YEAR 12 *Trial Exam Paper*

2020

MATHEMATICAL METHODS

Written examination 2

Reading time: 15 minutes

Writing time: 2 hours

STUDENT NAME:

QUESTION AND ANSWER BOOK

Structure of book

Section	*Number of questions*	*Number of questions to be answered*	*Number of marks*
A	20	20	20
B	5	5	60
			Total 80

- Students are permitted to bring into the examination room: pens, pencils, highlighters, erasers, sharpeners, rulers, a protractor, set squares, aids for curve sketching, one bound reference, one approved technology (calculator or software) and, if desired, one scientific calculator. Calculator memory DOES NOT need to be cleared. For approved computer-based CAS, full functionality may be used.
- Students are NOT permitted to bring into the examination room: blank sheets of paper and/or correction fluid/tape.

Materials supplied

- Question and answer book of 27 pages
- Formula sheet
- Answer sheet for multiple-choice questions

Instructions

- Write your **name** in the box provided above on this page, and on your answer sheet for multiple-choice questions.
- Unless otherwise indicated, the diagrams in this book are **not** drawn to scale.
- All written responses must be in English.

At the end of the examination

- Place the answer sheet for multiple-choice questions inside the front cover of this book.
- You may keep the formula sheet.

Students are NOT permitted to bring mobile phones and/or any other unauthorised electronic devices into the examination room.

SECTION A – Multiple-choice questions

Instructions for Section A

Answer **all** questions in pencil on the answer sheet provided for multiple-choice questions.

Choose the response that is **correct** for the question.

A correct answer scores 1; an incorrect answer scores 0.

Marks will **not** be deducted for incorrect answers.

No marks will be given if more than one answer is completed for any question.

Unless otherwise indicated, the diagrams in this book are **not** drawn to scale.

Question 1

Let $f: R \to R$, $f(x) = 3 - 2\sin\left(\dfrac{4\pi}{5}x\right)$.

The period and range of *f* are respectively

A. $\dfrac{2}{5}$ and $[1, 5]$

B. $\dfrac{5}{2}$ and $[-1, 3]$

C. $\dfrac{2}{5}$ and $[-5, 5]$

D. $\dfrac{5}{2}$ and $[1, 5]$

E. $\dfrac{5}{2}$ and $[-1, 5]$

Question 2

The simultaneous linear equations $2x + ky = 5$ and $kx + 8y = -10$ have no solution when

A. $k = -4$

B. $k = 4$

C. $k \in \{-4, 4\}$

D. $k \in R \setminus \{4\}$

E. $k \in R \setminus \{-4\}$

Question 3

If $x-a$ is a factor of $x^3-2x^2-(3a+1)x+a$, where $a \in R \setminus \{0\}$, then the value of a is

A. 0

B. 4

C. –5

D. 5

E. 10

Question 4

The average value of $1+2x^2$ for the interval $[1, b]$, where $b>1$, is 15.

The value of b is

A. 3

B. 4

C. 5

D. 6

E. 7

Question 5

The function $f: D \to R,\ f(x)=2x^3-x^2-20x+5$ will have an inverse function for

A. $D=R$

B. $D=(-\infty, 2]$

C. $D=\left[-\frac{5}{3}, \infty\right)$

D. $D=\left(-\infty, -\frac{5}{3}\right]$

E. $D=\left[-2, \frac{5}{3}\right]$

Question 6

Let $f: R \to R$, $f(x) = ax^2(3-x)$, $a \in R$. The y-intercept of the tangent to f at $x = 3$ is 9.

The value of a is

A. 3

B. 9

C. $\dfrac{1}{27}$

D. $\dfrac{1}{9}$

E. $\dfrac{1}{3}$

Question 7

A darts player with a 20% chance of hitting the bullseye attempts to hit a bullseye 50 times. Each attempt is independent of the previous attempts.

The probability that n or more attempts are successful is less than 30%.

The smallest value of n, where n is an integer, is

A. 11

B. 12

C. 13

D. 14

E. 15

Question 8

The transformation $T: R^2 \to R^2$, which maps the graph of $y = \dfrac{-2}{\sqrt{3x-1}} + 1$ onto the graph of $y = \dfrac{1}{\sqrt{x}}$, has the rule

A. $T\left(\begin{bmatrix} x \\ y \end{bmatrix}\right) = \begin{bmatrix} 3 & 0 \\ 0 & -\frac{1}{2} \end{bmatrix}\begin{bmatrix} x \\ y \end{bmatrix} + \begin{bmatrix} -1 \\ \frac{1}{2} \end{bmatrix}$

B. $T\left(\begin{bmatrix} x \\ y \end{bmatrix}\right) = \begin{bmatrix} 3 & 0 \\ 0 & -\frac{1}{2} \end{bmatrix}\begin{bmatrix} x \\ y \end{bmatrix} + \begin{bmatrix} -1 \\ -\frac{1}{2} \end{bmatrix}$

