



## *YEAR 12 Trial Exam Paper*

# 2020

# MATHEMATICAL METHODS

## Written examination 2

### *Worked solutions*

#### **This book presents:**

- worked solutions
- mark allocations
- tips.

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**SECTION A – Multiple-choice questions**

<b>Question</b>	<b>Answer</b>
1	<i>D</i>
2	<i>B</i>
3	<i>D</i>
4	<i>B</i>
5	<i>D</i>
6	<i>E</i>
7	<i>B</i>
8	<i>A</i>
9	<i>D</i>
10	<i>C</i>
11	<i>B</i>
12	<i>E</i>
13	<i>C</i>
14	<i>A</i>
15	<i>D</i>
16	<i>B</i>
17	<i>B</i>
18	<i>E</i>
19	<i>B</i>
20	<i>D</i>

**Question 1****Answer: D****Explanatory notes**

Period:

$$\frac{2\pi}{4\pi/5} = \frac{5}{2}$$

Range:

$$-1 \leq \sin\left(\frac{4\pi}{5}x\right) \leq 1$$

$$-2 \leq \sin\left(\frac{4\pi}{5}x\right) \leq 2$$

$$1 \leq 3 - 2 \sin\left(\frac{4\pi}{5}x\right) \leq 5$$

So the period and range is  $\frac{5}{2}$  and  $[1, 5]$  respectively.

**Tip**

- The period of both  $\sin(n\pi)$  and  $\cos(n\pi)$  is  $\frac{2\pi}{n}$ .

**Question 2****Answer: B****Explanatory notes**

Multiplying the first equation by  $k$  and the second by 2 gives

$$2kx + k^2y = 5k$$

$$2kx + 16y = -20$$

Subtracting the second equation from the first equation gives

$$(k^2 - 16)y = 5k + 20$$

If we let  $k = 4$ , then we have  $0y = 40$ , which is inconsistent. If we let  $k = -4$ , then we have  $0y = 0$ , which is consistent.

From this we conclude that no solution exists if  $k = 4$ .

Alternatively, we can use the determinant of the  $2 \times 2$  coefficient matrix

$$\begin{vmatrix} 2 & k \\ k & 8 \end{vmatrix} = 16 - k^2 = 0 \Rightarrow k = \pm 4$$

If  $k = 4$ , then the equations become

$$2x + 4y = 5$$

$$4x + 8y = -10$$

These equations have no solution.

If  $k = -4$ , then the equations become

$$2x - 4y = 5$$

$$-4x + 8y = -10$$

These equations have an infinite number of solutions.

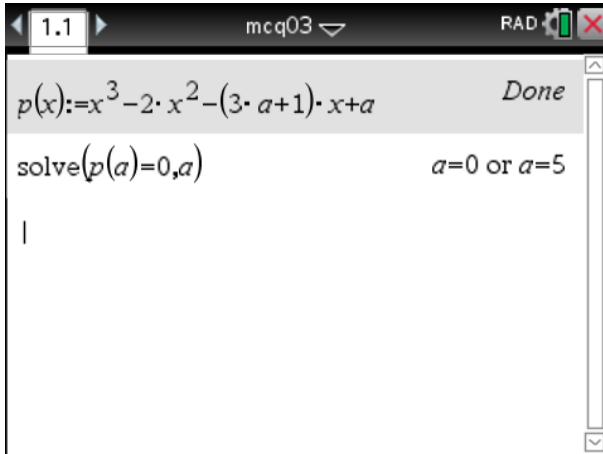
**Tip**

- *Using the determinant is an efficient way of solving these types of problem.*

**Question 3****Answer: D****Explanatory notes**

If  $x - a$  is a factor of  $p(x) = x^3 - 2x^2 - (3a + 1)x + a$ , then  $p(a) = 0$ .

Use CAS to solve  $p(a) = 0$ :



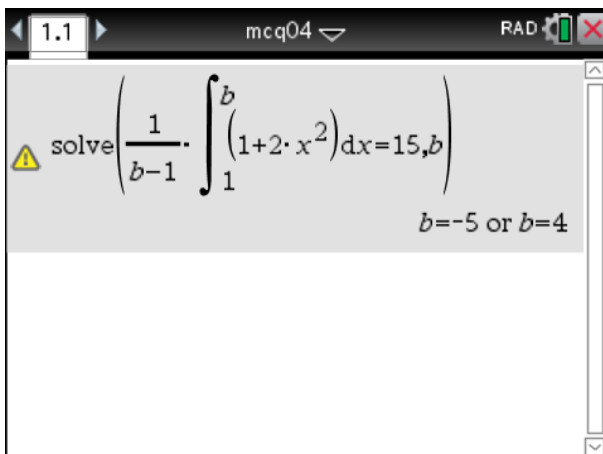
Since  $a \neq 0$ ,  $a = 5$ .

**Question 4****Answer: B****Explanatory notes**

The average value of  $1 + 2x^2$  for the interval  $[1, b]$ ,  $b > 1$  is

$$\frac{1}{b-1} \int_1^b (1 + 2x^2) dx$$

Use CAS to determine the value of  $b$ :



Since  $b > 1$ ,  $b = 4$ .

**Question 5****Answer: D****Explanatory notes**

We require  $f$  to be 1 – 1 for the inverse to exist. Use CAS to find the turning points:

A screenshot of a CAS interface. The title bar shows '1.1' and 'mcq05\_u...ted'. The main area contains the equation  $\text{solve}\left(\frac{d}{dx}(2 \cdot x^3 - x^2 - 20 \cdot x + 5) = 0, x\right)$ . The result is  $x = \frac{-5}{3}$  or  $x = 2$ .

The turning points of the cubic occur when  $x = -\frac{5}{3}$  and when  $x = 2$ .

Therefore,  $f$  is 1 – 1 on the intervals  $\left(-\infty, -\frac{5}{3}\right]$ ,  $\left[-\frac{5}{3}, 2\right)$  and  $[2, \infty)$ . Only the first interval is available as an option.

**Question 6****Answer: E****Explanatory notes**

Define  $f(x) = ax^2(3 - x)$  in CAS and use the **tangentLine** function to find that the equation of the tangent to  $f$  at  $x = 3$  is  $y = 27a - 9ax$ . Since this is in gradient–intercept form, the expression for the  $y$ -intercept can be seen.

If the  $y$ -intercept is 9, then  $27a = 9$  and  $a = \frac{1}{3}$ .

A screenshot of a CAS interface. The title bar shows '1.1' and 'mcq06'. The main area shows the following steps:
 

- $f(x) := a \cdot x^2 \cdot (3 - x)$  (Done)
- $\text{tangentLine}(f(x), x, 3)$   $27 \cdot a - 9 \cdot a \cdot x$
- $\text{solve}(27 \cdot a = 9, a)$   $a = \frac{1}{3}$

**Question 7****Answer: B****Explanatory notes**

Let  $X \sim \text{Bi}(50, 0.20)$ . We want to find the smallest value of  $n$  such that  $\Pr(X \geq n) = 0.3$ .

This can be done using trial and error.

Function	Result
<code>binomCdf(50,0.2,10,50)</code>	0.55626
<code>binomCdf(50,0.2,11,50)</code>	0.416441
<code>binomCdf(50,0.2,12,50)</code>	0.289332

It can be found that  $n = 12$ .

Alternatively, the inverse binomial function can be used. In order to use this function, the equation must first be rearranged:

$$\Pr(X \geq n) = 0.3 \Rightarrow \Pr(X \leq n-1) = 0.7.$$

Then  $n-1 = 11 \Rightarrow n = 12$ .

