Neap

Trial Examination 2020

VCE Mathematical Methods Units 3&4

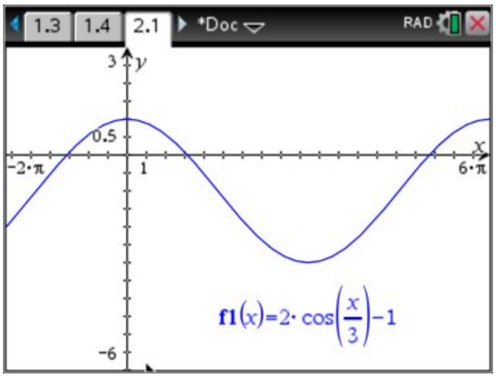
Written Examination 2

Suggested Solutions

SECTION A – MULTIPLE-CHOICE QUESTIONS

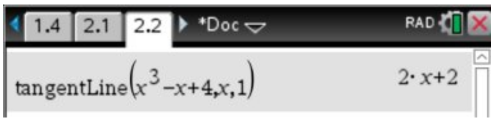
1	E
2	A
3	B
4	B
5	A
6	C
7	A
8	A
9	A
10	E
11	D
12	E
13	B
14	C
15	C
16	C
17	D
18	C
19	C
20	B

Question 1 **E**



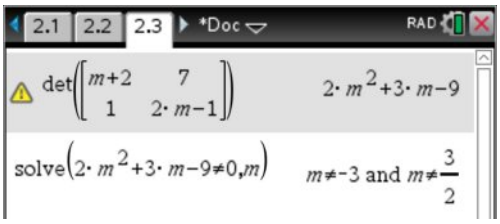
$$\text{period} = \frac{2\pi}{n}$$
$$= \frac{2\pi}{\frac{1}{3}}$$
$$= 6\pi$$

Question 2 **A**

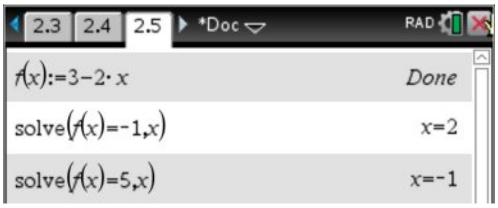


Question 3 **B**

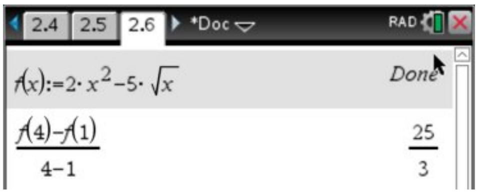
$$\begin{bmatrix} m+2 & 7 \\ 1 & 2m-1 \end{bmatrix}\begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} m+3 \\ 5 \end{bmatrix}$$



Question 4 **B**



Question 5 **A**



Question 6 **C**

$kx^2 - kx + \frac{1}{4} = 0$

$\Delta = b^2 - 4ac$

$\quad = k^2 - k$

$k^2 - k = 0$ for one solution

$k(k-1) = 0$

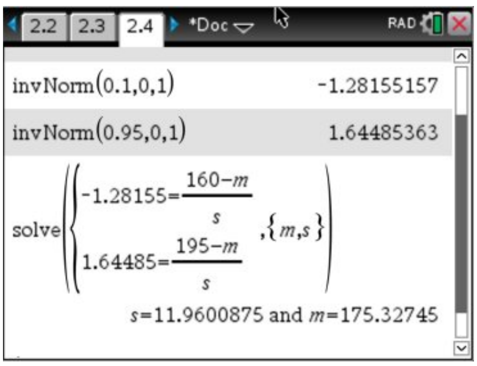
$k = 0$ or $k = 1$

However, when $k = 0$, the quadratic reduces to $\frac{1}{4} = 0$ and therefore $k \neq 0$.