C. $T\left(\begin{bmatrix} x \\ y \end{bmatrix}\right) = \begin{bmatrix} 3 & 0 \\ 0 & \frac{1}{2} \end{bmatrix}\begin{bmatrix} x \\ y \end{bmatrix} + \begin{bmatrix} 1 \\ \frac{1}{2} \end{bmatrix}$

D. $T\left(\begin{bmatrix} x \\ y \end{bmatrix}\right) = \begin{bmatrix} -\frac{1}{3} & 0 \\ 0 & 2 \end{bmatrix}\begin{bmatrix} x \\ y \end{bmatrix} + \begin{bmatrix} -1 \\ \frac{1}{2} \end{bmatrix}$

E. $T\left(\begin{bmatrix} x \\ y \end{bmatrix}\right) = \begin{bmatrix} \frac{1}{3} & 0 \\ 0 & -2 \end{bmatrix}\begin{bmatrix} x \\ y \end{bmatrix} + \begin{bmatrix} -1 \\ 1 \end{bmatrix}$

Question 9

The graph of $f(x) = x^3 - 6x^2 + b$ has exactly three x-intercepts for

A. $b > 0$

B. $b < 32$

C. $b \in R \setminus [0, 32]$

D. $b \in (0, 32)$

E. $b \in [0, 32]$

Question 10

The time taken to drive from Brunswick to Brighton in peak hour traffic is normally distributed with a mean of 48 minutes and a standard deviation of 14 minutes.

The proportion of trips from Brunswick to Brighton in peak hour that take less than 1 hour, given that they take more than 45 minutes, is closest to

A. 0.389

B. 0.804

C. 0.665

D. 0.335

E. 0.517

Question 11

The time John takes to walk to school in the morning is normally distributed, with a mean of μ minutes and a standard deviation of 3 minutes.

John is able to walk to school within 10 minutes 90% of the time.

The value of μ correct to one decimal place is

A. 6.1

B. 6.2

C. 6.5

D. 13.8

E. 13.9

Question 12

If the graph of $g(x) = f(2x) + 3$ passes through the point $(2, -4)$ then the graph of *f* must pass through the point

A. $(1, -7)$

B. $(1, -1)$

C. $(2, -7)$

D. $(4, -1)$

E. $(4, -7)$

Question 13

If $\int_2^6 f(x)\,dx = 5$, then $\int_1^3 \left(f(2x) + x\right) dx$ is equal to

A. 9

B. $\frac{9}{2}$

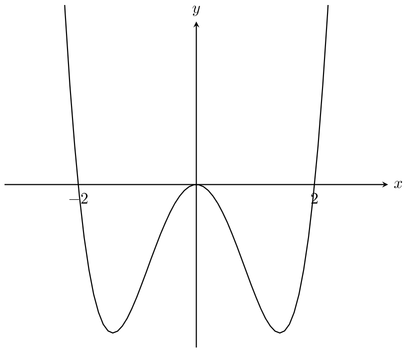
C. $\frac{13}{2}$

D. $\frac{15}{2}$

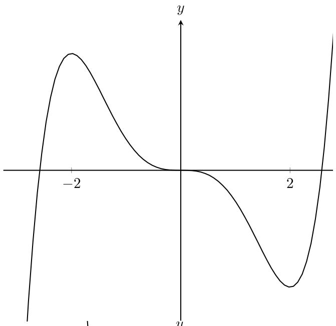
E. $\frac{5}{2}$

Question 14

The graph of $y = f'(x)$ is shown below.



The corresponding part of the graph of $y = f(x)$ is best represented by

A.


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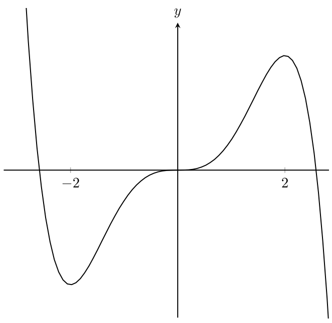

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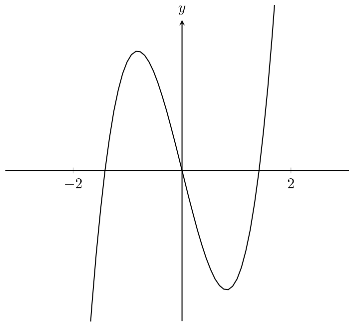

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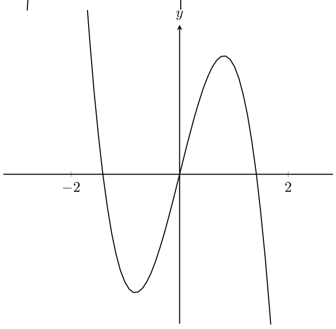
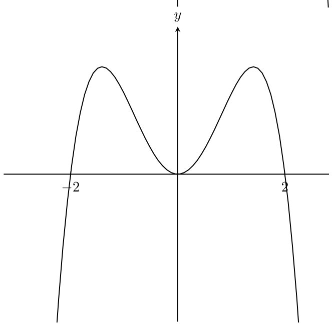


E.