Function	Result
<code>binomCdf(50,0.2,10,50)</code>	0.55626
<code>binomCdf(50,0.2,11,50)</code>	0.416441
<code>binomCdf(50,0.2,12,50)</code>	0.289332
<code>invBinom(0.7,50,0.2)</code>	11

**Question 8****Answer: A****Explanatory notes**

Rearrange  $y = \frac{-2}{\sqrt{3x-1}} + 1$  into the form  $\frac{y-1}{-2} = \frac{1}{\sqrt{3x-1}}$ .

The image is  $y' = \frac{1}{\sqrt{x'}}$ , so set

$$x' = 3x - 1$$

$$y' = \frac{y-1}{-2} = -\frac{1}{2}y + \frac{1}{2}$$

Therefore the transformation has the form

$$T\left(\begin{bmatrix} x \\ y \end{bmatrix}\right) = \begin{bmatrix} 3 & 0 \\ 0 & -\frac{1}{2} \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix} + \begin{bmatrix} -1 \\ \frac{1}{2} \end{bmatrix}$$



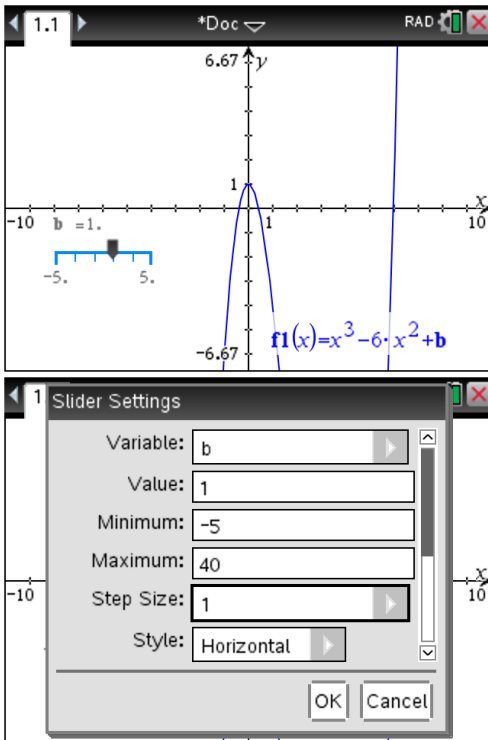
**Question 9****Answer: D****Explanatory notes**

If  $f(x) = x^3 - 6x^2 + b$ ,  $f'(x) = 3x^2 - 12x = 3x(x - 4)$ . Thus, the turning points occur when  $x = 0$  and  $x = 4$ .

Now  $f(0) = b$  and  $f(4) = b - 32$ , from which it follows that  $b > 0$  and  $b - 32 < 0 \Rightarrow b < 32$ .

Therefore there are exactly three  $x$ -intercepts if  $b \in (0, 32)$ .

Alternatively, sketch the function using sliders. Since the possible solutions include the numbers 0 and 32, set up your sliders to cover these values. The solution can then be found visually.

**Tip**

- *It can be helpful to draw a quick sketch of a typical cubic graph.*

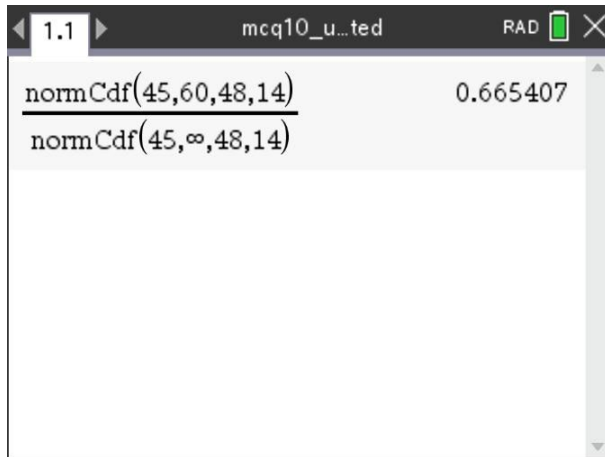
**Question 10****Answer: C****Explanatory notes**

This is a conditional probability question.

Let  $X \sim N(48, 14^2)$ .

The probability that a trip takes less than 60 minutes (1 hour), given that it takes more than 45 minutes, is

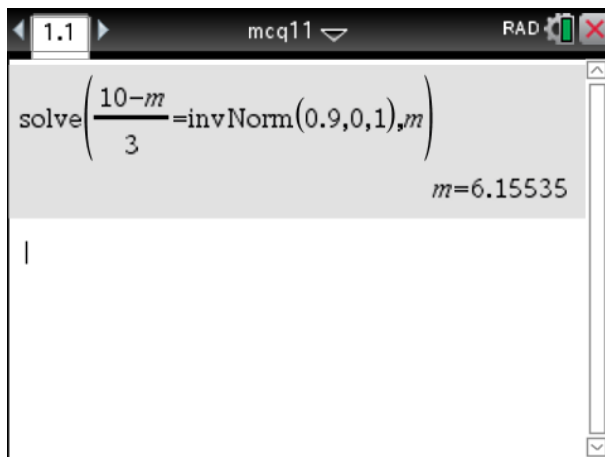
$$\Pr(X < 60 | X > 45) = \frac{\Pr(45 < X < 60)}{\Pr(X > 45)} = 0.6654$$

**Question 11****Answer: B****Explanatory notes**

Let  $X \sim N(\mu, 3^2)$  and  $Z \sim N(0,1)$ .

$$\Pr(X < 10) = 0.9 \text{ so } \Pr\left(Z < \frac{10 - \mu}{3}\right) = 0.9.$$

Use CAS to find that  $\mu = 6.2$  correct to one decimal place.



**Question 12****Answer: E****Explanatory notes**

We have that  $g(2) = f(4) + 3 = -4$  and so  $f(4) = -7$ . Therefore, the graph of  $f$  passes through the point  $(4, -7)$ .

**Question 13****Answer: C****Explanatory notes**

Since  $\int_2^6 f(x) dx = 5$ , the average value of  $f(x)$  on  $[2, 6]$  is  $\frac{5}{4}$  and the average value of  $f(2x)$  on  $[1, 3]$  is  $\frac{5}{4}$ . Therefore  $\int_1^3 f(2x) dx = 2 \times \frac{5}{4} = \frac{5}{2}$  and

$$\begin{aligned} \int_1^3 (f(2x) + x) dx &= \frac{5}{2} + \int_1^3 x dx \\ &= \frac{5}{2} + \left[ \frac{1}{2} x^2 \right]_1^3 \\ &= \frac{5}{2} + \frac{9}{2} - \frac{1}{2} \\ &= \frac{13}{2} \end{aligned}$$

**Question 14****Answer: A****Explanatory notes**

The graph of  $f(x)$  has stationary points when  $x = -2$ ,  $x = 0$  and  $x = 2$ . The stationary point at the origin is a stationary point of inflection.

Note that the gradient of  $f(x)$  is positive when  $x < -2$  and when  $x > 2$ . Therefore, option A is correct.

**Question 15****Answer: D****Explanatory notes**

One way to determine which option is correct is to test each one. This can be done quickly in CAS.

It can be found that  $f(x) = 2^{x-1}$  satisfies the relation.

**Question 16****Answer: B****Explanatory notes**

Use CAS to evaluate the integral and solve the equation.

It can be found that  $k = 12$ .

**Question 17****Answer: B****Explanatory notes**

The number of blue marbles in the bag is 8.

The screenshot shows a calculator window titled 'mcq17\_u...ted' with 'RAD' mode selected. The function  $f(n) := \frac{\text{binomCdf}\left(5, \frac{n}{20}, 3, 5\right)}{\text{binomCdf}\left(5, \frac{n}{20}, 2, 5\right)}$  is entered. Below the definition, the value  $f(8)$  is calculated as 0.478764. A 'Done' button is visible in the top right corner of the calculator display area.

**Question 18****Answer: E****Explanatory notes**

The gradient of the graph of the tangent to the function  $g$  at  $x = 3$  is  $g'(3) = \frac{1}{f'(5)} = -\frac{1}{2}$ .

