EXTRA
EXPERIENCED TEACHER RESOURCES & ASSESSMENTS

Mathematical Methods
EXAM 2 UNITS 3&4
2020 Written Trial Examination

Reading time: 15 minutes

Writing time: 2 hours

SOLUTIONS

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SECTION A – Multiple-choice questions

Question 1

The set of values of k for which $2x^2 + (k+1)x + k = 0$ has two real solutions is

A. $k \in \left(-\infty, 3+2\sqrt{2}\right)$

B. $k \in R$

C. $k \in \left(3-2\sqrt{2}, \infty\right)$

D. $k \in \left(-\infty, 3-2\sqrt{2}\right) \cup \left(3+2\sqrt{2}, \infty\right)$

E. $k \in \left\{3-2\sqrt{2}, 3+2\sqrt{2}\right\}$

For the quadratic to have 2 real solutions, the discriminant must be greater than zero.

$$\begin{aligned}
&\Delta = b^2 - 4ac \\
&(k+1)^2 - 4(2)k > 0 \\
&k^2 + 2k + 1 - 8k > 0 \\
&k^2 - 6k + 1 > 0 \\
&(k^2 - 6k + 9) - 9 + 1 > 0 \\
&(k-3)^2 - 8 > 0 \\
&(k - 3 - 2\sqrt{2})(k - 3 + 2\sqrt{2}) > 0 \\
&\therefore k \in (-\infty, 3-2\sqrt{2}) \cup (3+2\sqrt{2}, \infty)
\end{aligned}$$

Therefore, the answer is D.

Question 2

Let $g(x)=\frac{\log_e(ax)}{x^2}$, where a is non-zero real constant.

The derivative, $g'(x)$, is given by

A. $\frac{1}{2x^2}$

B. $\frac{2x\log_e(ax)-x}{x^4}$

C. $\frac{1-2\log_e(ax)}{x^3}$

D. $2x^2$

E. $x+2x\log_e(ax)$

Using the quotient rule,

$$Let\ u(x)=\log_e(ax)\ \ and\ \ v(x)=x^2$$

$$So\ \ u'(x)=\frac{1}{x}\ \ and\ \ v'(x)=2x$$

$$g'(x)=\frac{x^2\left(\frac{1}{x}\right)-2x\left(\log_e(ax)\right)}{\left(x^2\right)^2}$$

$$g'(x)=\frac{x-2x\left(\log_e(ax)\right)}{x^4}$$

$$g'(x)=\frac{x\left(1-2\log_e(ax)\right)}{x^4}$$

$$\therefore g'(x)=\frac{1-2\log_e(ax)}{x^3}$$

Therefore, the answer is C.

Question 3

A box contains five white tiles and six black tiles. Two tiles are drawn at random from the box without replacement. The probability that the tiles are the **same** colour is

A. $\frac{2}{11}$

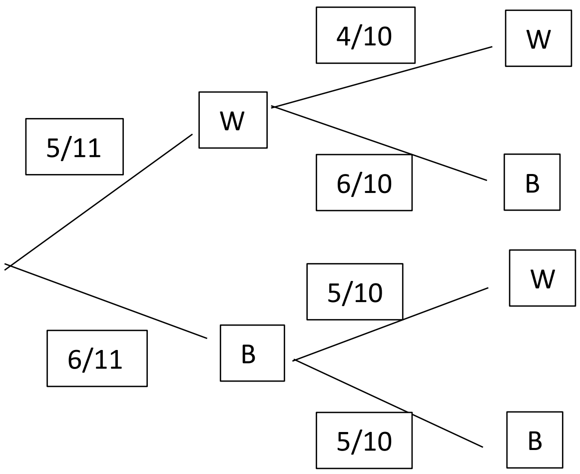
B. $\frac{5}{22}$

C. $\frac{3}{11}$

D. $\frac{5}{11}$

E. $\frac{6}{13}$

Using a tree diagram to model this sampling without replacement problem,



Two tiles the same colour means 2 whites or 2 blacks so,

$$\Pr(same\ colour) = \Pr(WW) + \Pr(BB)$$

$$\Pr(same\ colour) = \frac{5}{11} \times \frac{4}{10} + \frac{6}{11} \times \frac{5}{10} = \frac{5}{11}$$

Therefore, the answer is D.

Question 4

If $x-b$ is a factor of $3x^4-2x^3-x^2$, then the value of b could be

A. $\frac{-1}{3},0,1$

B. $0,1$

C. $\frac{1}{3},0$

D. $0,\frac{1}{3},1$

E. $0,1,3$

Using the factor theorem and substituting in the value *b*, hence using the Null Factor Law,

$$\begin{aligned}
&Let\ \ f(x)=3x^4-2x^3-x^2\\
&f(b)=3b^4-2b^3-b^2=0\\
&b^2(3b^2-2b-1)=0\\
&b^2(3b+1)(b-1)=0\\
&\therefore b=\frac{-1}{3},0,1
\end{aligned}$$

Therefore, the answer is A.