Question 15

A function f satisfies the relation $\left(f(x)\right)^2 - f(3x)f(-x) = 0$.

A possible rule for f is

A. $f(x) = \dfrac{1}{x-1}$

B. $f(x) = \sqrt{x+1}$

C. $f(x) = \log_e(x+1)$

D. $f(x) = 2^{x-1}$

E. $f(x) = x-1$

Question 16

If $\int_{k}^{2k} \frac{1}{\sqrt{2x+1}}\,dx = 2$ and $k > -\frac{1}{2}$, then k equals

A. $2\sqrt{3}+3$

B. 12

C. 10

D. 8

E. $\sqrt{2}+2$

Question 17

A bag contains 20 marbles, of which n are blue. The remainder of the marbles are red.

One marble is drawn from the bag, its colour is noted and then it is returned to the bag. Five marbles are inspected in this way.

The probability that three or more of the marbles inspected were blue, given that more than one of the marbles inspected was blue, is approximately 0.48.

The number of blue marbles in the bag is equal to

A. 5

B. 8

C. 10

D. 12

E. 15

Question 18

Let f be a one-to-one differentiable function such that $f(5) = 3$ and $f'(5) = -2$.

The function g is differentiable and $g(x) = f^{-1}(x)$ for all x.

The tangent to the graph of the function g at the point where $x = 3$ is

A. $y = 2x - 1$

B. $y = \frac{1}{2}x + \frac{1}{2}$

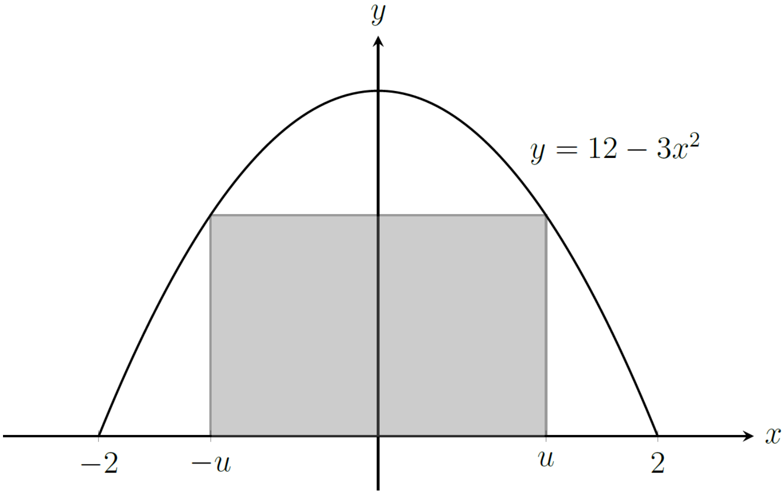
C. $y = \frac{1}{2}x + \frac{7}{2}$

D. $y = -\frac{1}{2}x + \frac{11}{2}$

E. $y = -\frac{1}{2}x + \frac{13}{2}$

Question 19

Consider the rectangle shaded in the diagram below.



The value of u for which the area is maximised and the maximum area, respectively, are equal to

A. $\frac{2}{\sqrt{3}}$ and $\frac{16}{\sqrt{3}}$

B. $\frac{2}{\sqrt{3}}$ and $\frac{32}{\sqrt{3}}$

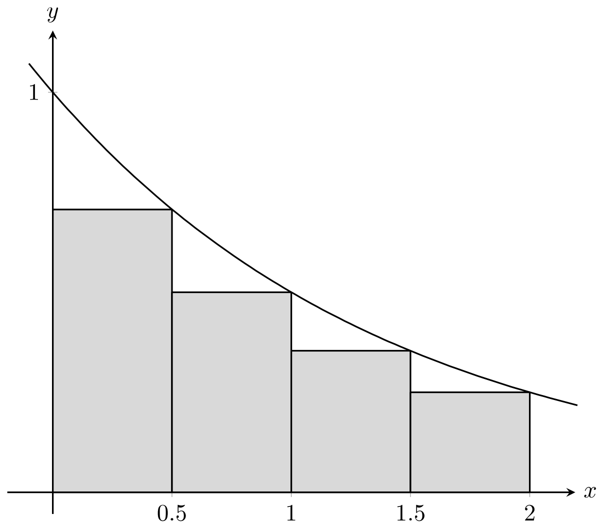
C. $\frac{1}{\sqrt{3}}$ and $\frac{11}{\sqrt{3}}$

D. $\frac{1}{\sqrt{3}}$ and $\frac{22}{\sqrt{3}}$

E. $\sqrt{3}$ and 6

Question 20

The area between the x-axis and the graph of $y = 2^{-x}$ over the interval [0, 2] is approximated using four bars as shown below.