$k = 1$ only

Question 7 **A**

$z = \frac{x - \bar{x}}{\mu}$



Question 8 **A**

$g(x) = 3f(2x - 4) + 1$

Transformations:

Dilation factor of 3 from x-axis: $(1, 3) \rightarrow (1, 9)$

Dilation factor of $\frac{1}{2}$ from y-axis: $(1, 9) \rightarrow \left(\frac{1}{2}, 9\right)$

Translate 2 units right: $(1, 9) \rightarrow \left(\frac{5}{2}, 9\right)$

Translate 1 unit down: $\left(\frac{5}{2}, 9\right) \rightarrow \left(\frac{5}{2}, 10\right)$

Question 9 **A**

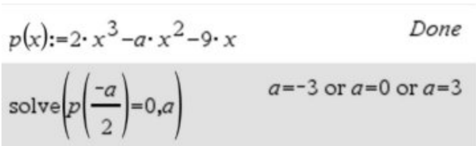
Let f and g be two functions such that $f(x + 1) = x$ and $g(x + 2) = f(x)$.

$f(x + 1) = x \rightarrow f(x) = x - 1$

$g(x + 2) = f(x) = x - 1 \rightarrow g(x) = x - 3$

$$\begin{aligned} f(g(x)) &= (x - 3) - 1 \\ &= x - 4 \end{aligned}$$

Question 10 **E**



Question 11 **D**

Let S = success, M = miss and A = any (success or miss).

Pr(at least 3 consecutive) = Pr($SSSAA$) + Pr($MSSSA$) + Pr ($AMSSS$)

Pr(at least 3 consecutive) = $\left(\frac{4}{5}\right)^3 \times 1 \times 1 + \left(\frac{1}{5}\right) \times \left(\frac{4}{5}\right)^3 \times 1 + 1 \times \frac{1}{5} \times \left(\frac{4}{5}\right)^3 = 0.7168$

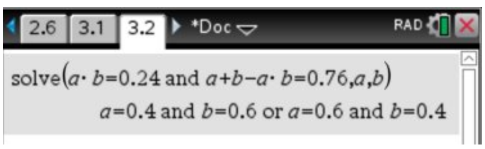
Question 12 **E**

Let $a = \Pr(A)$ and $b = \Pr(B)$.

$\Pr(A \cup B) = \Pr(A) + \Pr(B) - \Pr(A \cap B)$

$0.76 = a + b - ab$

$\Pr(A \cap B) = ab = 0.24$



Question 13 B

The derivative function f' is a positive cubic and matches the positive quartic in **B**.

Question 14 C

$2 - x > 0 \to x < 2$

$\log_e(2 - x) \neq 0 \to x \neq 1$

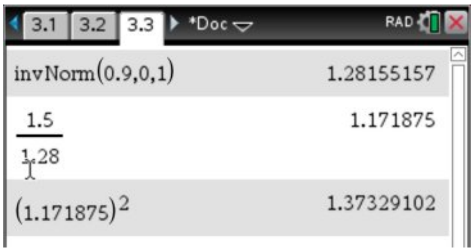
$\therefore x \in (-\infty, 2)\backslash\{1\}$

Question 15 C

$z = \dfrac{x - \mu}{\sigma}$

$\mu = 0 \to \sigma = \dfrac{x}{z}$

$\text{Var}(X) = \sigma^2$

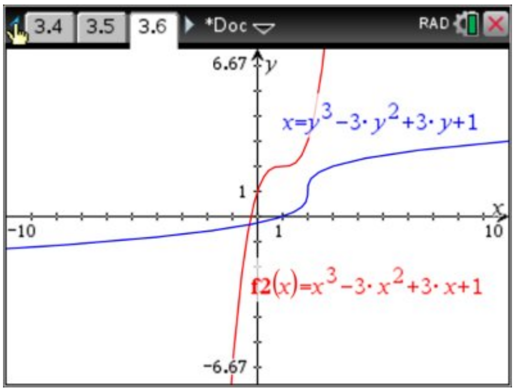


Question 16 C

$(x - 1)^3 = x^3 - 3x^2 + 3x - 1$

$\therefore g(x) = (x - 1)^3 + 2$

Thus the inverse will exist.