Question 5

The simultaneous linear equations $y-(m+2)x=4$ and $my-3x=k+1$ have no solutions when

A. $m=3$ and $k=-13$ or $m=1$ and $k\neq 3$

B. $m=3$ and $k\neq -13$ or $m\neq 1$ and $k\neq 3$

C. $m=1$ and $k=3$ or $m=1$ and $k=3$

D. $m=1$ and $k=-13$ or $m\neq 1$ and $k\neq 3$

E. $m=-3$ and $k\neq -13$ or $m=1$ and $k\neq 3$

For simultaneous linear equations to have no solutions, the must have equal gradients but different *y*-intercepts. Restate equations as

$$y=(m+2)x+4 \quad and \quad y=\frac{3}{m}x+\frac{k+1}{m}$$

Equating gradients

$$m+2=\frac{3}{m}$$
$$m^2+2m-3=0$$
$$(m+3)(m-1)=0$$
$$\therefore m=-3,1$$

Checking *y*-intercepts

$$When\ m=-3,$$
$$4=\frac{k+1}{-3}$$
$$-12=k+1$$
$$\therefore k=-13\ \ so\ we\ need\ k\neq -13$$
$$When\ m=1,$$
$$4=\frac{k+1}{1}$$
$$4=k+1$$
$$\therefore k=3\ \ so\ we\ need\ k\neq 3$$

Therefore, the answer is E.

Question 6

Let $f(x)=\sqrt{4-x}$ and $g(x)=x^2$.
For $f(g(x))$ to exist, the domain and range of $g(x)$ respectively, must be changed to

A. $[-\infty,0]$ and $[4,\infty]$

B. R and $[0,\infty]$

C. $[0,4]$ and $[-2,2]$

D. $[0,2]$ and $[-4,0]$

E. $[-2,2]$ and $[0,4]$

	Dom	Ran
f(x)	$(-\infty,4]$	$[0,\infty)$
g(x)	R	$[0,\infty)$

For $f(g(x))$ to exist, the range of $g(x)$ must be a subset or equal to the domain of $f(x)$. The range of $g(x)$ must be restricted for $f(g(x))$ to exist so

	Dom	Ran
f(x)	$(-\infty,4]$	$[0,\infty)$
g(x)	$[-2,2]$	$[0,4]$

Therefore, the answer is E.

Question 7

The point $A(4,-1)$ lies on the graph of the function $f(x)$. A transformation maps the graph of $f(x)$ to the graph of $g(x)$, where $g(x) = -3f(x+2)+1$.
If that same transformation maps the point A to the point P, the coordinates of the point P are

A. $(4,2)$

B. $(2,4)$

C. $(-2,4)$

D. $(-4,2)$

E. $(-1,4)$

Looking at the series of transformations performed on $f(x)$ as

- dilation by a factor of 3 from the *x*-axis
- reflected in the *x*-axis
- translated -2 units horizontally
- translated +1 unit vertically

then the initial point at $A(4,-1)$ will now also transform as

$$(4,-1) \to (4,-3) \to (4,3) \to (2,3) \to (2,4)$$

Therefore, the answer is B.

Question 8

A discrete random variable has a binomial distribution with a mean of 3.15 and a variance of 1.7325. The values of n (the number of independent trials) and p (the probability of success in each trial) are

A. $n = 9$ and $p = 0.45$

B. $n = 7$ and $p = 0.55$

C. $n = 7$ and $p = 0.45$

D. $n = 5$ and $p = 0.5$

E. $n = 7$ and $p = 0.65$

Set up two equations for mean and variance and solve simultaneously.

$$np = 3.15 \quad and \quad np(1-p) = 1.7325$$
$$1 - p = \frac{1.7325}{3.15}$$
$$1 - p = 0.55$$
$$\therefore p = 0.45$$
$$Substitute \ \ p$$
$$n(0.45) = 3.15$$
$$\therefore n = 7$$

Therefore, the answer is C.

Question 9

A six-sided die is loaded such that the chance of throwing a 1 is $\frac{x}{5}$, the chance of a 2 is $\frac{1}{5}$ and the chance of a 3 is $\frac{1}{5}(1+x)$. The chance of a 4, 5 or 6 is $\frac{1}{6}$ and the die is thrown twice. The probability of getting a sum of 6 on the dice is

A. $\frac{1}{75}\left(3x^2+11x+8\right)$

B. $\frac{1}{30}\left(3x^2+11x+8\right)$

C. $3x^2+11x+8$

D. $\frac{1}{75}\left(6x^2+17x+11\right)$

E. $\frac{1}{30}\left(6x^2+17x+11\right)$

Using a lattice diagram to show all the possibilities for a sum of 6, we can get (1,5) or (5,1), (2,4) or (4,2) and (3,3).