The error in this approximation is closest to

A. 0.15

B. 0.16

C. 0.17

D. 0.18

E. 0.19

END OF SECTION A
TURN OVER

SECTION B

Instructions for Section B

Answer **all** questions in the spaces provided. Write using blue or black pen.

In all questions where a numerical answer is required, an exact value must be given unless otherwise specified.

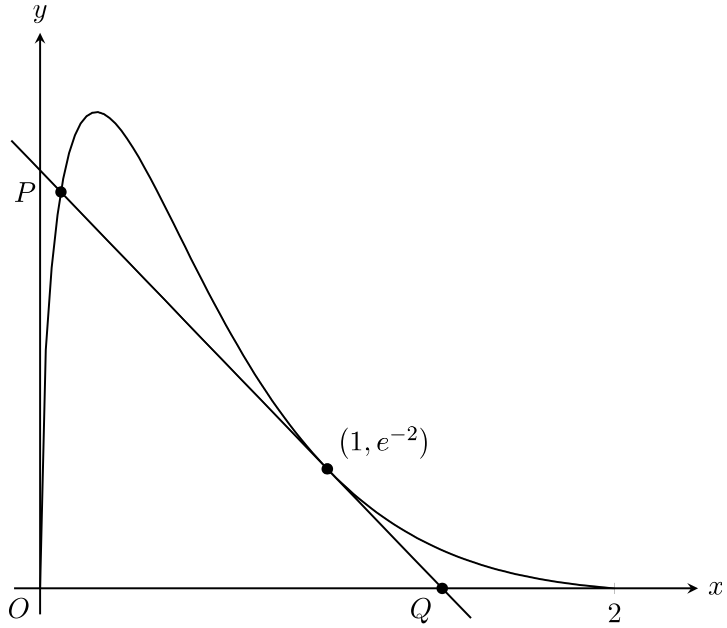
In questions where more than one mark is available, appropriate working **must** be shown.

Unless otherwise indicated, the diagrams in this book are **not** drawn to scale.

Question 1 (10 marks)

Let $f:[0,\ 2] \to R$, $f(x) = \sqrt{x}(2-x)e^{-2x}$.

The diagram below shows the graph of f and its tangent at $x = 1$. The tangent intersects the graph of f at P and the horizontal axis at Q.



a. Find $f'(x)$ in the form $\dfrac{\left(ax^2+bx+c\right)e^{-2x}}{2\sqrt{x}}$.

1 mark

b. **i.** Find the value of x at which the maximum value of the function f occurs.

1 mark

ii. Find the maximum value of the function f correct to four decimal places.

1 mark

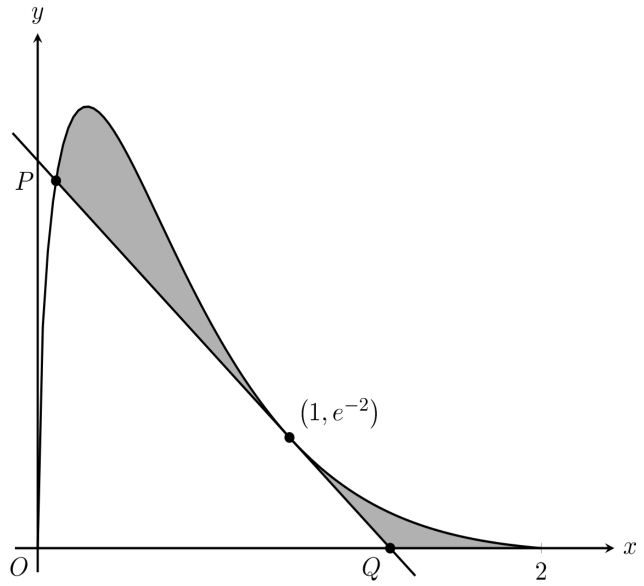
c. **i.** Find the equation of the tangent to the graph of f at $x = 1$ in the form $y = \dfrac{mx + n}{2e^2}$.

1 mark

ii. Find the x-coordinate of P correct to four decimal places.

1 mark

iii. Consider the shaded region in the diagram below.



Find the area of the shaded region correct to four decimal places.

2 marks

d. Let $R(r,s)$ be the point on the tangent to *f* at $x=1$ that is closest to the origin *O*.

i. Find the coordinates of *R*.

