (Probability and Statistics) were made by VCAA.

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SECTION A – Multiple-choice questions

Instructions for Section A

Answer all questions on the answer sheet provided for multiple-choice questions.

Choose the response that is **correct** for the question.

A correct answer scores 1; an incorrect answer scores 0.

Marks will **not** be deducted for incorrect answers.

No marks will be given if more than one answer is completed for any question.

Unless otherwise indicated, the diagrams in this book are **not** drawn to scale.

Question 1

Let $f : R \to R$, $f(x) = -2\cos\left(2\left(\frac{\pi}{5} - \frac{x}{6}\right)\right) + 4$.

The period and range of f are respectively

A. $\frac{\pi}{3}$ and $[-2,\ 4]$

B. $\frac{\pi}{3}$ and $[2,\ 6]$

C. 12π and $[2,\ 6]$

D. 6π and $[2,\ 6]$

E. 6π and $[-2,\ 2]$

Question 2

Let $f : R\backslash\{a\} \to R$, $f(x) = \frac{2}{x-a}$, where $a > 0$.

The average value of f from $x = -\frac{a}{2}$ to $x = \frac{a}{2}$ is

A. $\frac{2}{a}\log_e(3)$

B. $-\frac{2}{a}\log_e(3)$

C. $-\frac{2}{a}\log_e(3a)$

D. $-\frac{8}{3a^2}$

E. $\frac{8}{3a^2}$

Question 3

The graph of the function $f:(-\infty,\ -3]\to R$, where $f(x)=3-2\sqrt{1-3x}$, is translated +2 units from the *y*-axis, dilated by a factor of 2 from the *x*-axis and then reflected in the *y*-axis. Which one of the following is the equation of the transformed graph?

A. $y=6-4\sqrt{7+3x}$

B. $y=-6-4\sqrt{-5-3x}$

C. $y=-6+4\sqrt{-5-3x}$

D. $y=-6-4\sqrt{7-3x}$

E. $y=6-4\sqrt{-5+3x}$

Question 4

The point $A(-2,\ 4)$ lies on the graph of the function f. A transformation maps the graph of f to the graph of g, where $g(x)=-\dfrac{1}{2}f(2x+4)$. The point A is mapped to the point B.

The coordinates of the point B are

A. $(1,\ 2)$

B. $(1,\ -2)$

C. $(-2,\ -2)$

D. $(-6,\ -2)$

E. $(-3,\ -2)$

Question 5

The minimum distance of the turning point of the parabola $y=-x^2+2bx-2$, where $b\in R$, from the origin is

A. 2

B. $\dfrac{5}{3}$

C. $\dfrac{2\sqrt{6}}{3}$

D. $\dfrac{\sqrt{23}}{3}$

E. $\dfrac{\sqrt{7}}{2}$

Question 6

The inverse function of $f:[0,\ 2] \to R,\ f(x) = \dfrac{\sqrt{x+1}}{x-3}$ is

A. $f^{-1}:[0,\ 2] \to R,\quad f^{-1}(x) = \dfrac{x-3}{\sqrt{x+1}}$

B. $f^{-1}:\left[-\dfrac{1}{3},\ -\sqrt{3}\right] \to R,\quad f^{-1}(x) = \dfrac{6x^2+1+\sqrt{16x^2+1}}{2x^2}$

C. $f^{-1}:[0,\ 2] \to R,\quad f^{-1}(x) = \dfrac{6x^2+1+\sqrt{16x^2+1}}{2x^2}$

D. $f^{-1}:\left[-\dfrac{1}{3},\ -\sqrt{3}\right] \to R,\quad f^{-1}(x) = \dfrac{6x^2+1-\sqrt{16x^2+1}}{2x^2}$

E. $f^{-1}:[0,\ 2] \to R,\quad f^{-1}(x) = \dfrac{6x^2+1-\sqrt{16x^2+1}}{2x^2}$

Question 7

Let $f:(a,\ b] \to R,\ f(x) = \dfrac{2}{x^2}$, where a and b are positive real numbers.