As the tangent passes through the point  $(3, 5)$ , the equation of the tangent is

$$y = -\frac{1}{2}(x-3) + 5 = -\frac{1}{2}x + \frac{13}{2}$$

**Tip**

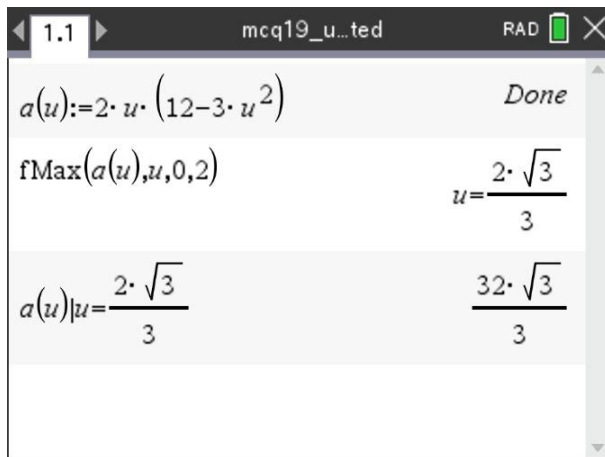
- Remember that the  $x$  values of the inverse function are the  $y$  values of the original function, and vice versa.

**Question 19****Answer: B****Explanatory notes**

Let  $a(u) = 2u(12 - 3u^2)$ . Use the fMax function of the calculator to find the value of  $u$  for which the area is maximised.

$$u = \frac{2\sqrt{3}}{3} = \frac{2}{\sqrt{3}}$$

The maximum area is  $\frac{32\sqrt{3}}{3} = \frac{32}{\sqrt{3}}$ .

**Question 20****Answer: D****Explanatory notes**

Let  $f(x) = y = 2^{-x}$ . The area of the bars is

$$0.5 \times (f(0.5) + f(1) + f(1.5) + f(2)) = 0.90533$$

The area between the  $x$ -axis and the graph of  $y = 2^{-x}$  over the interval  $[0, 2]$  is

$$\int_0^2 2^{-x} dx = \frac{3}{4 \log_e 2}$$

The error is

$$\frac{3}{4 \log_e 2} - 0.90533 = 0.176691.$$

## SECTION B

### Question 1a.

#### Worked solution

If a function will be used several times, it is always advisable to define it in your calculator.

Having done this, we can use CAS to differentiate the function.

The screenshot shows a CAS calculator interface with the following content:

- Top bar: 1.1, q1, RAD, and a close button.
- Function definition:  $f(x) := \sqrt{x} \cdot (2-x) \cdot e^{-2 \cdot x}$  with a "Done" button.
- Derivative calculation:  $\frac{d}{dx}(f(x))$  resulting in  $\frac{(4 \cdot x^2 - 11 \cdot x + 2) \cdot e^{-2 \cdot x}}{2 \cdot \sqrt{x}}$ .
- A vertical cursor is visible below the derivative result.

$$f'(x) = \frac{(4x^2 - 11x + 2)e^{-2x}}{2\sqrt{x}}$$

#### Mark allocation: 1 mark

- 1 mark for the correct answer in the required form

### Question 1b.i.

#### Worked solution

CAS can be used to find the function maximum.

The screenshot shows a CAS calculator interface with the following content:

- Top bar: 1.1, q1, RAD, and a close button.
- Function definition:  $f(x) := \sqrt{x} \cdot (2-x) \cdot e^{-2 \cdot x}$  with a "Done" button.
- Derivative calculation:  $\frac{d}{dx}(f(x))$  resulting in  $\frac{(4 \cdot x^2 - 11 \cdot x + 2) \cdot e^{-2 \cdot x}}{2 \cdot \sqrt{x}}$ .
- Maximum search:  $fMax(f(x), x, 0, 2)$  resulting in  $x = \frac{-(\sqrt{89} - 11)}{8}$ .
- A vertical cursor is visible below the maximum search result.

The function is maximum when  $x = \frac{11 - \sqrt{89}}{8}$ .

#### Mark allocation: 1 mark

- 1 mark for the correct answer

**Question 1b.ii.****Worked solution**

Evaluate the function at the value of  $x$  found previously.

The screenshot shows a CAS interface with the following results:

$$\frac{d}{dx}(f(x)) = \frac{(4 \cdot x^2 - 11 \cdot x + 2) \cdot e^{-2 \cdot x}}{2 \cdot \sqrt{x}}$$

$$\text{fMax}(f(x), x, 0, 2) \quad x = \frac{-(\sqrt{89} - 11)}{8}$$

$$f(x)|_{x = \frac{-(\sqrt{89} - 11)}{8}} = 0.539661$$

The function maximum is 0.5397 correct to four decimal places.

**Mark allocation: 1 mark**

- 1 mark for the correct answer

**Question 1c.i.****Worked solution**

The **tangentLine** function of CAS can be used to find the tangent.

The screenshot shows a CAS interface with the following results:

$$\text{tangentLine}(f(x), x, 1) = \frac{7 \cdot e^{-2}}{2} - \frac{5 \cdot e^{-2} \cdot x}{2}$$

$$\text{comDenom}\left(\frac{7 \cdot e^{-2}}{2} - \frac{5 \cdot e^{-2} \cdot x}{2}\right) = \frac{e^{-2} \cdot (7 - 5 \cdot x)}{2}$$

As the answer was required to be in a particular form, the common denominator command (**comDenom**) has been used.

The equation of the tangent is  $y = \frac{7 - 5x}{2e^2}$ .

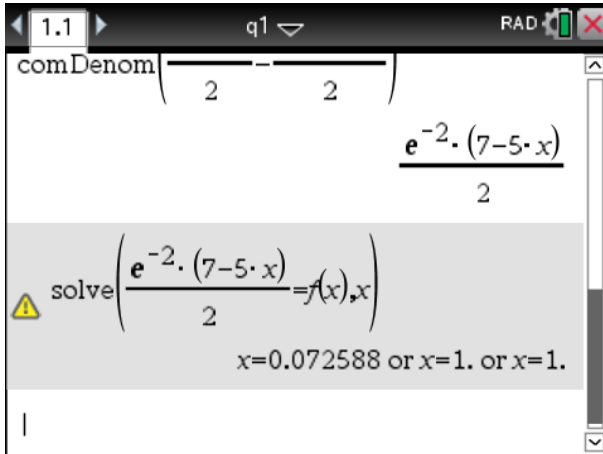
**Mark allocation: 1 mark**

- 1 mark for the correct answer in the required form



**Question 1c.ii.****Worked solution**

Use CAS to find the value of the  $x$ -coordinate of  $P$ .



The  $x$ -coordinate of  $P$  is 0.0726, correct to four decimal places.

**Mark allocation: 1 mark**

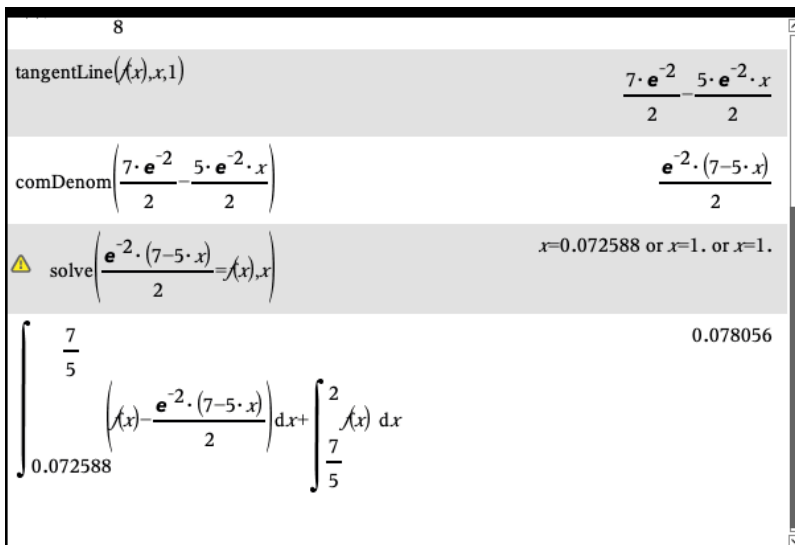
- 1 mark for the correct answer

**Question 1c.iii.****Worked solution**

Note that the tangent passes through the  $x$ -axis (point  $Q$ ) when  $x = \frac{7}{5}$ .

