NAME: ____________________________________ TEACHER: _______________

MELBOURNE HIGH SCHOOL

MATHEMATICAL METHODS UNIT 4

Formative Assessment Task

Dear Mathematical Methods students,

This is a formative-assessment task that forms part of your school-assessed coursework for Methods 3/4. It is designed to prompt you to investigate a section of the course in greater depth.

This is the **only** formative task for the Unit 4 SAC. We are anticipating the assessment data from this task may be validated after normal school operations resume. This validation could take the form of an assessment task run concurrently under exam condition for all students.

We encourage you to examine the material in detail – investigating, exploring, and testing conjectures – rather than completing the tasks in a rush with little thought. Do not copy somebody else's responses, because you will then not only break MHS and VCAA rules for school-assessed coursework (thereby exposing yourself to the consequences of doing so) but also forfeit the opportunities the task provides for you to learn concepts and practice mathematical and other skills that will be useful later in the course and in your university studies.

Please submit your completed task via Canvas. Either print a hard copy, complete it by hand, and submit a high-quality scan in pdf format (*not* a photograph taken with a mobile-phone camera), or else write in this document electronically using your touchscreen laptop and stylus and submit your work as a pdf via Canvas.

We encourage you to use any suitable technology. Consider using Mathematica (which is free to all Victorian students under licence from DET) and Desmos in addition to, or instead of, your CAS calculator. You may also choose to use the TI CAS computer software instead of the handheld calculator.

Kind regards,

Ms Rawson and the Mathematical Methods Team

IMPORTANT INFORMATION ABOUT THIS TASK

Please read the following notes carefully, which are specific to this task. Do not take the notes below as general assessment advice. In particular, VCAA mark their exams according to specific rules about exact values and decimal places that you should be familiar with.

1. In this task you will be working with some quantities that cannot be expressed as exact values. For every question in this task, unless otherwise stated, it is up to you to make decisions about how to express your answers. At certain times you may believe that an exact value is the most appropriate way to express your answer and other times you may believe that a rounded decimal value is more appropriate given the context of the question. For the latter, it is up to you to determine how many decimal places it is appropriate to show.

   There will be certain quantities you find during this task that will have to be expressed as a rounded decimal since it is not possible to find an exact value.

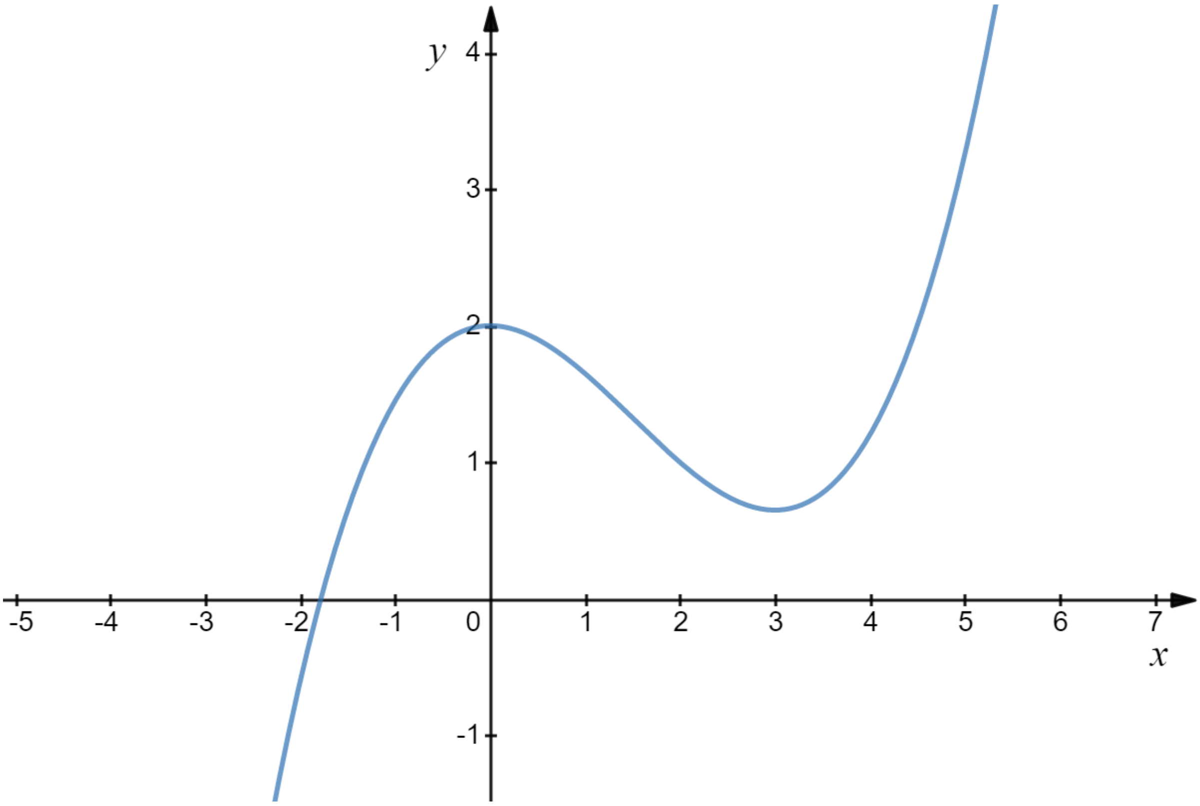
Do not contact your teacher or Ms Rawson to ask about decimal places or exact values in this task. All the relevant information is written above in note 1.