The range of f is

A. $\left[\dfrac{2}{b^2},\ \dfrac{2}{a^2}\right)$

B. $\left(\dfrac{2}{a^2},\ \dfrac{2}{b^2}\right]$

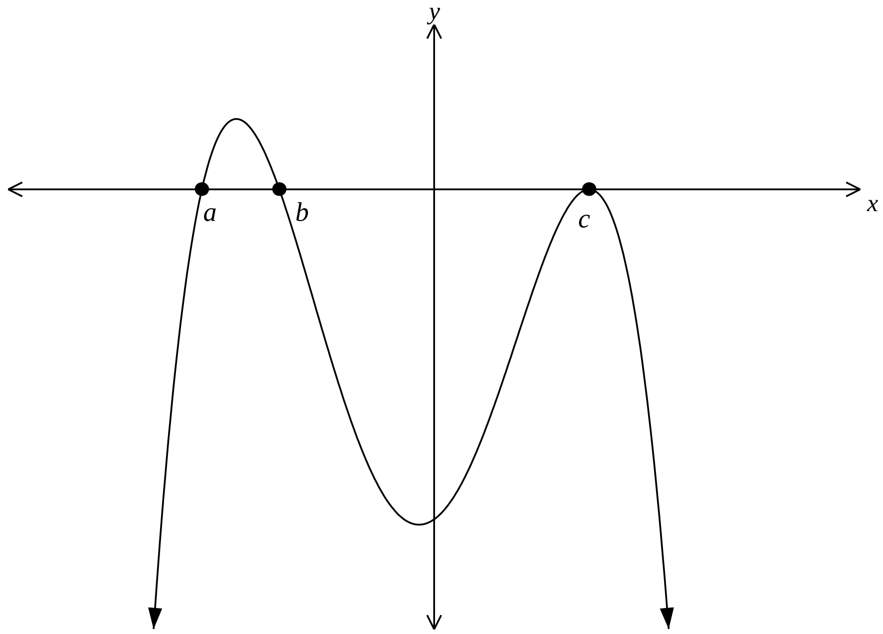
C. $\left[\dfrac{2}{a^2},\ \dfrac{2}{b^2}\right)$

D. $\left(\dfrac{2}{b^2},\ \dfrac{2}{a^2}\right]$

E. $(a,\ b]$

Question 8

Part of the graph of $y = f(x)$ is shown below.



A possible rule for $f(x)$ is

A. $(x-a)(x-b)(x-c)^2$

B. $(x+a)(x+b)(x-c)^2$

C. $(x-a)(x-b)(x-c)$

D. $(x-a)(b-x)(x-c)^2$

E. $(x-a)(x-b)(c-x)^2$

Question 9

The graphs of $y=(2-p)x^2+3x$ and $y=5x+p-1$ will have no points of intersection when

A. $p^2+3p+1<0$

B. $p^2+3p+1>0$

C. $p^2-3p+1>0$

D. $p^2-3p+1<0$

E. $p^2-3p-1>0$

Question 10

Let f and g be two functions defined such that $f(x) > \frac{3}{2}$ for $x \in [-1, 3]$ and $g(x) = x - 4f(x)$.
The area between the graph of g and the x-axis over the interval $x \in [-1, 3]$ is eleven units.
The area under the graph of f over the same interval is

A. 6

B. $\frac{15}{4}$

C. $\frac{11}{4}$

D. $\frac{7}{4}$

E. $-\frac{7}{4}$

Question 11

Let $f(x) = p\sin(3x) + q$ where p and q are real numbers and $p > 0$.
Then $f(x) < 0$ for all values of x if

A. $q < p$

B. $q > -p$

C. $q < -p$

D. $-p < q < p$

E. $q = 0$

Question 12

The transformation that maps the graph of $y = \left(4x^2 - 2\right)^{3/2}$ onto the graph of $y = \left(x^2 - 2\right)^{3/2}$ is a