$$For\ 6:(1,5)\ or\ (5,1) \quad 2\left(\tfrac{x}{5}\times\tfrac{1}{6}\right)=\tfrac{2x}{30}$$

$$(2,4)\ or\ (4,2) \quad 2\left(\tfrac{1}{5}\times\tfrac{1}{6}\right)=\tfrac{2}{30}$$

$$(3,3) \quad \tfrac{1}{5}(1+x)\times\tfrac{1}{5}(1+x)=\tfrac{1}{25}(1+x)^2$$

$$\Pr(6)=\tfrac{2x}{30}+\tfrac{2}{30}+\tfrac{1}{25}(1+x)^2$$

$$\Pr(6)=\frac{10x+10+6(1+x)^2}{150}$$

$$\Pr(6)=\frac{10x+10+6+12x+6x^2}{150}$$

$$\therefore \Pr(6)=\frac{3x^2+11x+8}{75}=\frac{1}{75}\left(3x^2+11x+8\right)$$

Therefore, the answer is A.

Question 10

The average value of the function with the rule $f(x) = 2x^3 + 3x$ over the interval $[0, m]$, where $m > 0$ is

A. $2m^2 + 3$

B. $\frac{2m^3 + 3}{m}$

C. $\frac{2}{m}(m + 3)$

D. $\frac{m}{2}(m^2 + 3)$

E. $6m^2 + 3$

The average value of the function is the integral which represents the rectangle equivalent to the area.

$$Average\ value = \frac{1}{m}\int_0^m 2x^3 + 3x\ dx$$

$$Avg\ value = \frac{1}{m}\left[\frac{2x^4}{4} + \frac{3x^2}{2}\right]_0^m$$

$$Avg\ value = \frac{1}{m}\left[\frac{x^4}{2} + \frac{3x^2}{2}\right]_0^m$$

$$Avg\ value = \frac{1}{m}\left[\left(\frac{m^4}{2} + \frac{3m^2}{2}\right) - (0)\right]$$

$$Avg\ value = \frac{m^3}{2} + \frac{3m}{2}$$

$$\therefore Avg\ value = \frac{m}{2}(m^2 + 3)$$

Therefore, the answer is D.

Question 11

A tangent line to the graph of $y = xe^{2x}$ can be found at $x = 1$.
This tangent will cross the y-axis at

A. $-2e^2$

B. e^2

C. $\frac{-2}{e^2}$

D. $3e^2$

E. $\frac{e^2}{2}$

The coordinate at $x = 1$ will be $y = (1)e^{2(1)}, \quad (1, e^2)$.

To find the gradient of the tangent

$$\frac{dy}{dx} = e^{2x}(1) + 2xe^{2x}$$
$$\frac{dy}{dx} = e^{2x} + 2xe^{2x}$$
$$When \ x = 1,$$
$$\frac{dy}{dx} = e^2 + 2(1)e^{2(1)} = 3e^2$$

For the equation of the tangent

$$y - e^2 = 3e^2(x-1)$$
$$y - e^2 = 3e^2x - 3e^2$$
$$\therefore y = 3e^2x - 2e^2$$

Therefore, the answer is A.

Question 12

The domain and range of $f(x) = 3\log_e(x+1) - x$, respectively, is given by

A. $(-1, \infty)$ *and* $\left(\log_e\left(\frac{27}{e^2}\right), \infty\right)$

B. $(-1, \infty)$ *and* $\left(-\infty, \log_e\left(\frac{27}{e^2}\right)\right)$

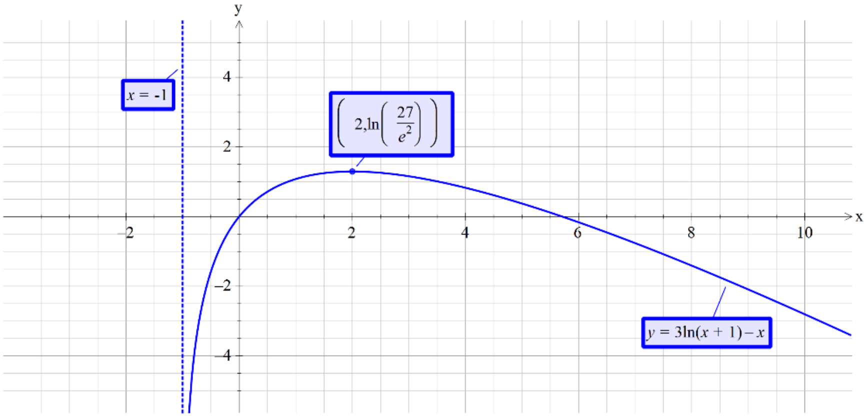
C. $(-1, \infty)$ *and* $\left(-\infty, \log_e\left(27e^2\right)\right)$

D. $(-1, \infty)$ *and* $\left(\log_e\left(27e^2\right), \infty\right)$

E. $\left(-\infty, \log_e\left(\frac{27}{e^2}\right)\right)$ *and* $(-1, \infty)$

The graph can be observed to have an asymptote at $x = -1$ and a turning point as a maximum, hence shown in the corresponding domain and range.

$$f(x) = 3\log_e(x+1) - x$$
$$f'(x) = 3\left(\frac{1}{x+1}\right) - 1 = 0$$
$$\frac{3}{x+1} = 1$$
$$x+1 = 3$$
$$\therefore x = 2$$
$$f(2) = 3\log_e(3) - 2$$
$$f(2) = \log_e(27) - 2\log_e(e)$$
$$f(2) = \log_e(27) - \log_e(e^2)$$
$$\therefore f(2) = \log_e\left(\frac{27}{e^2}\right)$$



Therefore, the answer is B.