Question 17 D

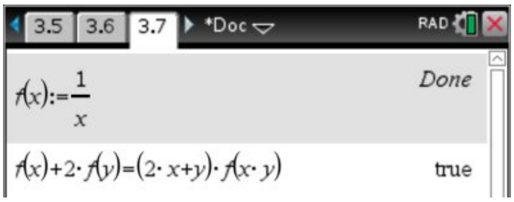
If $f(x) = \dfrac{1}{x}$:

$$\begin{aligned} \text{LHS} &= f(x) + 2f(y) \\ &= \frac{1}{x} + 2 \times \frac{1}{y} \\ &= \frac{1}{x} + \frac{2}{y} \\ &= \frac{2x + y}{xy} \end{aligned}$$

$$\begin{aligned} \text{RHS} &= (2x + y)f(xy) \\ &= (2x + y)\frac{1}{xy} \\ &= \frac{2x + y}{xy} \end{aligned}$$

$\therefore \text{LHS} = \text{RHS}$

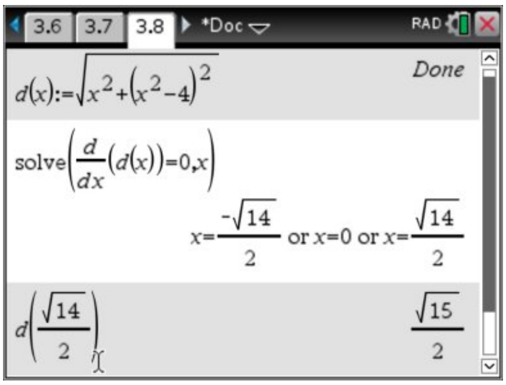
$\therefore f(x) + 2f(y) = (2x + y)f(xy)$



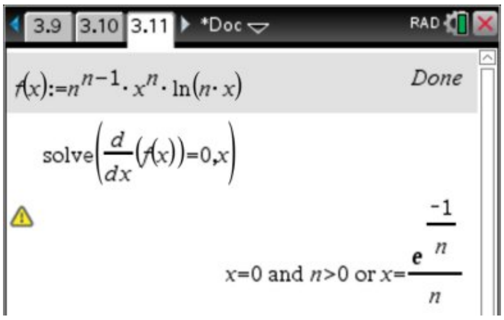
Question 18 C

$$\begin{aligned} \int_3^{10} f(x) + 1\,dx &= \int_3^{8} f(x)dx + \int_8^{10} f(x)dx + \int_3^{10} 1\,dx \\ &= 10 - 4 + 7 \\ &= 13 \end{aligned}$$

Question 19 C



Question 20 B



However $f(x)$ is undefined for $x = 0$ and therefore there is only one possible stationary point.

SECTION B

Question 1 (7 marks)

a.

$f(x)$	$x^3-6\cdot x^2+8\cdot x$
solve$(f(x)=0,x)$	$x=0$ or $x=2$ or $x=4$

(0, 0), (2, 0), (4, 0) A1

b. Let $f(x) = g(x)$.

$x^3 - 6x^2 + 8x = -ax$

$x^2 - 6x + 8 = -a \quad (1)$

Let $f'(x) = g'(x)$.

$3x^2 - 12x + 8 = -a \quad (2)$ M1

Equate (1) and (2).

$3x^2 - 12x + 8 = x^2 - 6x + 8$

$2x^2 - 6x = 0$

$2x(x - 3) = 0$

$x = 0$ or $x = 3$ M1

$\therefore p = 3$, as $p > 0$.

Substitute $p = 3$ into (1):

$(3)^2 - 6(3) + 8 = -a$ M1

$a = -1$

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c. Let $d(x)$ = vertical distance between $f(x)$ and $g(x)$.