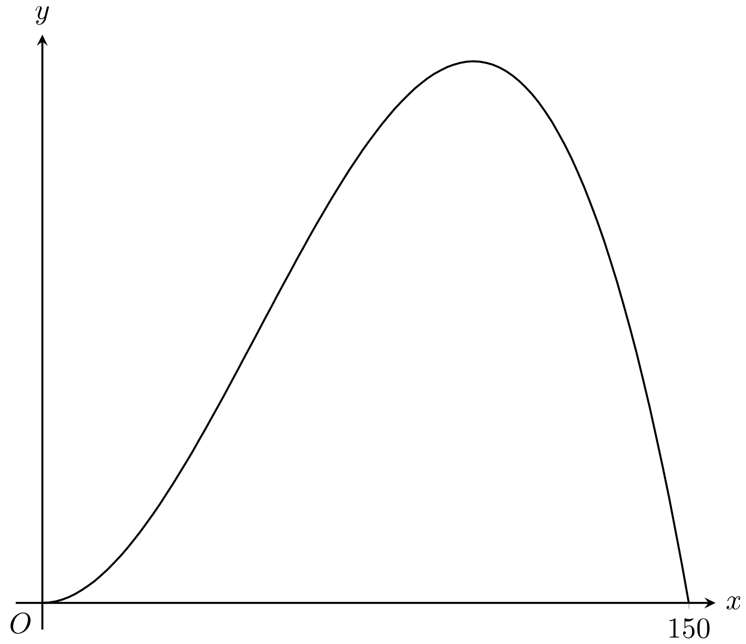
2 marks

ii. Find the distance from *R* to the origin.

1 mark

Question 2 (15 marks)

One side of the vertical stabiliser of a vintage aircraft needs to be painted. The vertical stabiliser is modelled by $y = \dfrac{x^2(150-x)}{2500}$, $x \in [0, 150]$, where x is the horizontal distance in centimetres and y is the vertical distance in centimetres, as shown in the diagram below.



a. Find $\dfrac{dy}{dx}$.

1 mark

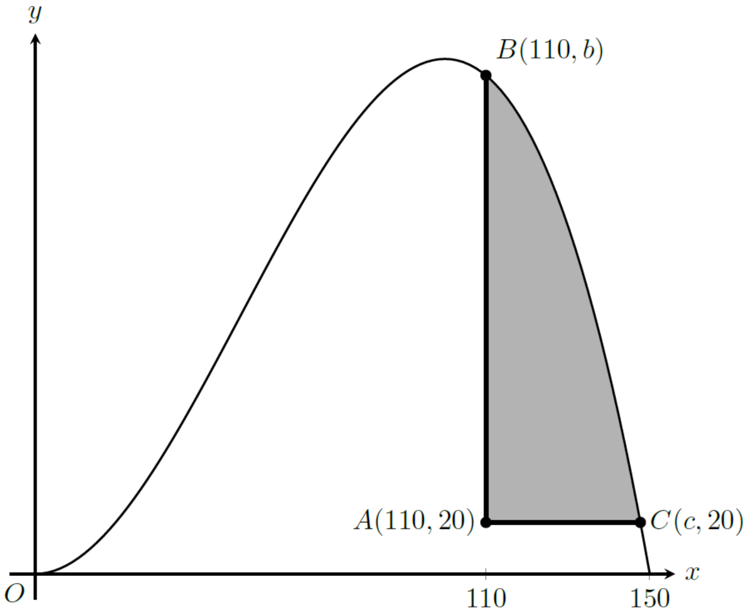
b. State the set of values for which y is strictly decreasing.

1 mark

c. Find the area of the vertical stabiliser.

1 mark

The rudder is part of the vertical stabiliser. It enables the pilot to move the aircraft from side to side. The bottom-left corner of the rudder is at $A(110, 20)$ and extends vertically and horizontally as shown in the shaded region in the diagram below.



d. **i.** State the value of c correct to three decimal places.

1 mark

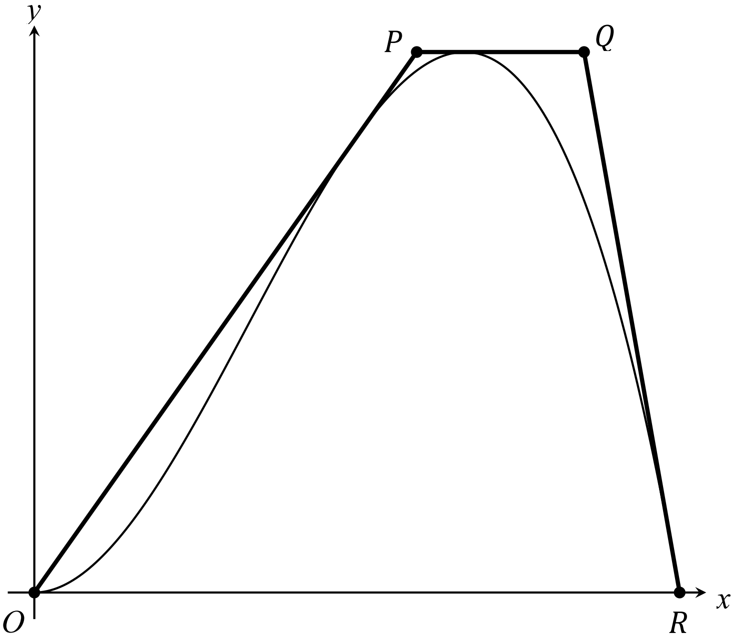
ii. State the value of b.