Question 13

If the point $(3a,-b)$ is transformed by

$$T\left(\begin{bmatrix} x \\ y \end{bmatrix}\right)=\begin{bmatrix} -3 & 0 \\ 0 & 2 \end{bmatrix}\begin{bmatrix} x \\ y \end{bmatrix}+\begin{bmatrix} 4 \\ -1 \end{bmatrix}$$

then the image coordinate is

A. $(4+9a, 2b-1)$

B. $(9a-4, 2b+1)$

C. $(4-9a, -2b-1)$

D. $(3a, -b)$

E. $(-9a, -2b)$

Substitute the initial coordinate into the initial matrix

$$\begin{bmatrix} x' \\ y' \end{bmatrix}=\begin{bmatrix} -3 & 0 \\ 0 & 2 \end{bmatrix}\begin{bmatrix} 3a \\ -b \end{bmatrix}+\begin{bmatrix} 4 \\ -1 \end{bmatrix}$$

$$\begin{bmatrix} x' \\ y' \end{bmatrix}=\begin{bmatrix} -9a \\ -2b \end{bmatrix}+\begin{bmatrix} 4 \\ -1 \end{bmatrix}$$

$$x'=-9a+4$$

$$y'=-2b-1$$

$$\therefore image\ coordinate\ (4-9a, -2b-1)$$

Therefore, the answer is C.

Question 14

Which one of the following is the inverse function of $f:(-\infty,-4]\to R,\ f(x)=x^2+8x-1$?

A. $f^{-1}:[-17,\infty)\to R,\ f^{-1}(x)=-4+\sqrt{x+17}$

B. $f^{-1}:[17,\infty)\to R,\ f^{-1}(x)=-4-\sqrt{x-17}$

C. $f^{-1}:[-17,17)\to R,\ f^{-1}(x)=4-\sqrt{x+17}$

D. $f^{-1}:[-17,\infty)\to R,\ f^{-1}(x)=-4-\sqrt{x+17}$

E. $f^{-1}:[0,\infty)\to R,\ f^{-1}(x)=-4-\sqrt{x+17}$

By completing the square, the $ran\ f(x)\in[-17,\infty)$.
To find the inverse function,

$$
\begin{aligned}
&Let\ \ y=f(x),\ swap\ \ x\,\&\,y\\
&x=y^2+8y-1\\
&x+1=(y^2+8y+16)-16\\
&x+17=(y+4)^2\\
&y+4=\pm\sqrt{x+17}\quad but\quad dom\ f(x)\in(-\infty,-4]\\
&y=-\sqrt{x+17}-4\\
&\therefore\ f^{-1}:[-17,\infty)\to R,\ f^{-1}(x)=-\sqrt{x+17}-4
\end{aligned}
$$

Therefore, the answer is D.

Question 15

Given that $\frac{d(2x\cos(3x))}{dx} = 2\cos(3x) - 6x\sin(3x)$, then $\int x\sin(3x)\,dx$ is equal to

A. $\frac{1}{9}\left(\int 2\cos(3x)\ dx - 2x\cos(3x)\right)$

B. $\frac{1}{3}\sin(3x) - \frac{1}{9}\cos(3x) + c$

C. $\frac{1}{3}\left(\int 2\cos(3x)\ dx - 2x\cos(3x)\right)$

D. $\frac{1}{3}\cos(3x) - \frac{1}{9}\sin(3x) + c$

E. $\frac{1}{6}\left(\int 2\cos(3x)\ dx - 2x\cos(3x)\right)$

Using integration by recognition,

$$\int 2\cos(3x) - 6x\sin(3x)\ dx = 2x\cos(3x)$$

$$\int 2\cos(3x)\ dx - \int 6x\sin(3x)\ dx = 2x\cos(3x)$$

$$\int 2\cos(3x)\ dx - 2x\cos(3x) = \int 6x\sin(3x)\ dx$$

$$\frac{1}{6}\int 6x\sin(3x)\ dx = \frac{1}{6}\left(\int 2\cos(3x)\ dx - 2x\cos(3x)\right)$$

$$\therefore \int x\sin(3x)\ dx = \frac{1}{6}\left(\int 2\cos(3x)\ dx - 2x\cos(3x)\right)$$

Therefore, the answer is E.