2. You should be using technology throughout the task, including to help you sketch graphs. However, all sketches in this task must be done by hand, either on your device using a stylus or on paper with a pen or pencil. You will be marked on the quality of the graphs you present. You can (and should) use the TI-Nspire, Desmos, or a similar graphics calculator to help you sketch by hand, but you cannot attach an image of a graph from one of these graphics calculators.

PART 1: APPROXIMATING AN AREA

A section of the graph of $y = f(x)$ is shown below where

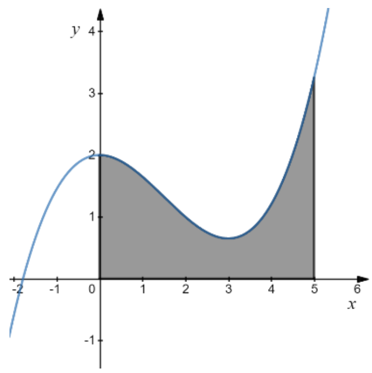
$$f : \mathbb{R} \to \mathbb{R}, \qquad f(x) = \frac{1}{20}(2x^3 - 9x^2 + 40)$$



Question 1: Calculating the exact area

Show that the area bound by the curve $y = f(x)$, the x-axis, and the vertical lines $x = 0$ and $x = 5$ is 6.875. Show full working out, including all relevant steps using algebra and calculus.

The desired area is shown in the image to the right.



$$Area = \int_0^5 \frac{1}{20}\left(2x^3 - 9x^2 + 40\right)dx$$

$$= \frac{1}{20}\left[\frac{x^4}{2} - 3x^3 + 40x\right]_0^5$$

$$= \frac{1}{20}\left(\frac{625}{2} - 375 + 200\right)$$

$$= \frac{55}{8} \quad or \quad 6.875$$

Question 2: Approximating the area

a) Fill in the missing values in the table below, writing each value as a decimal. This table will be useful in completing the rest of this question.

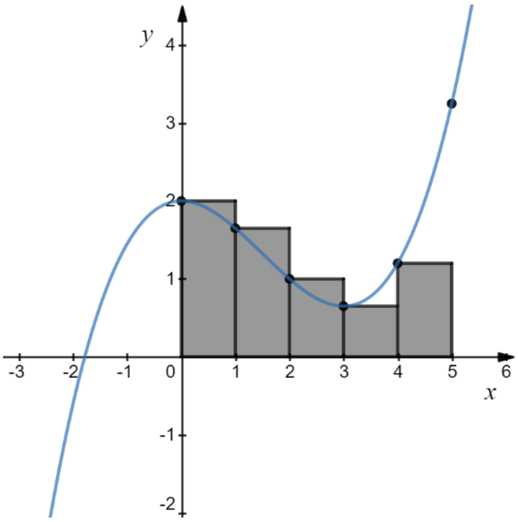
x	0	1	2	3	4	5
$f(x)$	2	1.65	1	0.65	1.2	3.25

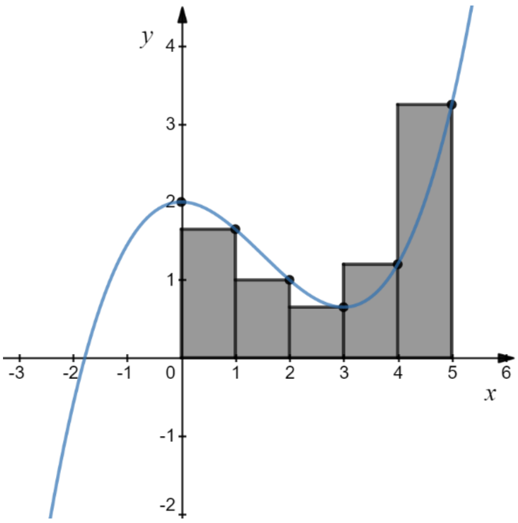
Note that each decimal in the table is also an exact value (that is, it is a decimal that terminates).

b) Areas under curves can be approximated with rectangles.

The left Riemann sum uses rectangles with heights determined by the lower value of x for each interval. The graph below and left shows the left Riemann sum (with rectangles of width 1) for approximating the area bound by the curve $y = f(x)$, the x-axis, and the vertical lines $x = 0$ and $x = 5$.

The right Riemann sum uses rectangles with heights determined by the higher value of x for each interval. The graph below and right shows the right Riemann sum (with rectangles of width 1) for approximating the area bound by the curve $y = f(x)$, the x-axis, and the vertical lines $x = 0$ and $x = 5$.





The left Riemann sum is the sum of the five rectangles shown on the graph above and left. It can be calculated as follows:

$$(1-0) \times f(0) + (2-1) \times f(1) + (3-2) \times f(2) + (4-3) \times f(3) + (5-4) \times f(4) = 6.5.$$

i) Calculate the right Riemann sum. Show your work by calculating the area of each individual rectangle.

ii) The right Riemann sum is greater than the left Riemann sum. Explain how you could determine this without calculating the left and right Riemann sums, but just by comparing the first rectangle of the left Riemann sum and the last rectangle of the right Riemann sum.

$$(1-0) \times f(1) + (2-1) \times f(2) + (3-2) \times f(3) + (4-3) \times f(4) + (5-4) \times f(5) = 7.75.$$