1 mark

e. Find the area of the rudder correct to three decimal places.

2 marks

A trapezium $OPQR$ is used to give an approximation of the area of the vertical stabiliser. The tangent to y which passes through the origin $O(0, 0)$ is used for the segment OP, and the tangent to y which passes through $R(150, 0)$ is used for the segment QR. The horizontal tangent to y is used for the line segment PQ, as shown in the diagram below.



f. **i.** Find the equation of the line segment OP.

1 mark

ii. Find the equation of the line segment QR.

1 mark

iii. Show that the x-coordinates of P and Q are $\frac{800}{9}$ and $\frac{1150}{9}$ respectively.

2 marks

iv. Find the area of the trapezium *OPQR*.

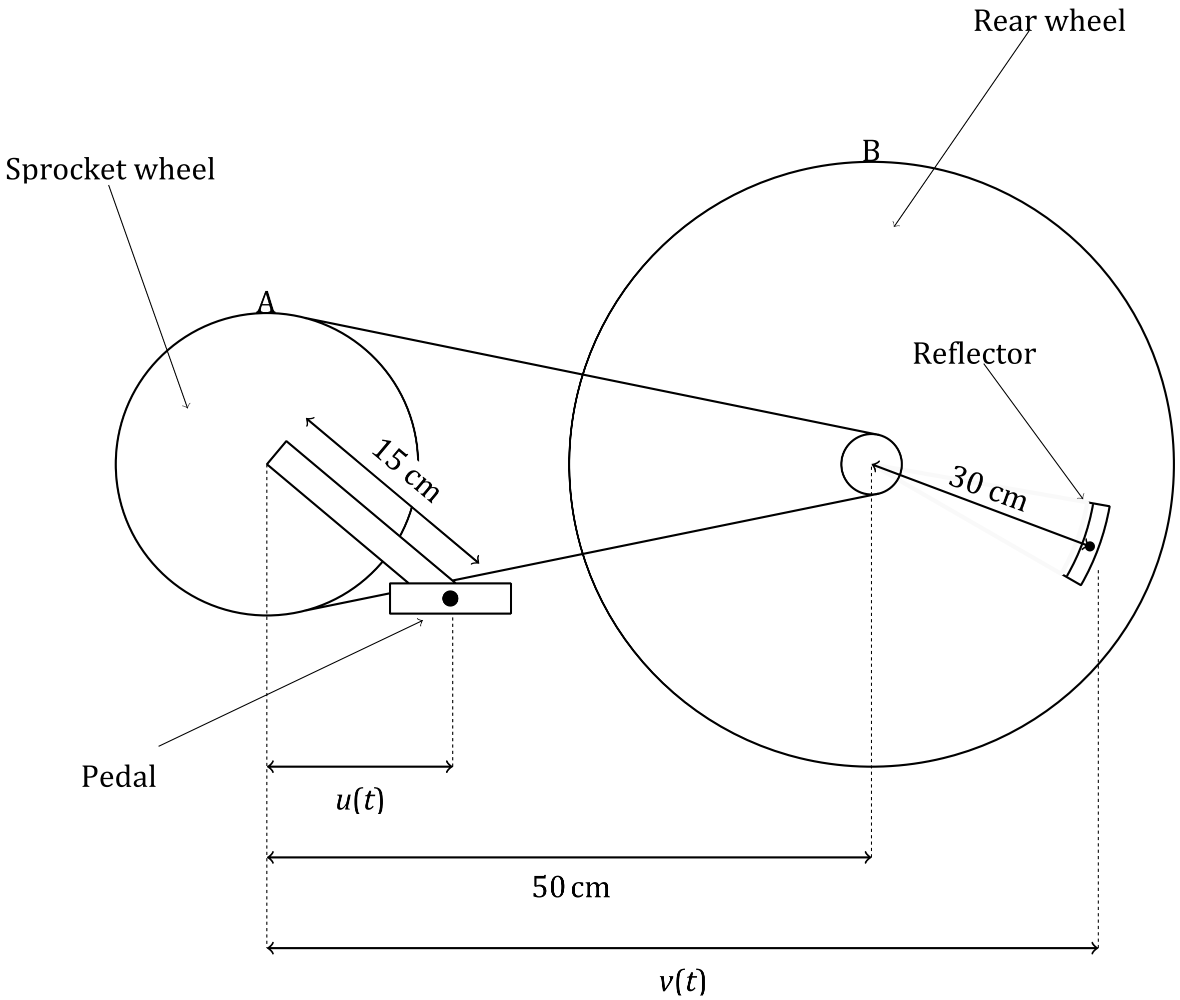
2 marks

v. Find the error in the approximation obtained in **part f.iv.** as a percentage of the actual area correct to two decimal places.