$d(x) = f(x) - g(x)$

$= x^3 - (6x^2 + 9x)$ M1

$d'(x) = 3x^2 - 12x + 9$

Let $d'(x) = 0$.

$x = 1$ or $x = 3$ M1

$d(1) = 4$ A1

maximum vertical distance = 4

$\frac{d}{dx}(d(x))$	$3 \cdot x^2 - 12 \cdot x + 9$
$\text{solve}(3 \cdot x^2 - 12 \cdot x + 9 = 0, x)$	$x = 1$ or $x = 3$
$d(x)$	$x^3 - 6 \cdot x^2 + 9 \cdot x$
$d(1)$	4

Question 2 (18 marks)

a. i.

1.4 | 1.5 | 2.1 | *Doc | RAD

$f(x) := 4 \cdot x + 5$	*Done*
$g(x) := x + 1$	*Done*
$\text{solve}(f(x) = g(x), x)$	$x = \frac{-4}{3}$
$f\left(\frac{-4}{3}\right)$	$\frac{-1}{3}$

$\left(-\frac{4}{3}, -\frac{1}{3}\right)$ A1

ii.

$\sqrt{\left(\frac{-4}{3}\right)^2 + \left(\frac{-1}{3}\right)^2}$	$\frac{\sqrt{17}}{3}$

$\frac{\sqrt{17}}{3}$ A1

iii.

$\tan^{-1}(4) - \tan^{-1}(1)$	30.9637565

$31°$ A1

b. i. There is a dilation factor of a from the y-axis and a dilation factor of $\frac{1}{a}$ from the x-axis. A1

ii. $x' = ax \to x = \dfrac{x'}{a}$

$y' = \dfrac{1}{a}y \to y = ay'$ M1

$y = x + 1$

$ay' = \dfrac{x'}{a} + 1$

$y' = \dfrac{1}{a^2}x + \dfrac{1}{a}$

$h(x) = \dfrac{1}{a^2}x + \dfrac{1}{a}$ A1

c. $f(x) = h(x)$

$4x + 5 = \dfrac{1}{a^2}x + \dfrac{1}{a}$

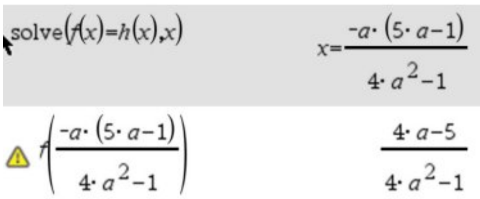
Unique solutions occur where $m_1 \neq m_2$.

$\dfrac{1}{a^2} \neq 4$

$a \neq \pm\dfrac{1}{2}$ M1

But $a \in R^+$ and therefore unique solutions occur for $a \in (0, \infty) / \left\{\dfrac{1}{2}\right\}$. A1

d.



$\left(\dfrac{-a(5a-1)}{4a^2-1}, \dfrac{4a-5}{4a^2-1}\right)$

x-coordinate A1
y-coordinate A1

e. Let distance = $d(a)$.

$$d(a) = \sqrt{\left(\frac{-a(5a-1)}{4a^2-1}\right)^2 + \left(\frac{4a-5}{4a^2-1}\right)^2}$$ M1

For min/max, let $d'(a) = 0$. M1

$$a = \frac{\pm\sqrt{21}+5}{4}$$

$$= \frac{\sqrt{21}+5}{4}$$ A1

$$d(a) := \sqrt{\left(\frac{-a\cdot(5\cdot a-1)}{4\cdot a^2-1}\right)^2 + \left(\frac{4\cdot a-5}{4\cdot a^2-1}\right)^2}$$

$$\text{solve}\left(\frac{d}{da}(d(a))=0,a\right)$$

$$a=\frac{-(\sqrt{21}-5)}{4} \text{ or } a=\frac{\sqrt{21}+5}{4}$$

$$\text{fMin}(d(a),a) \qquad a=\frac{\sqrt{21}+5}{4}$$

Note: Students may also determine the correct value of a from a graph of the distance function.

f. **i.** $\text{dom}_p = [-1, 1]$

Transformation T_2 represents a dilation factor of a from the y-axis and, as $a < 0$, there is also a reflection in the y-axis.