The left Riemann sum is $f(0) + f(1) + f(2) + f(3) + f(4)$

The right Riemann sum is $f(1) + f(2) + f(3) + f(4) + f(5)$

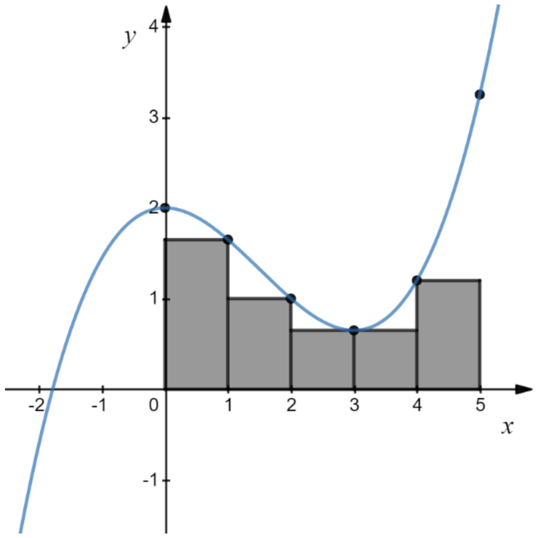
This means the only difference is that the right sum has $f(5)$ instead of $f(0)$ so we could tell that the right Riemann sum is greater simply because $f(5) > f(0)$.

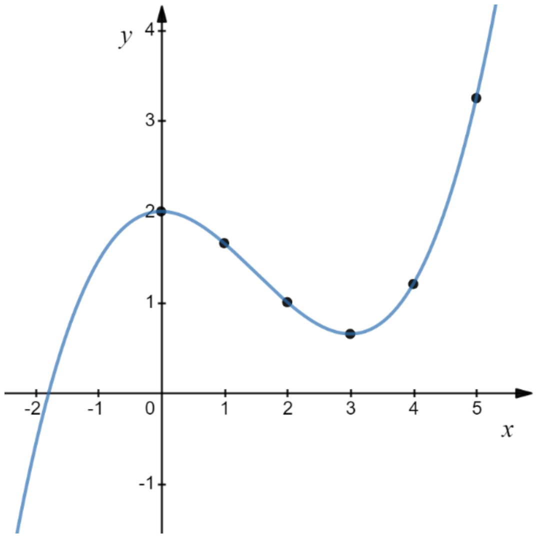
c) Overestimating and underestimating with rectangles.

When an exact area cannot be found, in certain contexts it may be useful to find an upper bound or a lower bound by deliberately overestimating or underestimating the area. This can be done with rectangles by always selecting the rectangle that overestimates or underestimates the area under the curve for each interval.

For example, the graph below and left would be an intentional underestimate of the area using rectangles of width 1.

i) On the graph below and right, draw in and shade the rectangles of width 1 that would intentionally overestimate the area.





ii) Calculate the area of the intentional underestimate. Show your work by calculating the area of each individual rectangle.

iii) Calculate the area of the intentional overestimate. Show your work by calculating the area of each individual rectangle.

i) $(1-0)\times f(1)+(2-1)\times f(2)+(3-2)\times f(3)+(4-3)\times f(3)+(5-4)\times f(4)=5.15.$

ii) $(1-0)\times f(0)+(2-1)\times f(1)+(3-2)\times f(2)+(4-3)\times f(4)+(5-4)\times f(5)=9.1.$

d) Improving your estimation

i) Find the average of the intentional overestimate and the intentional underestimate. Show your calculation.

ii) Find the average of the left Riemann sum and the right Riemann sum. Show your calculation.

iii) Your answers to i) and ii) should be the same. Will it always be the case that the average of the left and right Riemann sums is equal to the average of the overestimate and the underestimate? If yes, explain why, using words and diagrams. If not, provide a counterexample.
In your response, you only need to consider bounded areas that are entirely above the x-axis.

i) Over and under average $= \frac{1}{2}(9.1 + 5.15) = 7.125$

ii) Average of left and right Riemann sum $= \frac{1}{2}(6.5 + 7.755) = 7.125$

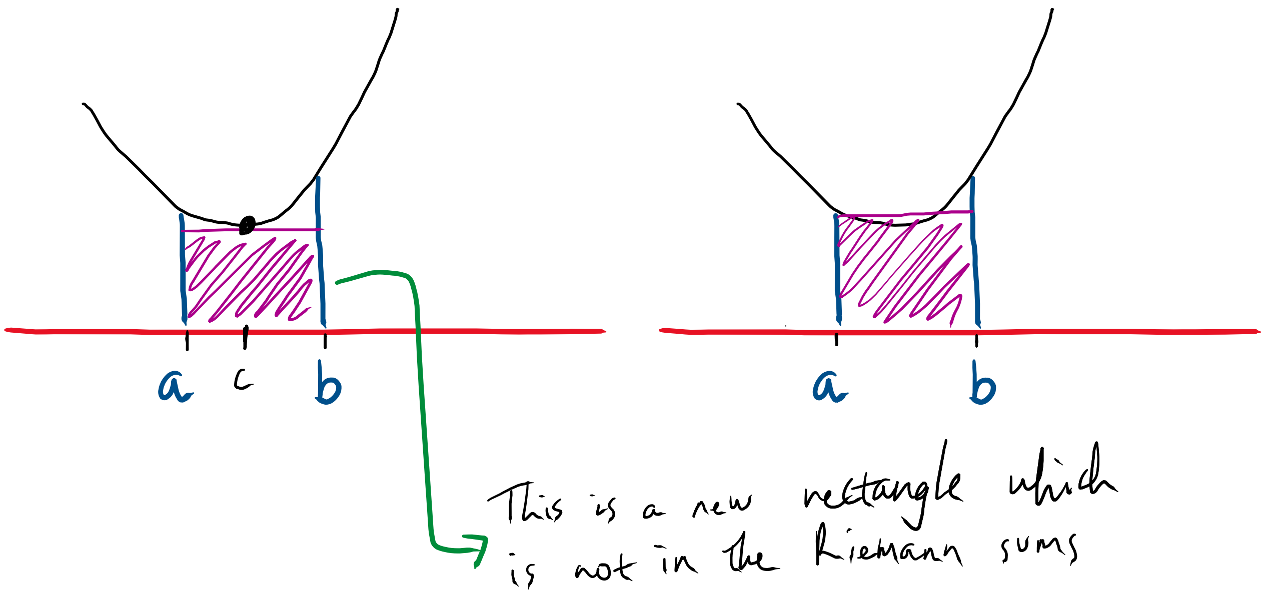
iii) For graphs like the one shown, the averages will be the same. That is, the turning point/s must be where the rectangles start and end. Then the averages will be the same because, in both the left and right Riemann sums and in the intentional over and underestimations, the middle rectangles will all be chosen twice and the first and last rectangle with each be chosen once.