A. dilation by a factor of 2 from the x-axis.

B. dilation by a factor of 2 from the y-axis.

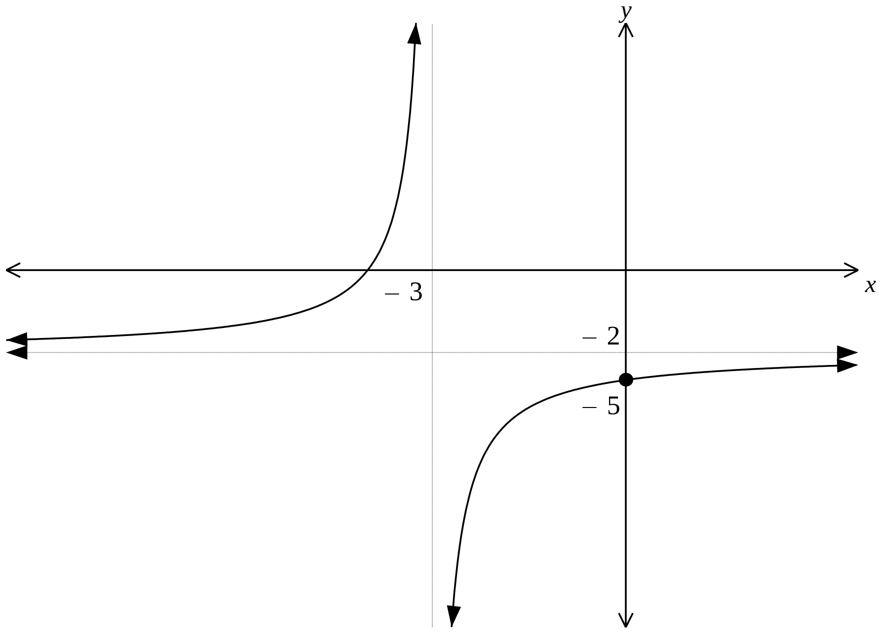
C. dilation by a factor of $\frac{1}{2}$ from the x-axis.

D. dilation by a factor of $\frac{1}{2}$ from the y-axis.

E. dilation by a factor of $\frac{1}{4}$ from the y-axis.

Question 13

Part of the graph of the function with equation $y = \frac{1}{Ax + B} - C$, where A, B and C are real constants, is shown below. The graph has a y-intercept at $y = -5$ and asymptotes with equations $x = -3$ and $y = -2$.



The values of A, B and C respectively are

A. $A = -\frac{1}{9}$, $B = -\frac{1}{3}$, $C = -2$

B. $A = -\frac{1}{9}$, $B = -\frac{1}{3}$, $C = 2$

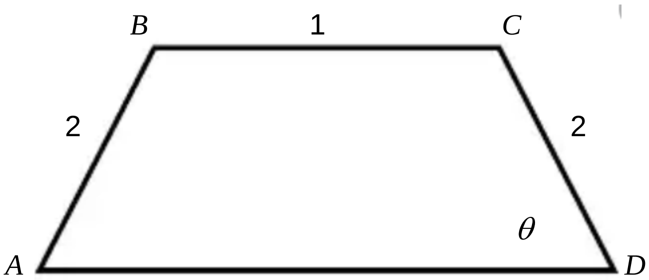
C. $A = \frac{1}{9}$, $B = -\frac{1}{3}$, $C = -2$

D. $A = \frac{1}{9}$, $B = \frac{1}{3}$, $C = 2$

E. $A = -\frac{1}{9}$, $B = \frac{1}{3}$, $C = -2$

Question 14

The trapezium $ABCD$ is shown below. The sides AB and CD are of length 2 and the side BC is also of length 1. The size of the acute angle ADC is θ radians.



The maximum possible area of the trapezium is closest to

A. 2.34

B. 2.44

C. 2.54

D. 2.64

E. 2.74

Question 15

The equation $3\cos(2x)-k=4$, where k is a real number, has two solutions in the interval $\left(0,\ \frac{5\pi}{4}\right)$. The maximal set of possible values of k is

A. $-7<k\leq -4$

B. $-7<k<-4$

C. $-4\leq k<-1$

D. $-4<k\leq -1$

E. $-7<k<-1$

Question 16

Let X be a discrete random variable with binomial distribution $X \sim \text{Bi}(n, p)$. The mean of X is twice its standard deviation. Given that $0<p<1$, the smallest number of trials, n, such that $p<0.1$ is