$\therefore \text{dom}_q = [a, -a]$ A1

ii. $q:[a,-a]\to R,\ q(x)=\frac{1}{a^2}x+\frac{1}{a}$

$f(x)=q(x)$

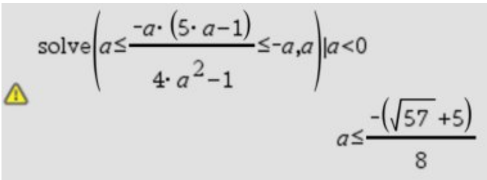
$4x+5=\frac{1}{a^2}x+\frac{1}{a}$

Solution is $\left(\frac{-a(5a-1)}{4a^2-1},\frac{4a-5}{4a^2-1}\right)$ from **part d.** M1

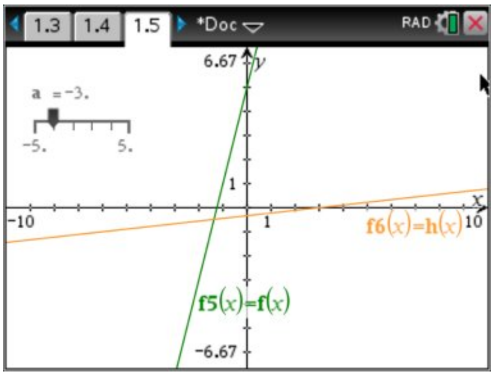
However, the restricted domain of $\text{dom}_q=[a,-a]$.

$\therefore a\le\frac{-a(5a-1)}{4a^2-1}\le -a$ for $a<0$ and $4a^2-1\ne 0$. M1

$a\in\left(-\infty,\frac{-5-\sqrt{57}}{8}\right]$ A1



Note: A graphical approach with sliders can be also used, as shown below.



iii. $q(x)=\frac{1}{a^2}x+\frac{1}{a}$

As $a\to-\infty$, $q(x)\to 0$.

$f(x)=q(x)$

$4x+5=0$

$x=-\frac{5}{4}$

$\therefore m_1=-\frac{5}{4}$ A1

Question 3 (13 marks)

a. **i.** average rate of change $= \dfrac{f(a) - 1}{a - 1}$

$$= \frac{\frac{1}{a} - 1}{a - 1}$$

$$= \frac{\frac{1 - a}{a}}{a - 1}$$

$$= -\frac{1}{a}$$ A1

ii. $f'(a) = -\dfrac{1}{x^2}$ A1

$$-\frac{1}{x^2} = -\frac{1}{a}$$

$$x^2 = a$$

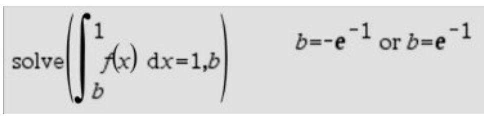
As $x > 0$, $x = \sqrt{a}$.

$\therefore$ point $Q = \left(\sqrt{a}, \dfrac{1}{\sqrt{a}}\right)$ A1

b. **i.**

$$\int_1^e f(x)dx = 1$$ A1

ii.