However, if we consider the case where a turning point exists inside an interval then we may not be able to choose a rectangle that underestimates or overestimates the area for that interval.

For example, let's say there is a minimum turning point inside the rectangle interval $a \leq x \leq b$. If a rectangle with height $f(a)$ or $f(b)$ could underestimate the area for that interval then it could be chosen. In this case, the two averages will be the same. However, we could have a scenario where rectangles with heights $f(a)$ and $f(b)$ both overestimate the area for the interval. Then we would need to resolve the problem, either by

- choosing rectangle widths such that the rectangle boundaries are at the turning point, or
- defining the height of the underestimation rectangle to be the height of the minimum point over the interval, regardless of whether that minimum is on the rectangle boundaries.

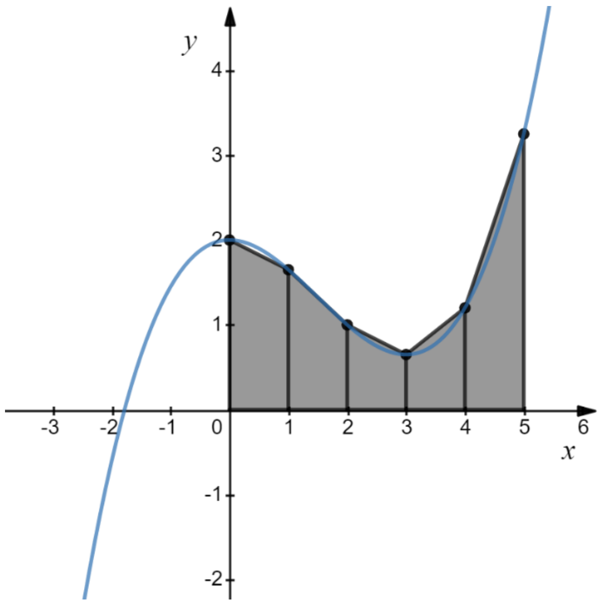
Both of these resolutions will mean that the two averages could be (almost always will be) different because new rectangles will be used. An illustration of the second resolution is shown below.



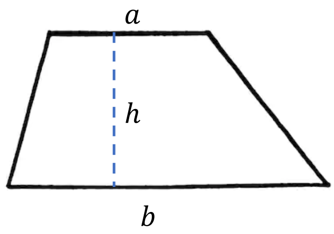
e) Trapezium estimation

Trapeziums are another shape that can be used to approximate areas under curves. A trapezium is a quadrilateral with a pair of parallel sides. Any quadrilateral that joins the points $(a, 0)$, $(b, 0)$, $(b, f(b))$ and $(a, f(a))$ where $f(a) > 0$ and $f(b) > 0$ will be a trapezium since it will include two vertical lines.

The graph below shows the trapezium method for approximating the area bound by the curve $y = f(x)$, the x-axis, and the vertical lines $x = 0$ and $x = 5$, using trapeziums with heights of 1 (we use the word "height" now, instead of "width," because the height of a trapezium is defined as the perpendicular distance between its parallel sides).



The **area of a trapezium** is $\frac{h(a+b)}{2}$ where a and b are the lengths of the parallel sides and h is the perpendicular distance between the two parallel sides.



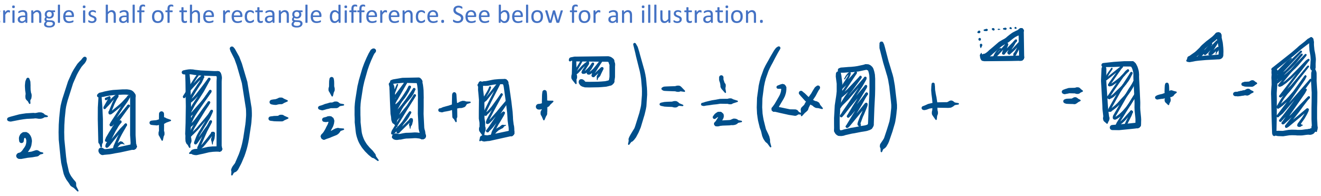
i) Calculate the trapezium estimate for the area bound by the curve $y = f(x)$, the x-axis, and the vertical lines $x = 0$ and $x = 5$ using trapeziums with heights of 1. Show your work by calculating the area of each of the five trapeziums.