A. 17

B. 18

C. 35

D. 36

E. 37

Question 17

A darts player is throwing darts at a target. Her probability of hitting the bullseye of the target is 0.3. The minimum number of shots needed by the darts player so that the probability of her hitting the bullseye at least six times is greater than 0.9 is equal to

A. 23

B. 25

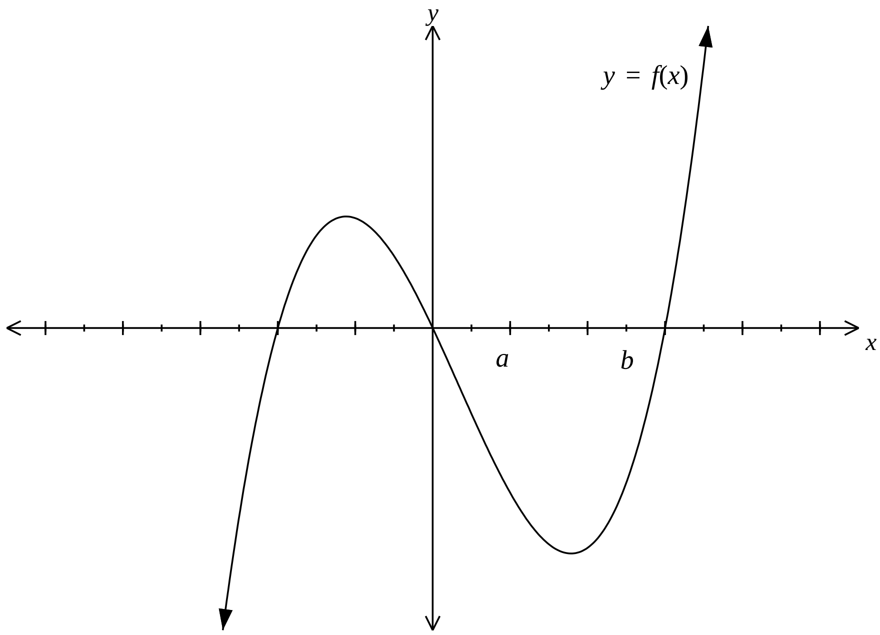
C. 27

D. 29

E. 31

Question 18

The graph of the function $y = f(x)$ is shown below.



Let g be a function such that $g'(x) = f(x)$. On the interval (a, b), the function g will

A. Have a local maximum value.

B. Have a local minimum value.

C. Have a zero gradient.

D. Be a decreasing function.

E. Be an increasing function.

Question 19

Consider the transformation T, defined as

$$T: R^2 \to R^2,\ T\left(\begin{bmatrix} x \\ y \end{bmatrix}\right) = \begin{bmatrix} -1 & 0 \\ 0 & -2 \end{bmatrix}\begin{bmatrix} x \\ y \end{bmatrix} + \begin{bmatrix} 0 \\ -3 \end{bmatrix}$$

The transformation T maps the graph of $y = f(x)$ onto the graph of $y = g(x)$.

If $\int_{-1}^{2} f(x)\,dx = 4$, then $\int_{-2}^{1} g(x)\,dx$ is equal to

A. 8

B. 1

C. -17

D. -8

E. -1

Question 20

The random variable X has a normal distribution with mean 16 and variance 9. If Z has the standard normal distribution, then the probability that the value of X lies between 13 and 22 is equal to

A. $1 - \Pr(Z > \frac{1}{2}) - \Pr(Z > 1)$

B. $1 + \Pr(Z > 1) - \Pr(Z > 2)$

C. $\Pr(Z < 2) + \Pr(Z > -1)$

D. $\Pr(Z < 1) + \Pr(Z > -\frac{1}{2})$

E. $1 - \Pr(Z > 1) - \Pr(Z > 2)$

SECTION B

Instructions for Section B

- Answer **all** questions in the spaces provided.
- In all questions where a numerical answer is required, an exact value must be given unless otherwise specified.
- In questions where more than 1 mark is available, appropriate working **must** be shown.
- Unless otherwise indicated, the diagrams in this book are **not** drawn to scale.