Question 16

The waiting time at a kiosk, in minutes, is normally distributed with a mean of 8 and a standard deviation of 1.2.
When a customer arrives at the kiosk, the probability that they wait longer than 10 minutes is closest to

A. 0.7408

B. 0.4780

C. 0.8748

D. 0.0478

E. 0.7804

Let X represent the waiting time at the kiosk with the parameters

$$\Pr(X > 10) = normCdf(10, \infty, 8, 1.2)$$
$$\therefore \Pr(X > 10) = 0.0478$$

Therefore, the answer is D.

Question 17

Let $h:[0,4] \to R, h(x) = \frac{-2}{1-x} + 3$

Which one of the following statements about h is true?

A. An endpoint is $\left(4, \frac{5}{3}\right)$

B. The y-intercept is $(0,1)$

C. The asymptotes are $x = -1$ and $y = 3$

D. The range of $h(x)$ is R^+

E. $h(x)$ has a stationary point at $x = 2$

Going through the options individually,

A.

$$h(4) = \frac{-2}{1-4} + 3$$

$$h(4) = \frac{11}{3} \quad coordinate \left(4, \frac{11}{3}\right)$$

B.

$$h(0) = \frac{-2}{1-0} + 3$$

$$h(4) = 1 \quad coordinate \ (0,1)$$

C.

The asymptotes are $x = 1$ and $y = 3$

D.

The range of $h(x)$ $\in R \setminus \{3\}$

E.

Hyperbolas do not have stationary points.

Therefore, the answer is B.

Question 18

The average rate of change of $g(x)=x^2\sin(2x)$ over the interval $\left[\frac{-\pi}{2},\frac{\pi}{4}\right]$ is

A. π

B. $\frac{\pi}{12}$

C. $\frac{\pi}{2}$

D. $\frac{12}{\pi}$

E. $\frac{3\pi}{4}$

Using the gradient equation to find the average rate of change,

$$g\left(\frac{-\pi}{2}\right)=\left(\frac{-\pi}{2}\right)^2\sin\left(\frac{-2\pi}{2}\right)$$

$$g\left(\frac{-\pi}{2}\right)=\frac{\pi^2}{4}\sin(-\pi)=0 \quad \left(\frac{-\pi}{2},0\right)$$

$$g\left(\frac{\pi}{4}\right)=\left(\frac{\pi}{4}\right)^2\sin\left(\frac{2\pi}{4}\right)$$

$$g\left(\frac{\pi}{4}\right)=\frac{\pi^2}{16}\sin\left(\frac{\pi}{2}\right)=\frac{\pi^2}{16} \quad \left(\frac{\pi}{4},\frac{\pi^2}{16}\right)$$

$$Average\ ROC=\frac{\pi^2/16-0}{\pi/4+\pi/2}$$

$$\therefore Average\ ROC=\frac{\pi^2/16}{3\pi/4}=\frac{\pi}{12}$$

Therefore, the answer is B.

Question 19

The area bounded by the graph of $f(x)$, the line $x = 0$, the line $x = 1$ and the x-axis, is $\frac{2}{e}+1$.

A possible equation for $f(x)$ is

A. $f(x)=-2e^{x-1}+3$

B. $f(x)=2e^{x+1}+3$

C. $f(x)=-2e^{x-1}-3$

D. $f(x)=-e^{x-1}+3$

E. $f(x)=-2e^{x}+3$

Going through the options individually,

A.

$$\int_0^1 -2e^{x-1}+3\,dx$$
$$=\left[-2e^{x-1}+3x\right]_0^1$$
$$=\left[\left(-2e^{0}+3(1)\right)-\left(-2e^{-1}+3(0)\right)\right]$$
$$=(-2+3)+2e^{-1}=1+\frac{2}{e}$$

B.

$$\int_0^1 2e^{x+1}+3\,dx$$
$$=\left[2e^{x+1}+3x\right]_0^1$$
$$=\left[\left(2e^{2}+3(1)\right)-\left(2e^{1}+3(0)\right)\right]$$
$$=(2e^{2}+3)-2e=2e^{2}-2e+3$$

C.

$$\int_0^1 -2e^{x-1}-3\,dx$$
$$=\left[-2e^{x-1}-3x\right]_0^1$$
$$=\left[\left(-2e^{0}-3(1)\right)-\left(-2e^{-1}-3(0)\right)\right]$$
$$=(-2-3)+2e^{-1}=-5+\frac{2}{e}$$

D.

$$\int_0^1 -e^{x-1}+3\ dx$$
$$=\left[-e^{x-1}+3x\right]_0^1$$
$$=\left[\left(-e^{0}+3(1)\right)-\left(-e^{-1}+3(0)\right)\right]$$
$$=(-1+3)+e^{-1}=2+\frac{1}{e}$$

E.

$$\int_0^1 -2e^{x}+3\ dx$$
$$=\left[-2e^{x}+3x\right]_0^1$$
$$=\left[\left(-2e^{1}+3(1)\right)-\left(-2e^{0}+3(0)\right)\right]$$
$$=(-2e+3)+2=5-2e$$

Therefore, the answer is A.