The area of the shaded region is therefore

$$\int_{0.0726}^{\frac{7}{5}} \left( f(x) - \frac{7-5x}{2e^2} \right) dx + \int_{\frac{7}{5}}^2 f(x) dx = 0.0781$$

**Mark allocation: 2 marks**

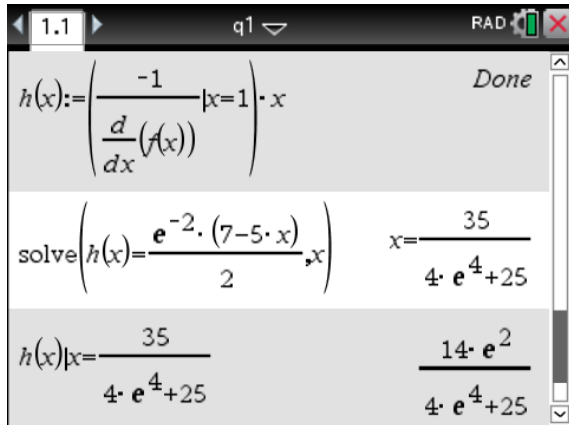
- 1 mark for setting up the two integrals
- 1 mark for the correct answer

**Question 1d.i.****Worked solution**

The shortest distance must be along a line perpendicular to the tangent passing through the origin. Since we know the gradient of the tangent, we can calculate the slope of a perpendicular line, and since it passes through the origin, its  $y$ -intercept must be 0. Therefore the equation of the line perpendicular to the tangent and that passes through the origin is

$y = \frac{2e^2x}{5}$ . This can be found using CAS or from inspecting the tangent.

Use CAS to find the point of intersection of the two lines.



The screenshot shows a CAS interface with the following steps:

$$h(x) := \left( \frac{-1}{\frac{d}{dx} f(x)} \right) \cdot x$$

$$\text{solve} \left( h(x) = \frac{e^{-2} \cdot (7 - 5 \cdot x)}{2}, x \right) \quad x = \frac{35}{4 \cdot e^4 + 25}$$

$$h(x) \Big|_{x = \frac{35}{4 \cdot e^4 + 25}} = \frac{14 \cdot e^2}{4 \cdot e^4 + 25}$$

Alternatively, you can use the distance formula to calculate the distance from the origin to a point on the tangent line, then use **fMin** in the CAS to find the point which minimises this distance.

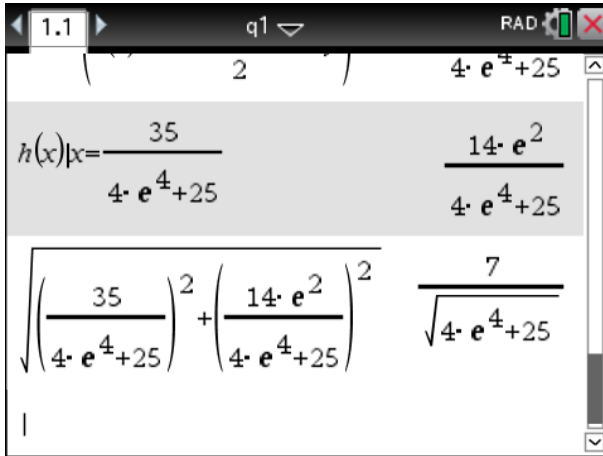
The coordinates of  $R$  are  $\left( \frac{35}{4e^4 + 25}, \frac{14e^2}{4e^4 + 25} \right)$ .

**Mark allocation: 2 marks**

- 0 marks for incorrect coordinates or no relevant working
- 1 mark for relevant working towards the solution with 0 or 1 correct coordinate given
- 2 marks for relevant working and correct coordinates given

**Question 1d.ii.****Worked solution**

Use the distance formula to find the required distance.



The screenshot shows a calculator interface with the following content:

- Top bar: 1.1, q1, RAD, and a close button.
- Function definition:  $h(x) \left|_{x=}$  followed by  $\frac{35}{4 \cdot e^4 + 25}$ .
- Point R coordinates:  $\frac{14 \cdot e^2}{4 \cdot e^4 + 25}$  and  $\frac{7}{\sqrt{4 \cdot e^4 + 25}}$ .
- Distance formula:  $\sqrt{\left(\frac{35}{4 \cdot e^4 + 25}\right)^2 + \left(\frac{14 \cdot e^2}{4 \cdot e^4 + 25}\right)^2}$ .
- Result:  $\frac{7}{\sqrt{4 \cdot e^4 + 25}}$ .

The distance from  $R$  to the origin is  $\frac{7}{\sqrt{4e^4 + 25}}$ .

**Mark allocation: 1 mark**

- 1 mark for the correct answer

**Question 2a.****Worked solution**

$$\frac{dy}{dx} = -\frac{3x(x-100)}{2500}$$

The screenshot shows a CAS calculator interface. At the top, it displays '1.1', 'q2', and 'RAD'. The main display area shows the function definition:  $f(x) := \frac{x^2 \cdot (150 - x)}{2500}$ . Below this, the derivative is calculated:  $\frac{d}{dx}(f(x)) = \frac{-3 \cdot x \cdot (x - 100)}{2500}$ . The word 'Done' is visible in the top right corner of the calculator window.

**Mark allocation: 1 mark**

- 1 mark for the correct answer

**Tip**

- *Although this is easy to find by hand, it is worth using CAS to find your answer so that you have the function defined in your calculator. In this case we have let  $y = f(x)$ .*

**Question 2b.****Worked solution**

Since  $\frac{dy}{dx} = 0$  when  $x = 100$ , the function is strictly decreasing for  $x \in [100, 150]$ .

**Mark allocation: 1 mark**

- 1 mark for the correct answer

**Tip**

- *The set of values for which this function is strictly decreasing includes the end points and the turning point.*

**Question 2c.****Worked solution**

Integrate using CAS to find that the area of the vertical stabiliser is  $16\,875\text{ cm}^2$  (units are not required).

The screenshot shows a CAS window with the following content:

$$f(x) := \frac{x^2 \cdot (150 - x)}{2500}$$

$$\frac{d}{dx}(f(x)) = \frac{-3 \cdot x \cdot (x - 100)}{2500}$$

$$\int_0^{150} f(x) \, dx = 16875$$

**Mark allocation: 1 mark**

- 1 mark for the correct answer

**Question 2d.i.****Worked solution**

The value of  $c$  is found by solving  $f(x) = 20$  for  $x$  using CAS.

The value of  $c$  is 147.708.