Total area = $\frac{1\times(f(0)+f(1))}{2} + \frac{1\times(f(1)+f(2))}{2} + \frac{1\times(f(2)+f(3))}{2} + \frac{1\times(f(3)+f(4))}{2} + \frac{1\times(f(4)+f(5))}{2} = 7.125$

ii) Your trapezium estimate should be equal to the average of the left Riemann sum and the right Riemann sum. Will this always be the case? If yes, explain why, using words and diagrams. If not, provide a counterexample. In your response, you only need to consider bounded areas that are entirely above the x-axis.

It is the same and yes, it will always be the case.

This is because each trapezium used in this method is a rectangle with a triangle at the top. If we average out the two rectangles used for each interval then the rectangle part of the trapezium is the common area of the two rectangles and the triangle is half of the rectangle difference. See below for an illustration.



f) Improving your trapezium approximation

i) Using trapeziums with heights of 0.5, find the trapezium approximation for the area bound by the curve $y = f(x)$, the x-axis, and the vertical lines $x = 0$ and $x = 5$. Show your work by calculating the area of each trapezium.

x	*0*	*0.5*	*1*	*1.5*	*2*	*2.5*	*3*	*3.5*	*4*	*4.5*	*5*
f(x)	*2*	*1.9*	*1.65*	*1.325*	*1*	*0.75*	*0.65*	*0.775*	*1.2*	*2*	*3.25*

$$Total\ Area = \frac{\frac{1}{2}\times\left(f(0)+f(0.5)\right)}{2} + \frac{\frac{1}{2}\times\left(f(0.5)+f(1)\right)}{2} + \frac{\frac{1}{2}\times\left(f(1)+f(1.5)\right)}{2} + \frac{\frac{1}{2}\times\left(f(1.5)+f(2)\right)}{2}$$
$$+ \frac{\frac{1}{2}\times\left(f(2)+f(2.5)\right)}{2} + \frac{\frac{1}{2}\times\left(f(2.5)+f(3)\right)}{2} + \frac{\frac{1}{2}\times\left(f(3)+f(3.5)\right)}{2} + \frac{\frac{1}{2}\times\left(f(3.5)+f(4)\right)}{2}$$
$$+ \frac{\frac{1}{2}\times\left(f(4)+f(4.5)\right)}{2} + \frac{\frac{1}{2}\times\left(f(4.5)+f(5)\right)}{2} = 6.9375$$

ii) Comment on the accuracy of your approximation in f), i). Compare it with your approximation from e), i). Also compare it with the exact area.

The approximation is extremely accurate. The exact area is 6.875 and this area is 6.9375 which is overestimating by less than 1%.

Using trapeziums with lower heights improved the approximation greatly, going from 7.125 when heights of 1 were used, to 6.9375 when heights of 0.5 were used.

Question 3: Transforming the area

A property of definite integrals is that $\int_a^b kg(x)dx = k\int_a^b g(x)dx$ for any continuous function g, and real constants a, b, and k.

a) Let m be a positive real number.
Using trapeziums with heights of 1, show that the trapezium approximation for the area under $y = mf(x)$ between $x = 0$ and $x = 5$ is equal to m times the trapezium approximation of the area under $y = f(x)$ between $x = 0$ and $x = 5$. Show your work by calculating the area of each individual trapezium. Include a diagram in your working out.

Total area = $\frac{1\times(m\times f(0)+m\times f(1))}{2}+\frac{1\times(m\times f(1)+m\times f(2))}{2}+\frac{1\times(m\times f(2)+m\times f(3))}{2}+\frac{1\times(m\times f(3)+m\times f(4))}{2}+\frac{1\times(m\times f(4)+m\times f(5))}{2}$

$$= m\times\left[\frac{1\times\left(f(0)+f(1)\right)}{2}+\frac{1\times\left(f(1)+f(2)\right)}{2}+\frac{1\times\left(f(2)+f(3)\right)}{2}+\frac{1\times\left(f(3)+f(4)\right)}{2}+\frac{1\times\left(f(4)+f(5)\right)}{2}\right]$$

$$= m\times 7.125$$

Another property of definite integrals is that $\int_a^b\left(g(x)\pm h(x)\right)dx = \int_a^b g(x)dx \pm \int_a^b h(x)dx$ for continuous functions g and h, and for real constants a and b.

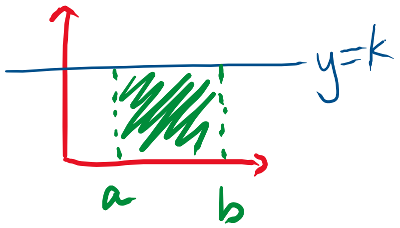
b) Area between curves

i) Show that $\int_a^b k\,dx = k(b-a)$ for real constants a, b, and k.

$$\int_a^b k\,dx = [kx]_a^b = k(b) - k(a) = k(b-a)$$

ii) Interpret the above integral geometrically. What does the bounded area look like when sketched on the Cartesian plane? For this question, assume that $k > 0$ and that $b > a$. Also, include a diagram.

$y = k$ is a horizontal line. Integrating between $x = a$ and $x = b$ gives a rectangle with a height of k and a width of $b - a$.