Question 20

Which one of the following statements is false for $f:(0,10]\rightarrow R,\ f(x)=\log_e(x)-\cos(x)$?

A. It has no y-intercept

B. It has an endpoint at $\frac{1}{\log_{10}(e)}-\cos(10)$

C. It has 3 turning points

D. $f'(x)=\frac{1+x\sin(x)}{x}$

E. It has a stationary point of inflection

Going through the options individually,

A.

The domain is exclusive at the lower bound, so no *y*-intercept is possible.

B.

$$f(10)=\log_e(10)-\cos(10)$$
$$f(10)=\frac{\log_{10}(10)}{\log_{10}(e)}-\cos(10)$$
$$\therefore f(10)=\frac{1}{\log_e(10)}-\cos(10)$$

C.

Sketching $f:(0,10]\rightarrow R,\ f(x)=\log_e(x)-\cos(x)$, you can observe 3 clear turning points, 2 maximums and 1 minimum.

D.

$$f'(x)=\frac{1}{x}+\sin(x)$$
$$f'(x)=\frac{1+x\sin(x)}{x}$$

E.

Sketching $f:(0,10]\rightarrow R,\ f(x)=\log_e(x)-\cos(x)$, you cannot observe any stationary points of inflection.

Therefore, the answer is E.

SECTION B – Extended Response questions

Question 1 (15 marks)

Let $f : R^+ \to R, f(x) = x^2 \log_e(x)$.

a. Find $\{x : f(x) = 0\}$.

2 marks

$x^2 \log_e(x) = 0$ **M1**

$Let\ x^2 = 0,\ x = 0\ but\ not\ possible,\ \therefore x \neq 0$

$Let\ \log_e(x) = 0,\ \therefore x = 1$ **A1**

b. **i.** Find the stationary point of $f(x)$.

3 marks

$f'(x) = \log_e(x)(2x) + x^2\left(\frac{1}{x}\right)$

$f'(x) = 2x\log_e(x) + x$

$f'(x) = x(2\log_e(x) + 1)$

$Let\ f'(x) = 0$

$x(2\log_e(x) + 1) = 0$ **M1**

$Let\ x = 0\ but\ not\ possible,\ \therefore x \neq 0$

$Let\ 2\log_e(x) + 1 = 0$

$2\log_e(x) = -1$

$\log_e(x) = \frac{-1}{2}$

$x = e^{-1/2} = \frac{1}{\sqrt{e}}$ **M1**

For the stationary point

$f\left(\frac{1}{\sqrt{e}}\right) = \left(\frac{1}{\sqrt{e}}\right)^2 \log_e\left(\frac{1}{\sqrt{e}}\right)$

$f\left(\frac{1}{\sqrt{e}}\right) = \frac{1}{e}\log_e\left(e^{-1/2}\right)$

$f\left(\frac{1}{\sqrt{e}}\right) = \frac{-1}{2e}$

$\therefore coordinate\ \left(\frac{1}{\sqrt{e}}, \frac{-1}{2e}\right)$ **A1**

ii. Show that the nature of the turning point is a minimum.

1 mark

x	$\frac{1}{2}$	$\frac{1}{\sqrt{e}}$	1
$f'(x)$	-0.19	0	1
slope	\	——	/

A1

Hence, the turning point is a local minimum.

c. Find the equation of the tangent line at $x = e$.