The screenshot shows a CAS window with the following content:

$$\frac{d}{dx}(f(x)) = \frac{-3 \cdot x \cdot (x - 100)}{2500}$$

$$\int_0^{150} f(x) \, dx = 16875$$

$$\text{solve}(f(x)=20,x)$$

$$x = -17.2883 \text{ or } x = 19.58 \text{ or } x = 147.708$$

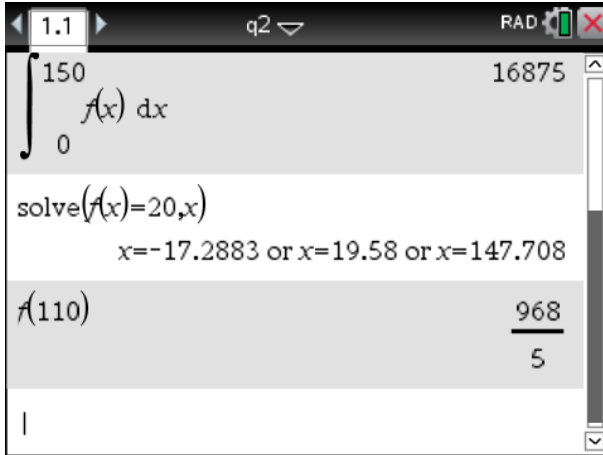
**Mark allocation: 1 mark**

- 1 mark for the correct answer

**Question 2d.ii.****Worked solution**

The value of  $b$  is found by evaluating  $f(110)$ .

$$b = \frac{968}{5} = 193.6$$



1.1 q2 RAD

$\int_0^{150} f(x) dx$  16875

solve( $f(x)=20,x$ )  
 $x=-17.2883$  or  $x=19.58$  or  $x=147.708$

$f(110)$   $\frac{968}{5}$

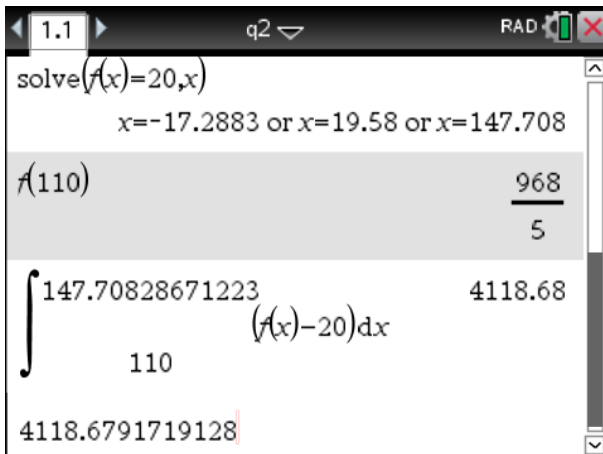
**Mark allocation: 1 mark**

- 1 mark for the correct answer

**Question 2e.****Worked solution**

The area of the rudder is found by evaluating the integral

$$\int_{110}^{147.708} (f(x) - 20) dx = 4118.679$$



1.1 q2 RAD

solve( $f(x)=20,x$ )  
 $x=-17.2883$  or  $x=19.58$  or  $x=147.708$

$f(110)$   $\frac{968}{5}$

$\int_{110}^{147.708} (f(x)-20) dx$  4118.68

4118.6791719128

**Mark allocation: 2 marks**

- 1 mark for formulating the integral
- 1 mark for the correct answer, to three decimal places

**Question 2f.i.****Worked solution**

The line segment  $OP$  lies on the tangent to the curve that passes through the origin. To find this tangent, first find the general equation for the tangent to the curve (at a point  $x = a$ , for example) and then determine the value of  $a$  when the tangent passes through the origin (that is, when  $x = 0$ ).

This can be done quickly using CAS. We find that we require the tangent to the curve when  $x = 75$ .

```

1.1 | q2 | RAD
-----
110
4118.6791719128 | 4118.68
solve(tangentLine(f(x),x,a)=0,a)|x=0
                                     a=0 or a=75
tangentLine(f(x),x,75) | 9*x
                                     4
  
```

The equation of the tangent is  $y = \frac{9x}{4}$ .

**Mark allocation: 1 mark**

- 1 mark for the correct answer

**Question 2f.ii.****Worked solution**

The tangent to the curve when  $x = 150$  is  $y = 1350 - 9x$ .

```

1.1 | q2 | RAD
-----
4118.6791719128 | 4118.68
solve(tangentLine(f(x),x,a)=0,a)|x=0
                                     a=0 or a=75
tangentLine(f(x),x,75) | 9*x
                                     4
tangentLine(f(x),x,150) | 1350-9*x
  
```

**Mark allocation: 1 mark**

- 1 mark for the correct answer

**Tip**

- *If the function is defined on a restricted domain, then the tangent line does not exist at the endpoints. To avoid this problem, do not define the function on a restricted domain.*

**Question 2f.iii.****Worked solution**

Note that  $\frac{9x}{4} = f(100) = 200 \Rightarrow x = \frac{800}{9}$

$$1350 - 9x = f(100) = 200 \Rightarrow x = \frac{1150}{9}$$

The screenshot shows a TI-84 Plus calculator interface. At the top, it displays '1.1', 'q2', and 'RAD'. The main screen shows the following steps:

- The tangent line equation:  $\text{tangentLine}(f(x), x, 150) \quad 1350 - 9 \cdot x$
- The first solve command:  $\text{solve}\left(\frac{9 \cdot x}{4} = f(100), x\right) \quad x = \frac{800}{9}$
- The second solve command:  $\text{solve}(1350 - 9 \cdot x = f(100), x) \quad x = \frac{1150}{9}$

**Mark allocation: 2 marks**

- 1 mark for each equation leading to the required solution

**Tip**

- *This is a 'show that' question, evidence needs to be given to justify each mark.*



**Question 2f.iv.****Worked solution**

The area of the trapezium  $OPQR$  is

$$\frac{1}{2} \left( 150 + \left( \frac{1150}{9} - \frac{800}{9} \right) \right) \times f(100) = \frac{170000}{9}$$

1.1    q2    RAD

solve( $\frac{9 \cdot x}{4} = f(100)$ , x)     $x = \frac{800}{9}$

solve( $1350 - 9 \cdot x = f(100)$ , x)     $x = \frac{1150}{9}$

$\frac{1}{2} \cdot \left( 150 + \frac{1150}{9} - \frac{800}{9} \right) \cdot f(100)$      $\frac{170000}{9}$

**Mark allocation: 2 marks**

- 1 mark for using the formula for the area of a trapezium
- 1 mark for the correct answer

**Question 2f.v.****Worked solution**

The error is the true value minus the approximate value. To find this as a percentage of the actual area, we divide the error by the actual area and multiply by 100.

$$\frac{16875 - \frac{170000}{9}}{16875} \times 100 = -11.93$$

Therefore the error in the approximation is 11.93%.

1.1    \*q2    RAD

solve( $1350 - 9 \cdot x = f(100)$ , x)     $x = \frac{1150}{9}$

$\frac{1}{2} \cdot \left( 150 + \frac{1150}{9} - \frac{800}{9} \right) \cdot f(100)$      $\frac{170000}{9}$

$\frac{16875 - \frac{170000}{9}}{16875} \cdot 100$     -11.9342

**Mark allocation: 2 marks**

- 1 mark for calculating the error as the actual area minus the approximate area
- 1 mark for the correct answer (expressed as a positive value)

**Question 3a.****Worked solution**

The amplitude is 15 and the period is 1. Therefore  $a = 15$ .

$$\frac{2\pi}{n} = 1 \Rightarrow n = 2\pi$$

**Mark allocation: 1 mark**

- 1 mark for both values (providing that justifications are given)

**Question 3b.****Worked solution**

The centre of wheel B is 50 cm horizontally from the centre of wheel A.

The amplitude is 30 and the period is  $\frac{2}{3}$ .

Therefore,

$$b = 50$$

$$c = 30$$

$$\frac{2\pi}{m} = \frac{2}{3} \Rightarrow m = 3\pi$$

**Mark allocation: 1 mark**

- 1 mark for all values (providing that justifications are given)

**Question 3c.****Worked solution**

The period is 2.