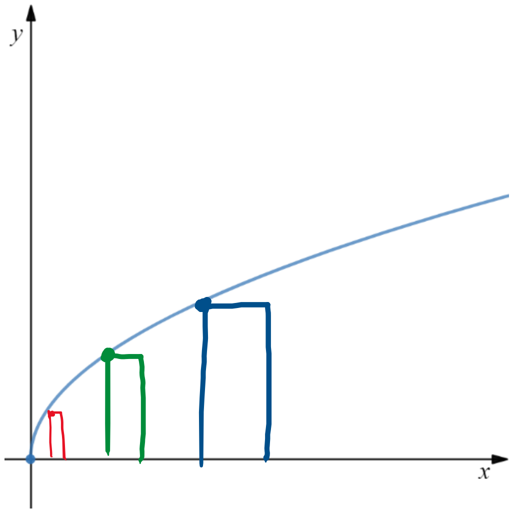


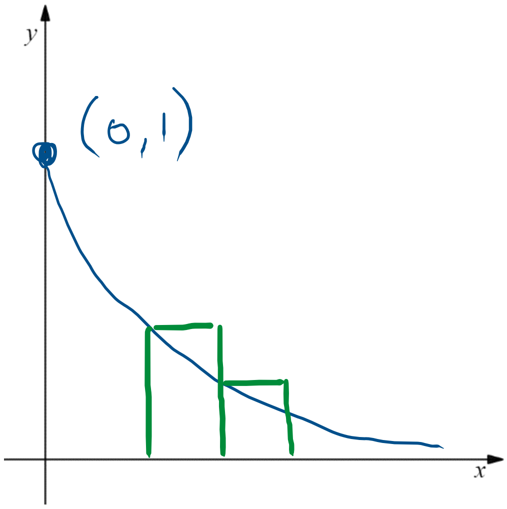
iii) Explain, in words and diagrams, why the trapezium approximation for the area under $y = f(x) + 3$ between $x = 0$ and $x = 5$ is 15 more than the trapezium approximation for the area under $y = f(x)$ between $x = 0$ and $x = 5$.

First find the trapezium approximation for $y = f(x)$ between $x = 0$ and $x = 5$. Then the graph of $y = f(x) + 3$ is the graph of $y = f(x)$ shifted up by 3. Therefore, the area under the curve for $y = f(x)$ between $x = 0$ and $x = 5$ is going to be shifted up 3 units, creating a 3 by 5 rectangle underneath it. Then the area under $y = f(x) + 3$ will be the area under $y = f(x)$ plus that rectangle which has area 15.

Question 4: The effect of graph shape on area approximations

Below and left is the graph of $y = 2\sqrt{x}$





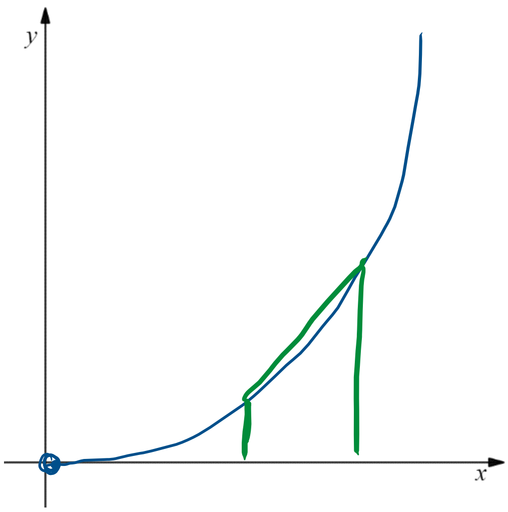
a) Explain, in words and diagrams, why the left Riemann sum will always underestimate the area under the curve for $y = 2\sqrt{x}$, regardless of the size of the rectangles.

The graph is increasing, so the left x-value of any interval will have a corresponding y-value that is lower than for the right x-value.

b) On the blank axes above and right, sketch a continuous graph such that the area under the curve will always be overestimated by the left Riemann sum, regardless of the size of the rectangles. For the sake of simplicity, you must choose a graph which is always above or on the x-axis.
Write the rule for the equation you chose in the space provided below.

$y = e^{-x}$

c) On the blank axes below, sketch a continuous graph such that the area under the curve will always be overestimated by the trapezium method, regardless of the heights of the trapeziums. For the sake of simplicity, you must choose a graph which is always above or on the x-axis.
Beside the graph, write the rule for the equation you chose and explain what features of a curve must be present for the trapezium method to overestimate the area.



$y = x^2$

The curve must be concave up. That means the graph is either

- Increasing with the gradient increasing, or
- Decreasing with the gradient increasing

PART 2: TRANSCENDENTAL GRAPHS

Certain functions that are studied in Methods are "transcendental", meaning they cannot be expressed as a finite series of algebraic operations involving x. Examples include e^x and $\sin(x)$.

It is not always possible to solve for the intersections between polynomials and transcendentals using algebraic methods, and instead, graphical methods are used. For example, the equation $2^x = x^2$ has three solutions: $x = 2$ and $x = 4$, which can be found by inspection or through trial and error, and $x \approx -0.7$, which cannot be expressed as an exact value and therefore can only be approximated using algorithms.

Question 1: Finding intersections and areas

For a quadratic and a linear equation, the intersections, if they exist, can always be expressed as an exact value. However, for a polynomial and a transcendental function, this is not necessarily true.

a) The graphs of $y = e^x + 4$ and $y = 2x + e^2$ intersect twice.

i) Find the coordinates of one of these points of intersection by inspection (or trial and error). Show or explain your reasoning.