2 marks

$f(e) = e^2 \log_e(e)$

$f(e) = e^2 \quad \therefore (e, e^2)$

$f'(e) = 2e\log_e(e) + e$

$f'(e) = 3e$ **M1**

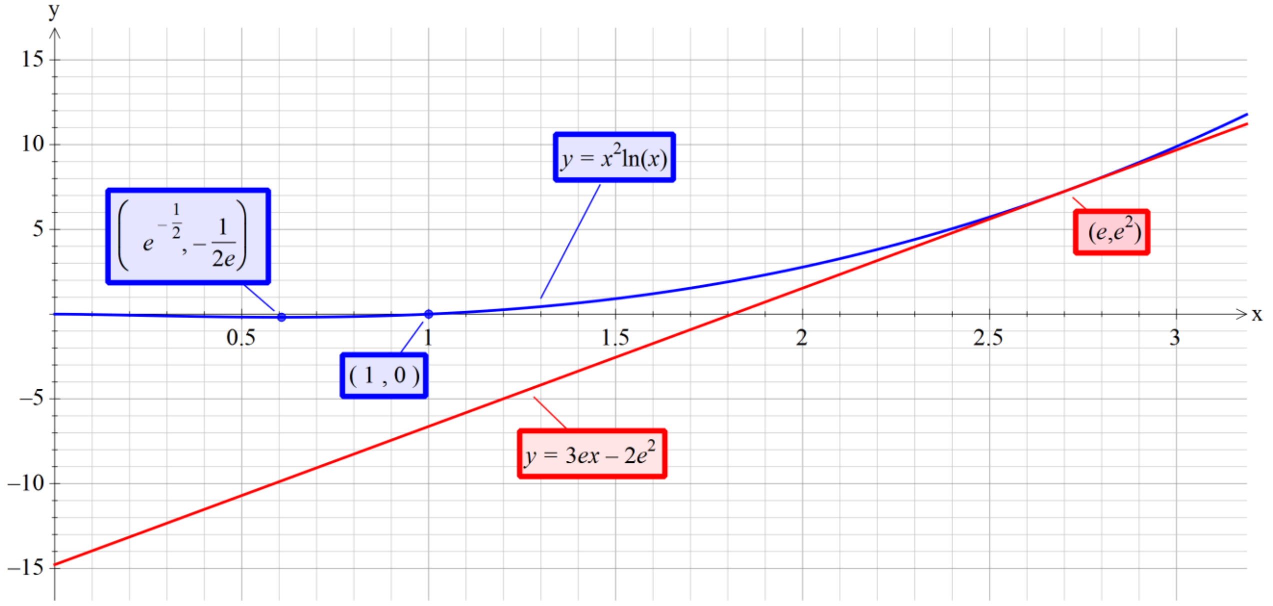
$y - e^2 = 3e(x - e)$

$y - e^2 = 3ex - 3e^2$

$\therefore y = 3ex - 2e^2$ **A1**

d. Sketch $f(x)$ and the tangent line at $x=e$, for $x \in (0,3]$.

3 marks



1 mark – $f(x)$ shape with endpoint $(3, 9\log_e(3))$

1 mark – $f(x)$ intercepts and turning point

1 mark – Tangent with endpoint $(3, 9e-2e^2)$

e. **i.** If $y = x^3\log_e(x)$, find $\dfrac{dy}{dx}$.

1 mark

$$\frac{dy}{dx} = \log_e(x)\left(3x^2\right) + x^3\left(\frac{1}{x}\right)$$

$$\frac{dy}{dx} = 3x^2\log_e(x) + x^2$$ **A1**

ii. Hence, algebraically find $\int_1^2 x^2 \log_e(x)\,dx$.

3 marks

$$\int_1^2 3x^2 \log_e(x) + x^2 \ dx = x^3 \log_e(x)$$

$$\int_1^2 3x^2 \log_e(x) \ dx + \int x^2 \ dx = x^3 \log_e(x)$$ **M1**

$$\int_1^2 3x^2 \log_e(x) \ dx = x^3 \log_e(x) - \int x^2 \ dx$$

$$\int_1^2 x^2 \log_e(x) \ dx = \frac{1}{3}\left[x^3 \log_e(x) - \frac{x^3}{3} \right]_1^2$$ **M1**

$$\int_1^2 x^2 \log_e(x) \ dx = \frac{1}{3}\left[\left(8\log_e(2) - \frac{8}{3} \right) - \left(\log_e(1) - \frac{1}{3} \right) \right]$$

$$\int_1^2 x^2 \log_e(x) \ dx = \frac{1}{3}\left(8\log_e(2) - \frac{8}{3} + \frac{1}{3} \right)$$

$$\therefore \int_1^2 x^2 \log_e(x) \ dx = \frac{1}{3}\left(8\log_e(2) - \frac{7}{3} \right)$$ **A1**

Question 2 (16 marks)

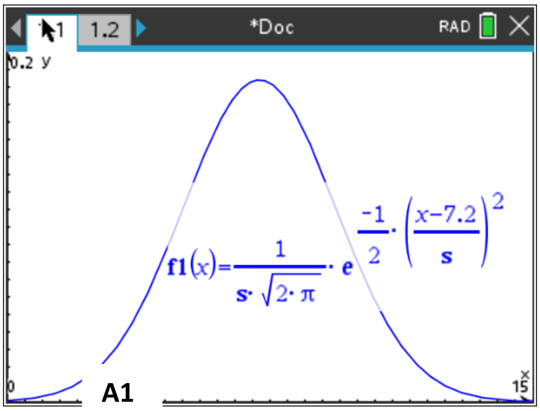
An airport has been analysing their departure processes and have some information on three main stages: taking carry-on luggage, selection for random swabbing and proceeding through customs.

Carry-on luggage statistics has shown to follow a normal distribution with a mean of 7.2 kg and a variance of 4.70.

a. If the top 10% of carry-on luggage is rejected as too heavy and needs to be placed in the cargo hold, what is the maximum acceptable weight for carry-on luggage, correct to 2 decimal places?

2 marks

The top 10% will occur at the upper end of the Normal Distribution as shown. This means that the lower 90% can be used as the area in our working.



Let X represent the weight of carry-on luggage.

$\Pr(X > a) = 0.1$

$Use\ invNorm(0.9, 7.2, \sqrt{4.70})$ **M1**

$\therefore a = 9.98$

Hence, the maximum acceptable weight is 9.98 kgs. **A1**

b. Find the value of c if $\Pr(-c < X < c) = 0.95$ for carry-on luggage weights, correct to 3 decimal places?