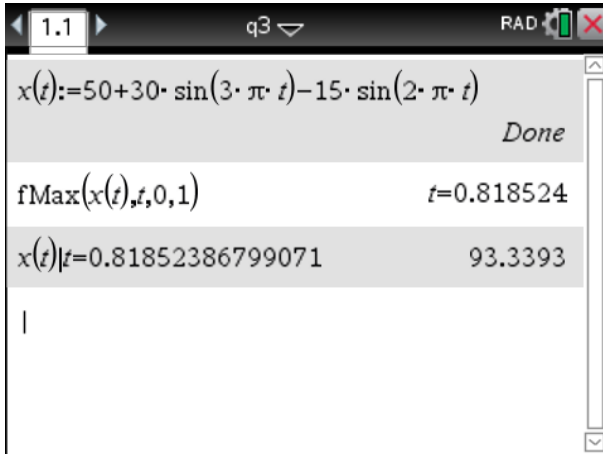
**Note:** Wheel A makes 2 complete revolutions in 2 seconds and wheel B makes 3 complete revolutions in 2 seconds.

**Mark allocation: 1 mark**

- 1 mark for the correct answer

**Question 3d.****Worked solution**

Use the function maximum function (**fMax**) in CAS.



The maximum value of  $x(t)$  is 93.34 cm and this first occurs when  $t = 0.82$  seconds.

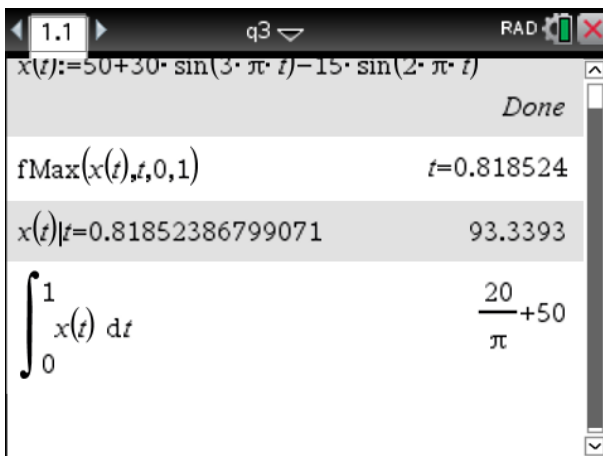
**Mark allocation: 2 marks**

- 1 mark for correctly calculating the maximum value
- 1 mark for correctly calculating the time

**Question 3e.****Worked solution**

The area is found by evaluating the integral

$$\int_0^1 x(t) dt = \frac{20}{\pi} + 50$$

**Mark allocation: 2 marks**

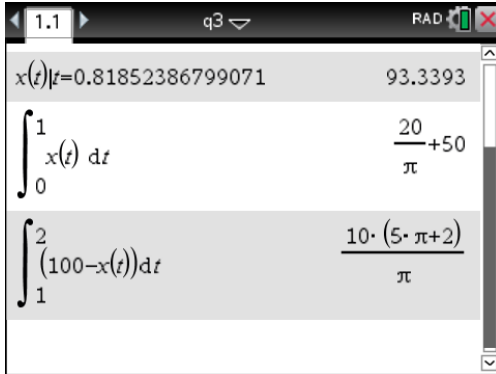
- 1 mark for formulating the correct integral
- 1 mark for the correct answer

**Question 3f.****Worked solution**

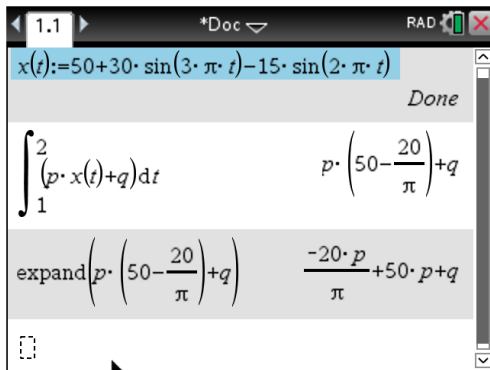
By inspecting the graph, it can be seen that a reflection of the function  $x(t)$  in the  $t$  axis followed by a shift upwards of 100 units produces a function with the required property.

$$\int_1^2 (100 - x(t)) dt = \frac{20}{\pi} + 50.$$

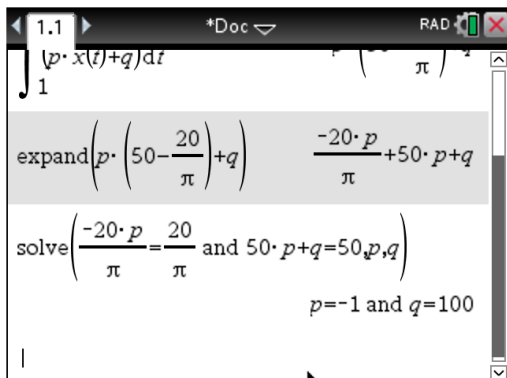
Therefore  $p = -1$  and  $q = 100$ .



Alternatively, calculate the integral using CAS.



Then equate the terms containing  $\pi$  and separately equate the remaining terms (since the pronumerals cannot be irrational).

**Mark allocation: 2 marks**

- 1 mark for correctly calculating  $p$
- 1 mark for correctly calculating  $q$

**Question 3g.i.****Worked solution**

$$\frac{dx}{dt} = 90\pi \cos(3\pi t) - 30\pi \cos(2\pi t)$$

CAS may be used to check that the differentiation has been performed correctly.

The screenshot shows a CAS window with the following content:

- Top bar: 1.1, q3, RAD, and a close button.
- Input:  $\int_1^2 (100 - x(t)) dt$
- Output:  $\frac{10 \cdot (5 \cdot \pi + 2)}{\pi}$
- Input:  $\frac{d}{dt}(x(t))$
- Output:  $90 \cdot \pi \cdot \cos(3 \cdot \pi \cdot t) - 30 \cdot \pi \cdot \cos(2 \cdot \pi \cdot t)$

**Mark allocation: 1 mark**

- 1 mark for the correct derivative

**Tip**

- *It is not necessary to factorise the result.*

**Question 3g.ii.****Worked solution**

The least value of  $t$  such that  $t > 0$  and  $\frac{dx}{dt} = 0$  is  $t = 0.14$ .

This may be found by inspecting the graph of the function and finding the first local maximum.

The screenshot shows a CAS window with the following content:

- Top bar: 1.1, q3, RAD, and a close button.
- Input:  $\int_1^2 (100 - x(t)) dt$
- Output:  $\frac{10 \cdot (5 \cdot \pi + 2)}{\pi}$
- Input:  $\frac{d}{dt}(x(t))$
- Output:  $90 \cdot \pi \cdot \cos(3 \cdot \pi \cdot t) - 30 \cdot \pi \cdot \cos(2 \cdot \pi \cdot t)$
- Input:  $fMax(x(t), t, 0, 0.3)$
- Output:  $t = 0.1448$

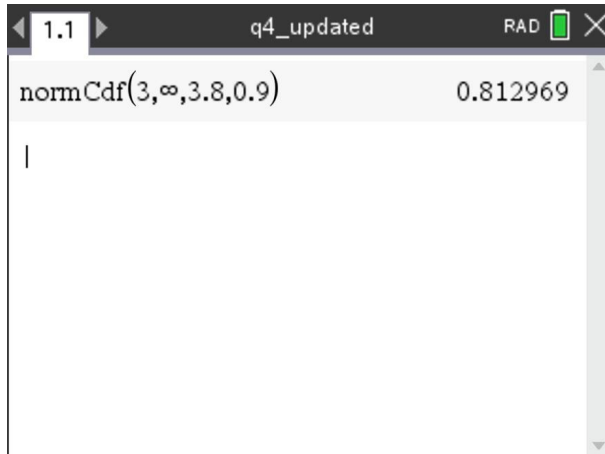
**Mark allocation: 1 mark**

- 1 mark for the correct answer

**Question 4a.****Worked solution**

$$X \sim (3.8, 0.9^2)$$

Then  $\Pr(X > 3) = 0.813$ .