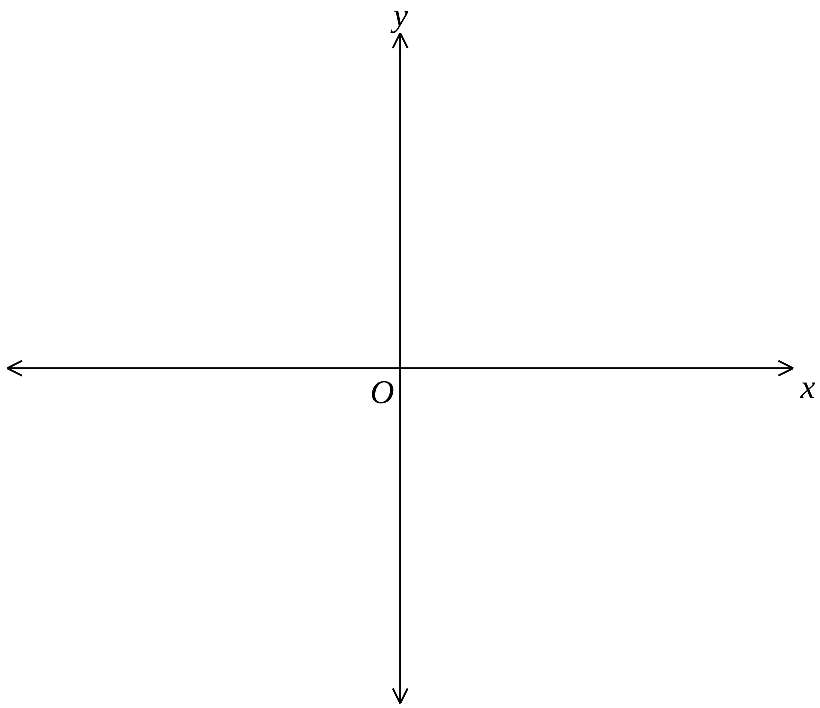
Question 1 (11 marks)

a. Solve the equation $e^{2x} - 5e^{x} + 5 = 0$. 3 marks

b. Let $f : R \rightarrow R$, $f(x) = e^{2x} - 5e^{x} + 5$.

i. State the minimum value of the function f and the value of x for which the minimum value occurs. 2 marks

ii. Sketch a graph of $y = e^{2x} - 5e^{x} + 5$. Label any axes intercepts and turning points with their coordinates and any asymptotes with their equation. 3 marks



c. Find the values of k for which the equation $e^{2x} + ke^{x} - k = 0$ has exactly one real solution. 3 marks

Question 2 (13 marks)

Galaxian Highscore, the famous adventurer, is searching for the gratuitous Gem of Trigonomia when he is captured by natives. The natives will only release Galaxian if he can answer the ancient Puzzles of Trigonomia.

a. Galaxian is given the function $f: R \to R$, $f(x) = a\sin(b\pi x - c) + d$ where a, b, c and d are real numbers.

He is told that $-\frac{\pi}{2} < c < \frac{\pi}{2}$ and that the graph of f has range $[-1, 3]$, a period of 3 and passes through the point $(0, 2)$. Find the values of a, b, c and d. 3 marks

Galaxian is next given the function $g: [-4, 4] \to R$, $g(x) = -2\sin\left(\frac{\pi}{3}(x+2)\right) + 3$.

b. Find the maximal set of values of x for which g is strictly decreasing. 2 marks

c. Find in the form $y = mx + c$ the equation of the normal to g at the point where $x = 0$. 2 marks

d. The line $y=-x+1$ is translated vertically upwards by k units so that it is a tangent to g. Find, correct to four decimal places, the minimum value of k. 3 marks

Finally, Galaxian Highscore is given the functions

$$h_1: [0, 40] \to R, h_1(x) = p^2 \cos\left(\frac{x}{5}\right) + 4$$

$$h_2: [0, 40] \to R, h_1(x) = (3p-1)\cos\left(\frac{x}{5}\right) + 8$$

where $p \in R^+$. He is told that the graphs of h_1 and h_2 never intersect when $p \in (0, w)$.

e. Find the largest value of w. 3 marks

Question 3 (14 marks)

Consider the equation $x^4 - ax^3 + (7-2a)x^2 - ax + 1 = 0$ where a is a real number.