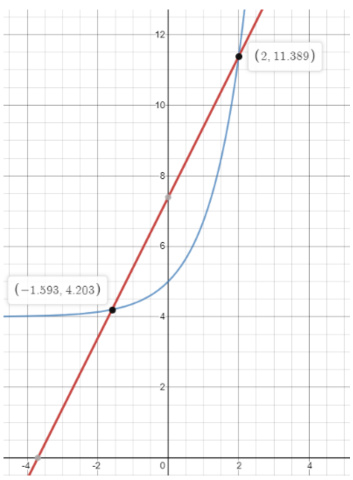
$$(2, e^2 + 4)$$

By inspection, we can see that if $x = 2$ then each y-value will include e^2. Then we can check that the constants match, too, which they do.

ii) Find the coordinates of the other point of intersection using technology and write it in the space provided below.

$$(1.593, 4.203)$$

iii) Sketch the two equations below on the same set of axes, labelling all relevant points and features.



Note: students must draw their graph

iv) Find the area bound between these two curves. Write the integral expression/s that you used to calculate the area.

$$\int_{-1.593}^{2} \left(2x + e^2 - (e^x + 4)\right)dx \approx 6.4535$$

b) The graphs of $y = e^x$ and $y = -x^2 + 2x + 1$ intersect twice.

i) Find the coordinates of one of these points of intersection by inspection (or trial and error). Show or explain your reasoning.

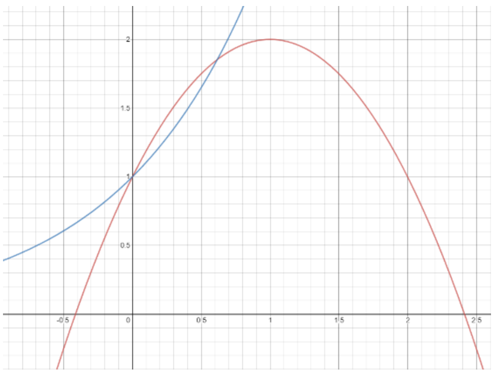
$$(0,1)$$

By inspection, we can see that the only way to "disappear" the e is if $x = 0$, so we check $x = 0$ and it gives the same y-value for each equation.

ii) Find the coordinates of the other point of intersection on your calculator and write it in the space provided below.

$$(0.617, 1.853)$$

iii) Sketch the two equations below on the same set of axes, labelling all relevant points and features.



Note : students must draw their graph

iv) Find the area bound between these two curves. Write the integral expression/s that you used to calculate the area.

$$\int_0^{0.617} (-x^2 + 2x + 1 - e^x)dx \approx 0.0660$$

Question 2: Approximating an unknowable area

Consider the relation $(x+2)\sqrt{4-x}-\sqrt{y}=2\log_e(y+1)$

a) The axes intercepts

i) The two x-intercepts are $(-2,0)$ and $(4,0)$. Show that these can be found with algebra.

Let $y=0$

$$(x+2)\sqrt{4-x}-\sqrt{0}=2\log_e(0+1)$$
$$(x+2)\sqrt{4-x}=0$$
$$x+2=0 \quad or \quad \sqrt{4-x}=0$$
$$x=-2 \quad or \quad x=4$$

ii) Use technology to find the y-intercept. Write it in the space provided below.

$(0,2.4036)$

b) Finding points on the graph

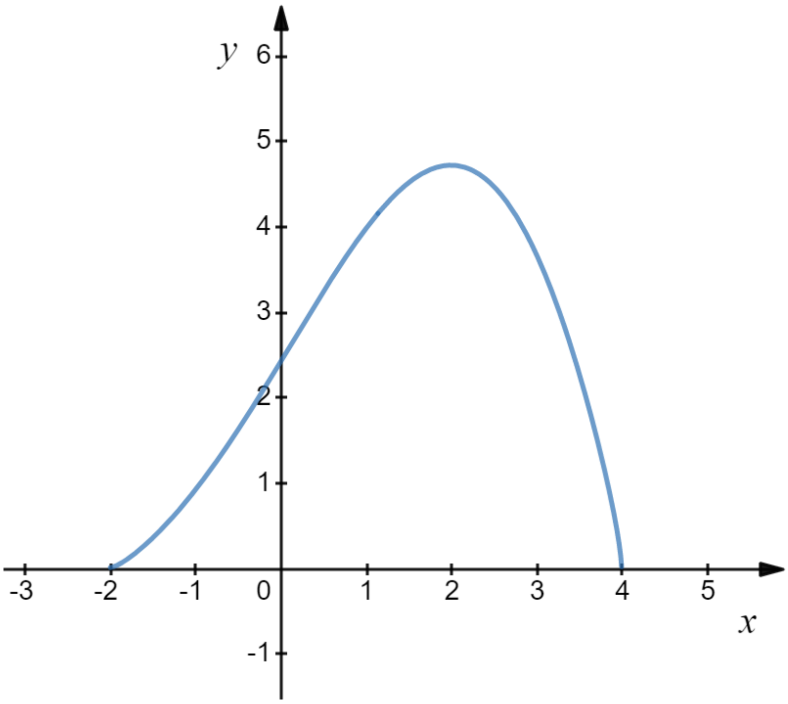
i) Use technology to find the value of y when $x=1$

ii) Use technology to find the values of x when $y=1$

$(1,3.965)$
$(-0.925,1)$
$(3.833,1)$

The curve of $(x+2)\sqrt{4-x}-\sqrt{y}=2\log_e(y+1)$ is shown below.