A screenshot of a calculator window titled 'q4\_updated' with a 'RAD' indicator. The input field contains the formula `normCdf(3,∞,3.8,0.9)` and the output field shows the result `0.812969`.

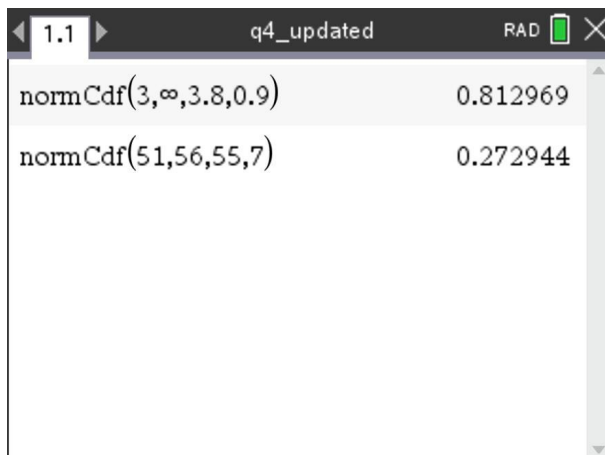
**Mark allocation: 1 mark**

- 1 mark for the correct answer

**Question 4b.****Worked solution**

$$\text{Let } A \sim N(55, 7^2).$$

Then  $\Pr(51 < A < 56) = 0.273$ .



A screenshot of a calculator window titled 'q4\_updated' with a 'RAD' indicator. The input field contains the formula `normCdf(3,∞,3.8,0.9)` and the output field shows the result `0.812969`. Below this, the input field contains the formula `normCdf(51,56,55,7)` and the output field shows the result `0.272944`.

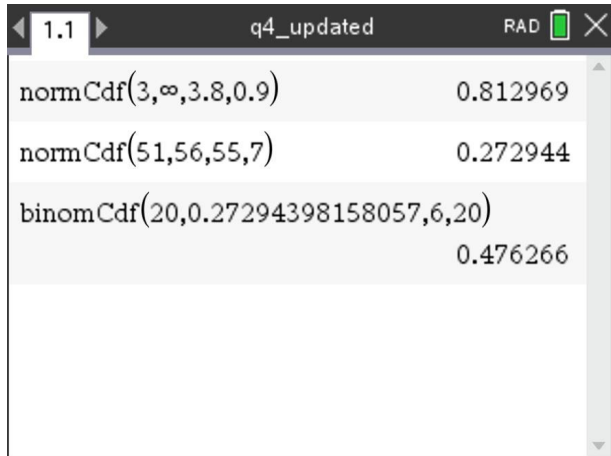
**Mark allocation: 2 marks**

- 1 mark for two probability statements
- 1 mark for the correct answer

**Question 4c.****Worked solution**

Let  $W \sim \text{Bi}(20, 0.273)$ .

Then  $\Pr(W > 5) = 0.476$ .



The screenshot shows a calculator interface with the following data:

Function	Result
normCdf(3,∞,3.8,0.9)	0.812969
normCdf(51,56,55,7)	0.272944
binomCdf(20,0.27294398158057,6,20)	0.476266

**Mark allocation: 2 marks**

- 1 mark for binomial distribution, 20 trials, using the answer from **part b**.
- 1 mark for the correct answer

**Question 4d.****Worked solution**

$$\Pr(B < 59) = 0.15 \text{ and } \Pr(B > 68) = 0.2 \Rightarrow \Pr(B < 68) = 0.8$$

Let  $Z \sim N(0, 1)$  be the standard normal distribution. Then

$$\Pr\left(Z < \frac{59 - \mu}{\sigma}\right) = 0.15$$

$$\Pr\left(Z < \frac{68 - \mu}{\sigma}\right) = 0.8$$

This gives two equations that can be solved to find  $\mu$  and  $\sigma$ .

$$\mu = 63.967$$

$$\sigma = 4.792$$

The screenshot shows a calculator window titled 'q4\_updated' with the following content:

```

normCdf(51,56,55,7)      0.272944
binomCdf(20,0.27294398158057,6,20)  0.476266
solve({ (59-m)/s = invNorm(0.15,0,1), (68-m)/s = invNorm(0.8,0,1) })
s=4.79219 and m=63.9668
  
```

**Mark allocation: 4 marks**

- 1 mark for writing probability statements
- 1 mark for converting to the standard normal
- 1 mark for solving to find  $\mu$  and  $\sigma$
- 1 mark for the correct answer

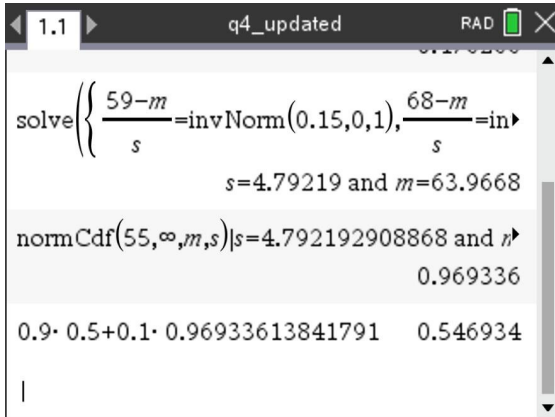


**Question 4e.****Worked solution**

We have  $\Pr(A > 55) = 0.5$  and  $\Pr(B > 55) = 0.969$ .

Since 90% of the trees are type A trees and 10% of the trees are type B trees, the probability that a randomly selected tree has a diameter that is greater than 55 cm is

$$0.9 \times 0.5 + 0.1 \times 0.969 = 0.547$$

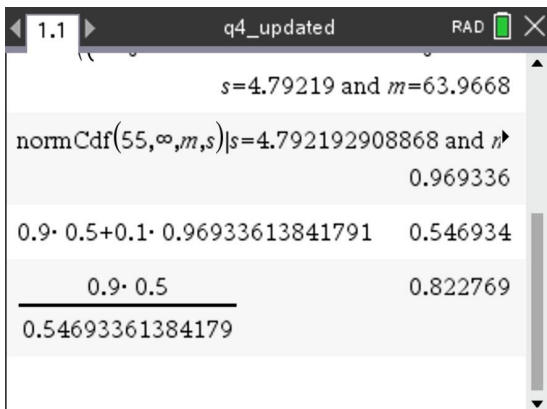
**Mark allocation: 1 mark**

- 1 mark for finding  $\Pr(B > 55) = 0.969$  and using this in the formula to find the given answer

**Question 4f.****Worked solution**

This is a conditional probability question:

$$\begin{aligned} & \Pr(\text{type A tree} \mid \text{diameter} > 55) \\ &= \frac{\Pr(\text{type A tree} \cap \text{diameter} > 55)}{\Pr(\text{diameter} > 55)} \\ &= \frac{0.9 \times 0.5}{0.547} \\ &= 0.823 \end{aligned}$$

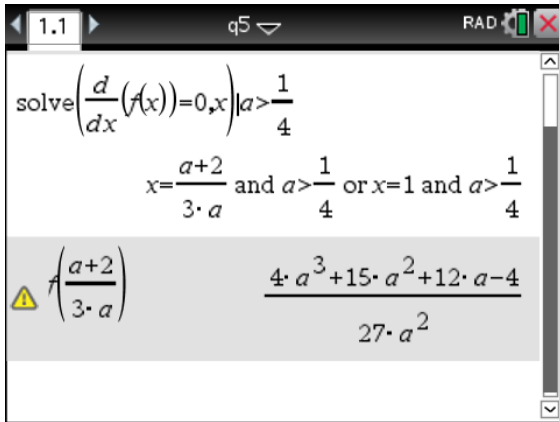
**Mark allocation: 2 marks**

- 1 mark for correct conditional probability
- 1 mark for the correct answer

**Question 5a.****Worked solution**

The turning point at  $P$  occurs when  $x = \frac{a+2}{3a}$ . Therefore the coordinates of the point  $P$  are

$$\left( \frac{a+2}{3a}, \frac{4a^3 + 15a^2 + 12a - 4}{27a^2} \right)$$



1.1 q5 RAD

solve( $\frac{d}{dx}(f(x))=0,x$ )| $a>\frac{1}{4}$

$x = \frac{a+2}{3 \cdot a}$  and  $a > \frac{1}{4}$  or  $x = 1$  and  $a > \frac{1}{4}$