$b = \dfrac{1}{e}$ as $0 < b < 1$ A1

c. **i.** The area is a trapezium.

$$\frac{1}{2}\cdot\left(1+\frac{1}{a}\right)\cdot(a-1) \qquad \frac{(a-1)\cdot(a+1)}{2\cdot a}$$

$$A = \frac{1}{2}\left(1+\frac{1}{a}\right)\times(a-1)$$ M1

$$= \frac{(a-1)(a+1)}{2a}$$ A1

ii.

$$\text{solve}\left(\frac{(a-1)\cdot(a+1)}{2\cdot a}=1,a\right) \qquad a=-(\sqrt{2}-1) \text{ or } a=\sqrt{2}+1$$

$$\frac{(a-1)(a+1)}{2a} = 1$$ M1

$$a = \sqrt{2}+1$$ A1

iii. The line PA is above the graph of f. The area below the line (the trapezium) has an area of 1 square unit so that $\int_1^a f(x)dx$ must be less than this. The integral $\int_1^e f(x)dx = 1$ so that $\int_1^a f(x)dx$ must be less than 1 and hence $a < e$. A1

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d. If $\int_{k}^{m} f(kx)dx = \frac{1}{k}$, then $\int_{1}^{\frac{m}{k}} f(x)dx = 1$, as k represents a dilation factor of $\frac{1}{k}$ units from the y-axis.

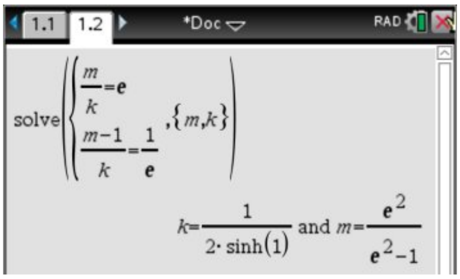
If $\int_{m-1}^{k} f(kx)dx = \frac{1}{k}$, then by the same logic $\int_{\frac{m-1}{k}}^{1} f(x)dx = 1$. M1

In **part b.** it was found that and $\int_{1}^{e} f(x)dx = 1$ and $\int_{\frac{1}{e}}^{1} f(x)dx = 1$.

Therefore $\frac{m}{k} = e$ and $\frac{m-1}{k} = \frac{1}{e}$. M1

Solve simultaneously to give $m = \frac{e^2}{e^2 - 1}$ and $k = \frac{e}{e^2 - 1}$. A1

Note: CAS gives k using hyperbolic sine, which is not permitted in the VCAA exams. Students can use the CAS but then should be able to substitute m back into the equation $\frac{m}{k} = e$ *to find the acceptable version for k.*



Question 4 (10 marks)

a. **i.** $X \sim Bi\left(5, \frac{1}{5}\right)$

$$\Pr(X = 0) = \left(\frac{4}{5}\right)^5$$

$$= \frac{1024}{3125}$$ A1

ii. $$\Pr(X \geq 3 \mid X \geq 1) = \frac{\Pr(X \geq 3 \cap X \geq 1)}{\Pr(X \geq 1)}$$

$$= \frac{\Pr(X \geq 3)}{1 - \Pr(X = 0)}$$ M1

$\text{binomCdf}\left(5, \frac{1}{5}, 3, 5\right)$ 0.05792

$$\Pr(X \geq 3 \mid X \geq 1) = \frac{0.05792}{\left(1 - \frac{1024}{3125}\right)}$$