2 marks

Question 3 (11 marks)

The horizontal distance between the centre of a bike pedal and a reflector on the back wheel of the bike is to be investigated.



The pedals are attached to a sprocket wheel (wheel A) and the distance between the centre of the sprocket wheel and the centre of the back wheel (wheel B) is 50 centimetres. The horizontal distance at any time $t \geq 0$ between the centre of wheel A and the centre of the pedal is $u(t)$ where t is the time in seconds.

When $t = 0$ the pedal is directed straight down. Wheel A makes a complete revolution once every second.

a. $u(t)$ is modelled by the formula $u(t) = a\sin(nt)$.

Justify that $a = 15$ and $n = 2\pi$.

1 mark

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Wheel B makes three complete revolutions every two seconds. When $t = 0$ the reflector is at its lowest point. The horizontal distance between the centre of wheel A and the centre of the reflector is $v(t)$ centimetres, where $t \geq 0$ and measured in seconds.

b. $v(t)$ is modelled by $v(t) = b + c\sin(mt)$.

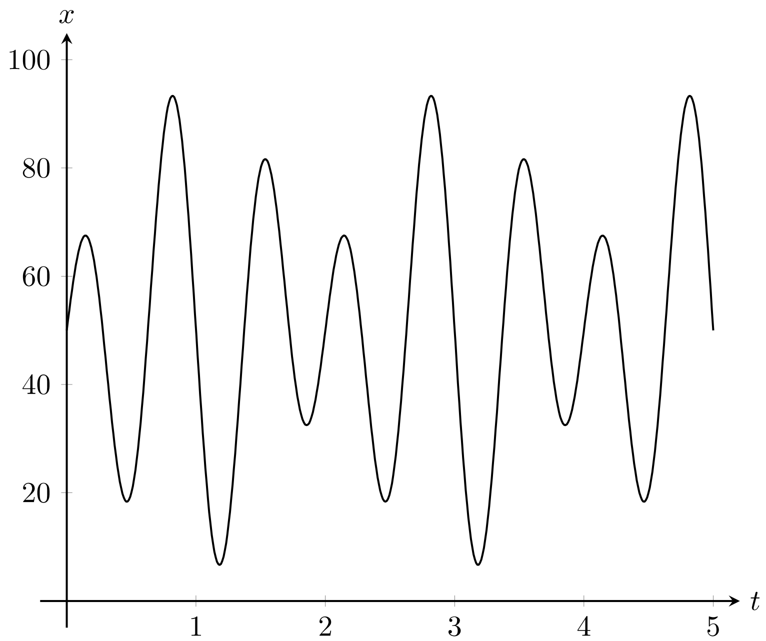
Justify that $b = 50$, $c = 30$ and $m = 3\pi$.

1 mark

The horizontal distance between the centre of the pedal and the centre of the reflector is

$$x(t) = 50 + 30\sin(3\pi t) - 15\sin(2\pi t)$$

Part of the graph of $x(t)$ is shown below.



c. State the period of $x(t)$.

1 mark

d. Find the maximum value of $x(t)$ in centimetres and the value of t at which it first occurs correct to two decimal places.

2 marks

e. Find the area between the graph of $x(t)$, the horizontal axis and the lines where $t = 0$ and $t = 1$.

2 marks

f. The integral $\int_1^2 \left(p \cdot x(t) + q\right) dt$, where p and q are integers, has the same value as the area found in **part e.**

Find the value of p and q.

2 marks

g. **i.** Find $\frac{dx}{dt}$.

1 mark

ii. Find the least value of t, where $t > 0$, such that $\frac{dx}{dt} = 0$ correct to two decimal places.

1 mark

Question 4 (12 marks)

A certain temperate rainforest contains two species of trees, which scientists have denoted type A and type B.

Type A trees tend not to grow close to one another. The distance between a randomly selected type A tree and its nearest type A neighbour is normally distributed with a mean of 3.8 m and a standard deviation of 0.9 m.

a. Find the probability that a distance between a randomly selected type A tree and its nearest type A neighbour is greater than 3 m, correct to three decimal places.

1 mark

The diameter (at its widest) of mature type A trees is normally distributed with a mean of 55 cm and a standard deviation of 7 cm.

b. Find the probability that a randomly selected mature type A tree has a diameter that is between 51 cm and 56 cm, correct to three decimal places.

2 marks

c. Twenty type A trees are selected at random.

Determine the probability that more than five trees have a diameter that is between 51 cm and 56 cm, correct to three decimal places.