$f\left(\frac{a+2}{3 \cdot a}\right) = \frac{4 \cdot a^3 + 15 \cdot a^2 + 12 \cdot a - 4}{27 \cdot a^2}$

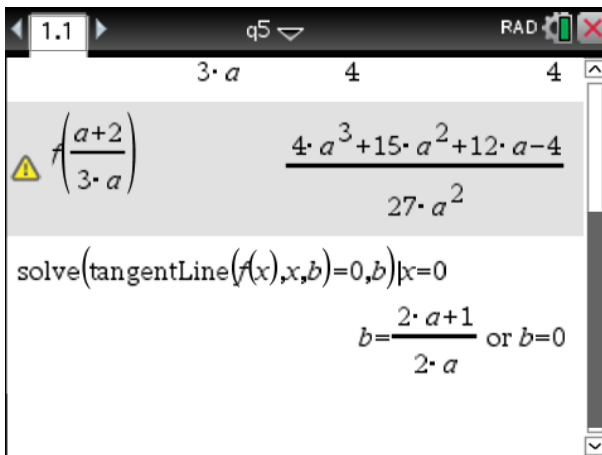
**Mark allocation: 1 mark**

- 1 mark for the correct answer

**Question 5b.****Worked solution**

Find the tangent line to  $f$  at any point on the curve, say  $x = b$ . The tangent line passes through the origin for certain values of  $b$ .

This can be obtained quickly using CAS.



1.1 q5 RAD

$f\left(\frac{a+2}{3 \cdot a}\right) = \frac{4 \cdot a^3 + 15 \cdot a^2 + 12 \cdot a - 4}{27 \cdot a^2}$

solve(tangentLine( $f(x),x,b$ )=0, $b$ )| $x=0$

$b = \frac{2 \cdot a + 1}{2 \cdot a}$  or  $b = 0$

The  $x$ -coordinate of  $Q$  is  $\frac{2a+1}{2a}$ .

**Mark allocation: 1 mark**

- 1 mark for the correct answer

**Question 5c.****Worked solution**

The gradient of the tangent is  $\frac{f(b)}{b}$  where  $b = \frac{2a+1}{2a}$ .

A screenshot of a CAS calculator interface. The top bar shows '1.1', 'q5', and 'RAD'. The main display area shows the following steps:
 

- Input:  $\sqrt{3 \cdot a}$
- Equation:  $27 \cdot a^2$
- Command:  $\text{solve}(\text{tangentLine}(f(x), x, b) = 0, b) | x = 0$
- Result:  $b = \frac{2 \cdot a + 1}{2 \cdot a}$  or  $b = 0$
- Result:  $\frac{f(b)}{b} | b = \frac{2 \cdot a + 1}{2 \cdot a}$  and  $\frac{4 \cdot a - 1}{4 \cdot a}$

Therefore the equation of the tangent to  $f$  that passes through the origin is  $y = \left(\frac{4a-1}{4a}\right)x$ .

The area of the shaded region is

$$\int_0^{\frac{2a+1}{2a}} \left( f(x) - \left(\frac{4a-1}{4a}\right)x \right) dx = \frac{(2a+1)^2(4a^2+4a+1)}{192a^3} = \frac{(1+2a)^4}{192a^3}$$

A screenshot of a CAS calculator interface. The top bar shows '1.1', 'q5', and 'RAD'. The main display area shows the following steps:
 

- Integral setup:  $\int_0^{\frac{2 \cdot a + 1}{2 \cdot a}} \left( f(x) - \frac{4 \cdot a - 1}{4 \cdot a} \cdot x \right) dx$
- Result:  $\frac{(2 \cdot a + 1)^2 \cdot (4 \cdot a^2 + 4 \cdot a + 1)}{192 \cdot a^3}$

**Mark allocation: 3 marks**

- 1 mark for the equation of the tangent
- 1 mark for the integral with correct terminals
- 1 mark for the correct answer

**Tip**

- Note that CAS does not automatically simplify the result.

**Question 5d.****Worked solution**

$$\text{Solve } \frac{d}{da} \left( \frac{(1+2a)^4}{192a^3} \right) = 0.$$

$$\text{If } a > \frac{1}{4}, \text{ then } a = \frac{3}{2}.$$

The screenshot shows a calculator window with the following content:

- Top bar: "1.1" on the left, "q5" in the center, and "RAD" on the right.
- Input area: 
$$\frac{(2 \cdot a + 1)^2 \cdot (4 \cdot a^2 + 4 \cdot a + 1)}{192 \cdot a^3}$$
- Operation area: 
$$\text{solve} \left( \frac{d}{da} \left( \frac{(2 \cdot a + 1)^2 \cdot (4 \cdot a^2 + 4 \cdot a + 1)}{192 \cdot a^3} \right) = 0, a \right)$$
- Output area: 
$$a = \frac{3}{2}$$

**Mark allocation: 2 marks**

- 1 mark for differentiating the result from **part c.** and setting it to zero
- 1 mark for the correct value of  $a$

**Question 5e.****Worked solution**

The area of  $B$  is equal to

$$\int_0^{\frac{a+2}{3a}} \left( \frac{4a^3 + 15a^2 + 12a - 4}{27a^2} - g(x) \right) dx = \frac{(a+2)(5a^3 + 18a^2 + 12a - 8)}{324a^3}$$

$$= \frac{(a+2)^3(5a-2)}{324a^3}$$

The image shows two screenshots of a calculator interface. The top screenshot displays the integral expression:  $\int_0^{\frac{a+2}{3 \cdot a}} \left( \frac{4 \cdot a^3 + 15 \cdot a^2 + 12 \cdot a - 4}{27 \cdot a^2} - f(x) \right) dx$ . Below the integral, the result is shown as  $\frac{(a+2) \cdot (5 \cdot a^3 + 18 \cdot a^2 + 12 \cdot a - 8)}{324 \cdot a^3}$ . The bottom screenshot shows the same result with a 'factor' function applied to the numerator, resulting in  $\frac{(a+2)^3 \cdot (5 \cdot a - 2)}{324 \cdot a^3}$ .

**Mark allocation: 3 marks**

- 2 marks for the integral with the difference between the functions, and for the correct terminals
- 1 mark for the correct answer

**Question 5f.****Worked solution**

The area of  $A$  is

$$\int_0^{\frac{a+2}{3a}} (g(x) - x) dx$$

Equating this area to the area found in **part e.** gives

$$\int_0^{\frac{a+2}{3a}} (g(x) - x) dx = \frac{(a+2)^3(5a-2)}{324a^3} \Rightarrow a = 1$$

The screenshot shows a TI-84 Plus calculator interface. At the top, the expression  $\frac{(a+2)^3 \cdot (5a-2)}{324 \cdot a^3}$  is displayed. Below it, the text 'solve(' is followed by the integral equation  $\int_0^{\frac{a+2}{3a}} (f(x)-x) dx = \frac{(a+2)^3 \cdot (5a-2)}{324 \cdot a^3}, a$ . The calculator returns the solutions  $a = -2$  or  $a = 1$ .

**Mark allocation: 2 marks**

- 1 mark for equating the integral for the area  $A$  with the area found in **part e.**
- 1 mark for the correct answer

**END OF WORKED SOLUTIONS**

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