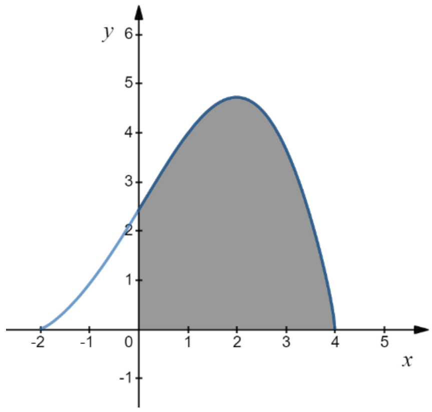
c) Plot the points you found in part b on the curve, along with their coordinates.



d) Approximating an area

There is no way to find an exact area under this curve.

i) Use an appropriate method involving geometry (rectangles, triangles, trapeziums) to estimate the area bound between the curve and the x and y axes (as shown). Write one or two sentences explaining your method and clearly show the calculations you used to find your approximation.



x	0	0.5	1	1.5	2	2.5	3	3.5	4
y	2.40358	3.22394	3.96515	4.50529	4.71373	4.46760	3.67284	2.28365	0

Description of method

(I used a spreadsheet on the TI, with a column for x-values and another column for y-values, and then did the relevant arithmetic on the cells to calculate the area of each trapezium and then sum all of them)

Trapezium method, with heights of 0.5

$$Area = \frac{0.5 \times (y_0 + y_{0.5})}{2} + \frac{0.5 \times (y_{0.5} + y_1)}{2} + \cdots + \frac{0.5 \times (y_{3.5} + y_4)}{2}$$
$$= 14.017$$

(Trapeziums with heights 0.1 gives 14.192)

(Trapeziums with heights 0.01 gives 14.203)

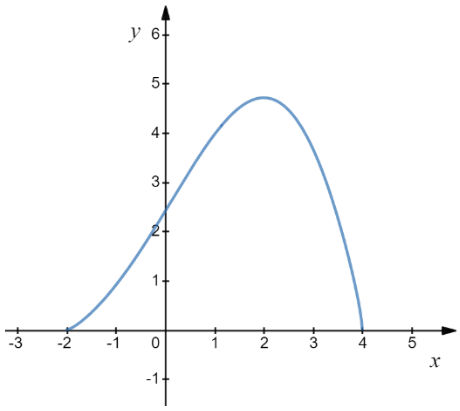
ii) Comment on the accuracy of your approximation from i). Include whether you believe your approximation is an overestimation or underestimation and why.

Estimation is quite accurate because of the short heights of the trapeziums. However, since the exact area is unknown, it is not possible to compare it and get a good sense of the accuracy.

The approximation is probably an underestimation because every trapezium will underestimate the area in that interval. This is because before the TP, the graph looks like its gradient is decreasing while the graph is increasing, and after the TP, the graph is decreasing while the gradient is decreasing.

The maximum point on the curve is approximately $(2, 4.7)$.

iii) Given this information, use one or more quadratics to find an approximation for the same shaded area from part i). Write one or two sentences explaining your method and show your work by writing the rule(s) for the quadratic(s) you used. Also, show the calculations you used to decide on the coefficients of your quadratic(s) and sketch the quadratic(s) you use on top of the relation on the graph provided.



Method

Try to match two quadratics to the curve, one on each side of the TP.

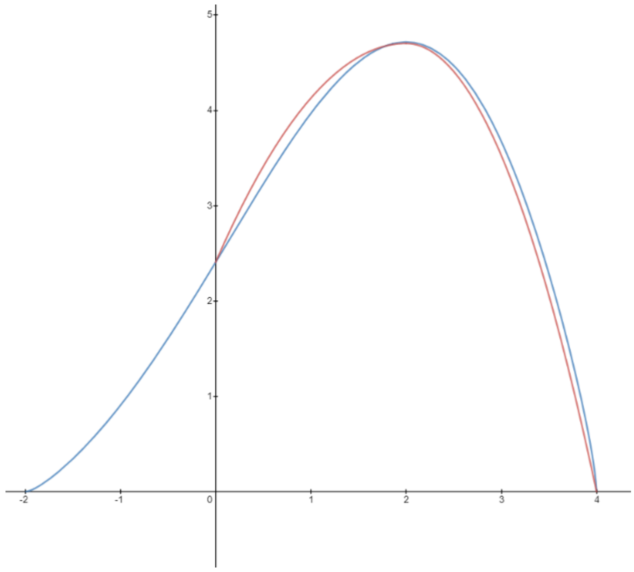
Left quadratic: $y = a(x-2)^2 + 4.7$

To make it go through the y-int, $2.40358 = a(0-2)^2 + 4.7$ gives $a = -0.574$

Note : students must draw their graphs

Right quadratic: $y = b(x-2)^2 + 4.7$

To make it go through the x-int, $0 = b(4-2)^2 + 4.7$ gives $b = -1.175$



Approximated area is therefore

$$\int_0^2 (-0.574(x-2)^2 + 4.7)dx + \int_2^4 (-1.175(x-2)^2 + 4.7)dx$$

14.138

iv) Comment on the accuracy of your approximation from iii). Include whether you believe your approximation is an overestimation or underestimation and why.

The approximation is similar to the trapezium approximation, but the method isn't as strong. This is because the quadratic doesn't follow the curve very well.

The approximation looks to be a slight underestimation but it is hard to tell. The left quadratic slightly overestimates the area and the right quadratic slightly underestimates the area so they look to possibly balance each other out well.