2 marks

d. It is known that 15% of type B trees have a diameter that is less than 59 cm and 20% of type B trees have a diameter that is greater than 68 cm. Assume that the diameter of type B trees are normally distributed.

Determine the mean and standard deviation of the diameter of type B trees, correct to three decimal places.

4 marks

In the temperate rainforest 90% of the trees are type A trees and 10% are type B trees.

e. Show that the probability that a randomly selected tree has a diameter that is greater than 55 cm is 0.547, correct to three decimal places.

1 mark

f. A single tree was examined and found to have a diameter that is greater than 55 cm.

Determine the probability that the tree was a type A tree, correct to three decimal places.

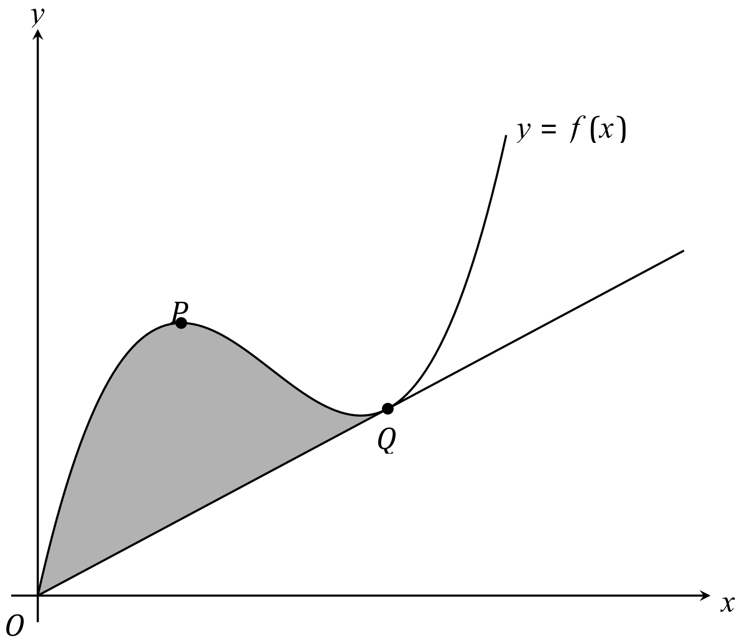
2 marks

Question 5 (12 marks)

Let $f:[0,\infty)\to R$, $f(x)=(ax-1)(x-1)^2+1$ where $a>\frac{1}{4}$.

The shaded region in the diagram below shows the area bounded by the graph of *f* and the tangent to the graph of *f* which passes through the origin.

The point *P* is the local maximum of *f* and the point *Q* is the point of intersection of the graph of *f* and the tangent to the graph of *f* that passes through the origin.



a. Find the coordinates of *P,* in terms of *a*.

1 mark

b. Find the *x*-coordinate of *Q,* in terms of *a*.

1 mark

c. Find the area of the shaded region, in terms of a.

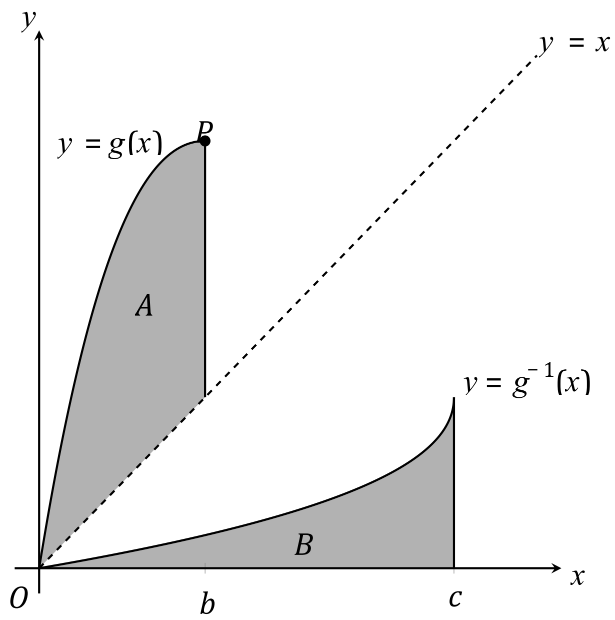
3 marks

d. Find the value of a, where $a > \frac{1}{4}$, for which the area of the shaded region is the minimum.

2 marks

Let $g:[0, b] \rightarrow R$, $g(x)=(ax-1)(x-1)^2+1$ where $a>\frac{1}{4}$ and b be the x-coordinate of P, where P is the local maximum of f.

The diagram below shows the graphs of g, g^{-1} and $y = x$. The region denoted by A is bounded by the graph of g, the line $y = x$ and $x = b$. The region denoted by B is bounded by g^{-1}, the x-axis and $x = c$ where c is the maximum value in the domain of g^{-1}.



e. Find the area of B.

3 marks

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f. Find the value of a such that the areas of A and B are equal.

2 marks

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END OF QUESTION AND ANSWER BOOK