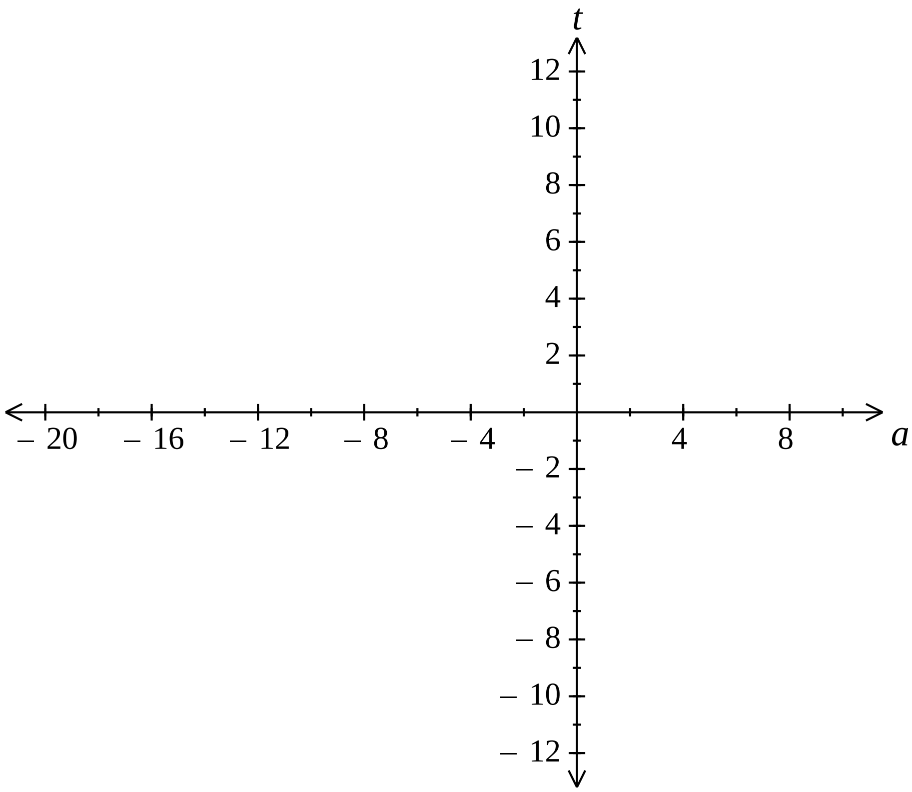
a. Let $t = x + \frac{1}{x}$.

Show that $t^2 - at + 5 - 2a = 0$. 2 marks

b. i. Find all values of a for which t has real solutions. 2 marks

ii. Hence find all values of a for which $x^4 - ax^3 + (7-2a)x^2 - ax + 1 = 0$ has real solutions. 2 marks

c. Draw a graph of $t = \dfrac{a \pm \sqrt{a^2 + 8a - 20}}{2}$ over its maximal domain. Label all endpoints with their coordinates and any asymptotes with their equation. 3 marks



d. Find the value(s) of a for which $x^4 - ax^3 + (7-2a)x^2 - ax + 1 = 0$ has exactly:

i. four real solutions. 1 mark

ii. two repeated real solutions. 2 marks

e. Find the value(s) of a for which $x^4 - ax^3 + (7-2a)x^2 - ax + 1 = 0$ has exactly:

i. two non-repeated real solutions. 1 mark

ii. one real solution. 1 mark

Question 4 (14 marks)

The amount of the anaesthetic *Outlikealight* (OLAL) required to achieve surgical anaesthesia in a 78 kg adult is a normally distributed random variable with a mean of 60 milligrams and a standard deviation of 15 milligrams.

a. **i.** Find the probability that a 78 kg adult will require less than 68 mg of OLAL to achieve surgical anaesthesia. Give the answer correct to four decimal places. 1 mark

ii. The probability that a 78 kg adult will require more than a milligrams of OLAL to achieve surgical anaesthesia is equal to 0.65.