$$= 0.0861$$ A1

b. $$\Pr(Y = 4) + \Pr(Y = 5) = 11\Pr(Y = 5)$$

$$\Pr(Y = 4) = 10\Pr(Y = 5)$$

$${}^4C_4 p^4 (1 - p)^1 = 10\,{}^5C_4 p^5 (1 - p)^0$$

$$5p^4(1 - p) = 10p^5$$ M1

$$p^4(1 - p) = 2p^5$$

$1 - p = 2p$ as $p \neq 0$

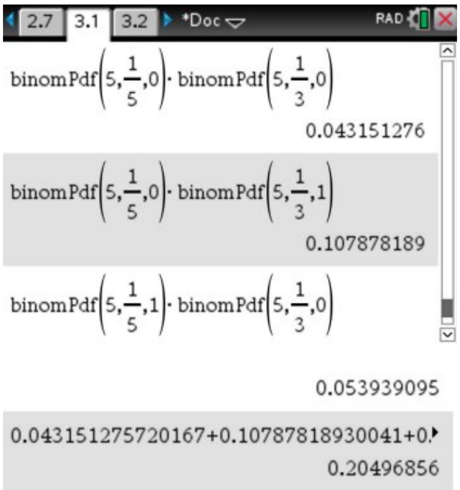
$$3p = 1$$

$$p = \frac{1}{3}$$ A1

c. $\Pr((X+Y)<2) = \Pr((X+Y)=0) + \Pr((X+Y)=1)$

$= \Pr(X=0)\times\Pr(Y=0) + \Pr(X=0)\times\Pr(Y=1) + \Pr(X=1)\times\Pr(Y=0)$ M1

$= 0.2050$ A1



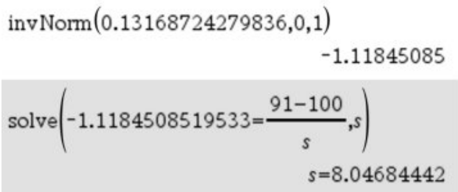
d. $\Pr(Y \geq 1) = 0.8683$ M1

$\Pr(T>91) = 0.8683\ldots$

$\Pr(T<91) = 1 - 0.8683\ldots$

$= 0.1316\ldots$

$z = \dfrac{x-\bar{x}}{\sigma}$ M1



$\sigma = 8.0468$ A1

Question 5 (12 marks)

a. **i.** $x \in (-1, \infty)$ A1

ii. $y \in [-1, 1]$ A1

b. Let $\sin(\log_e(x+1)) = 0$.

$\log_e(x+1) = n\pi,\ \ n \in Z$ M1

$n = 0 \rightarrow \log_e(x+1) = 0 \rightarrow x = 0$

$n = 1 \rightarrow \log_e(x+1) = \pi \rightarrow x + 1 = e^{\pi} \rightarrow x = e^{\pi} - 1$ A1

Therefore $(e^{\pi} - 1, 0)$ is the first positive x-intercept.

c. **i.** $g(x) = f(x-1)$

$f(x) = \sin(\log_e(x+1))$

$g(x) = \sin(\log_e(x-1+1))$ M1

$\therefore g(x) = \sin(\log_e(x))$ as required

ii. $g(x) = f(x-1)$

The graph of $y = g(x)$ is a translation of 1 unit right from the graph of $y = f(x)$.

Therefore $(1, 0)$ and $(e^{\pi}, 0)$ are the required x-intercepts of $y = g(x)$. A1

d. $\frac{d}{dx}x(\sin(\log_e(x) - \cos(\log_e(x))))$

$= 1 \times (\sin(\log_e(x)) - \cos(\log_e(x)) + x \times \left(\cos(\log_e(x)) \times \frac{1}{x} + \sin(\log_e(x)) \times \frac{1}{x}\right)$

$= 2\sin(\log_e(x))$ A1

$\rightarrow \int 2\sin(\log_e(x))\,dx = x(\sin(\log_e(x)) - \cos(\log_e(x)))$ M1

$\rightarrow \int \sin(\log_e(x))\,dx = \frac{x}{2}(\sin(\log_e(x)) - \cos(\log_e(x)))$ A1

e. $\int_0^{e^{\pi}-1} f(x)\,dx = \int_1^{e^{\pi}} g(x)\,dx$ M1

$\int_1^{e^{\pi}} g(x)\,dx = \left[\frac{x}{2}(\sin(\log_e(x)) - \cos(\log_e(x))\right]_1^{e^{\pi}}$ M1

$\int_0^{e^{\pi}-1} f(x)\,dx = \frac{1}{2}(e^{\pi} + 1)$ A1

$$\int_1^{e^{\pi}} g(x)\,dx \qquad \frac{e^{\pi}}{2} + \frac{1}{2}$$

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