Find the value of a, correct to one decimal place. 1 mark

A 78 kg adult is considered OLAL-*sensitive* if less than 18 milligrams of OLAL is required to achieve surgical anaesthesia, OLAL-*typical* if between 18 and b milligrams is required and OLAL-*resistant* otherwise. The probability of a patient being OLAL-*typical* is equal to 0.94.

b. **i.** Find the value of b, correct to one decimal place. 2 marks

ii. Find the probability that a 78 kg adult who is OLAL-*typical* will require less than 60 milligrams to achieve surgical anaesthesia. Give the answer correct to four decimal places. 2 marks

c. i. Find the probability that three adults in a random group of nine 78 kg adults will require between 46 and 57 milligrams of OLAL to achieve surgical anaesthesia. Give the answer correct to four decimal places. 2 marks

ii. The probability that more than one adult is OLAL-*sensitive* in a random group of 78 kg adults is greater than 0.4. Find the smallest size of the group. 2 marks

The *numb-tongue dose* of OLAL is the amount of OLAL that will cause the tongue to become numb for 24 hours. For a 78 kg adult the numb-tongue dose is a normally distributed random variable with a mean of 130 milligrams and a standard deviation of 14 milligrams.

d. Find the percentage of 78 kg adults whose tongue will become numb for 24 hours if the amount of OLAL required to achieve surgical anaesthesia in 90% of 78 kg adults is used. Give the answer correct to four decimal places. 2 marks

The amount of OLAL required to achieve surgical anaesthesia in a 30 kg child is a normally distributed random variable with a mean of 31 milligrams. The probability that more than 27 milligrams is required is equal to 0.83.

e. Find the standard deviation of the amount of OLAL required to achieve surgical anaesthesia in a 30 kg child. Give the answer in milligrams, correct to two decimal places. 2 marks

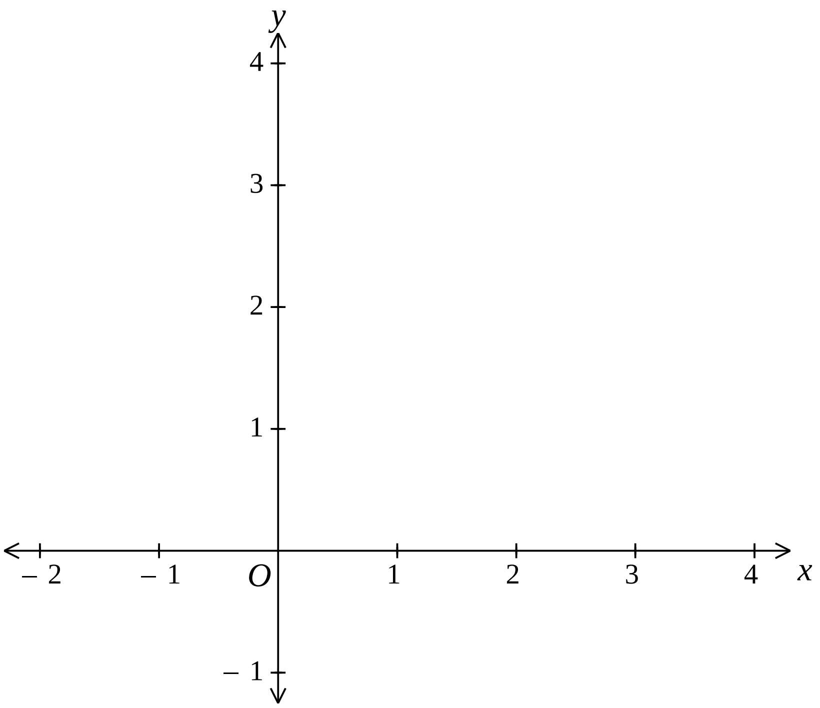
Question 5 (8 marks)

Let $f:[k, +\infty) \to R$, $f(x) = 4xe^{-x}$ and k is the smallest value for which the inverse function f^{-1} of f exists.

a. Find the value of k. 2 marks

b. Draw a graph of $y = f^{-1}(x)$.

Label all endpoints with their coordinates and all asymptotes with their equation. 3 marks



c. Find, correct to four decimal places, the value of $f^{-1}\left(\frac{2}{3}\right)$. 3 marks

END OF QUESTION AND ANSWER BOOK