1 mark

For a Normal Distribution, for $\Pr(-c < X < c) = 0.95$ occurs 2 standard deviations from the mean, so

$$\mu - 2\sigma \le X \le \mu + 2\sigma$$

$$7.2 - 2\left(\sqrt{4.70}\right) \le X \le 7.2 + 2\left(\sqrt{4.70}\right)$$

$$2.864 \le X \le 11.536$$

So carry-on luggage is between 2.864 and 11.536 kgs, with 95% confidence. **A1**

c. A low-cost carrier has a different policy on carry-on luggage. Their limits of acceptability are imposed on the lowest 5% and highest 10%, being 3.1327 kg and 7.5223 kg respectively. Find the mean and standard deviation of this normally distributed policy, correct to 2 decimal places.

3 marks

Show the upper and lower bounds graphically, and find the equivalent standard Z-scores to the X values.

$$\textit{Lower bound}, 5\%$$

$$Z_1 = \frac{3.1327 - \mu}{\sigma}$$

$$-1.6449 = \frac{3.1327 - \mu}{\sigma}$$ **M1**

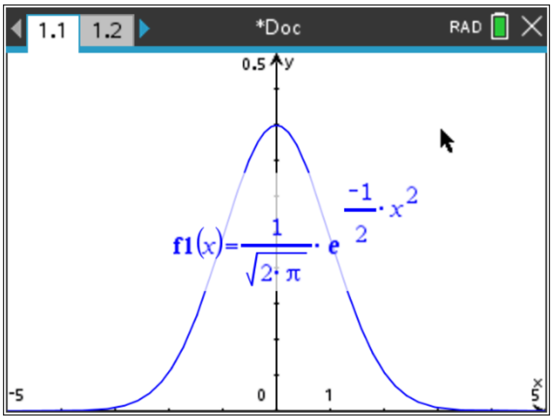
$$\textit{Upper bound}, 10\%$$

$$Z_2 = \frac{7.5223 - \mu}{\sigma}$$

$$1.2816 = \frac{7.5223 - \mu}{\sigma}$$ **M1**

$$\frac{-1.6449}{1.2816} = \frac{3.1327 - \mu}{7.5223 - \mu}$$

$$\therefore \mu = 5.60 \quad \textit{and} \quad \sigma = 1.50$$



Hence, the mean is 5.60 kgs and the standard deviation is 1.50 kgs. **A1**

Random swabbing is a way of checking for traces of explosives and other potential threats. The airport has set a target of checking 22% of the passengers as they pass through security.

d. If a group of 10 travellers pass through security, find the probability that less than half are swabbed, correct to 4 decimal places.

2 marks

Let X represent the number of travellers swabbed. Using the Binomial Distribution where n = 10 and p = 0.22.

$$\Pr(X \le 4) = \textit{binomCdf}(10, 0.22, 0, 4)$$ **M1**

$$\Pr(X \le 4) = 0.9521$$ **A1**

e. If the airport claims that there is more than a 98% chance of at least 2 passengers being swabbed when n travellers pass through security, find the smallest possible value of n.

2 marks

Setup the Binomial Distribution which represents this situation

$$\Pr(X \geq 2) > 0.98$$
$$1 - \Pr(X = 0) - \Pr(X = 1) > 0.98$$
$$0.02 < \Pr(X = 0) + \Pr(X = 1)$$
$$0.02 < \binom{n}{0}(0.22)^0(0.78)^n + \binom{n}{1}(0.22)(0.78)^{n-1}$$ **M1**
$$0.02 < (0.78)^n + n(0.22)(0.78)^{n-1}$$
$$n > 23.9959...$$
$$\therefore n = 24$$

Hence, 24 passengers will need to pass through security for this claim to be true. **A1**

From the group of 10 travellers previously mentioned, 6 of them are locals and the rest are internationals. As they pass through the next stage and approach customs, they are asked to approach the desk 4 at a time.

f. Find the probability that the 4 chosen are all internationals.

1 mark

Let X represent the number of local travellers. This is a sampling without replacement situation.

$$\Pr(X = 0) = \frac{\binom{6}{0}\binom{4}{4}}{\binom{10}{4}}$$
$$\therefore \Pr(X = 0) = \frac{1}{210}$$ **A1**

g. Find the probability that more locals are chosen than internationals in the selected 4 travellers to approach the desk.

2 marks

If there are more locals than internationals, it means that there can be 3 or 4 locals included in the selected 4 people.

$$\Pr(X \geq 3) = \frac{\binom{6}{3}\binom{4}{1} + \binom{6}{4}\binom{4}{0}}{\binom{10}{4}} \quad \textbf{M1}$$

$$\Pr(X \geq 3) = \frac{(20)(4) + (15)(1)}{210}$$

$$\therefore \Pr(X \geq 3) = \frac{95}{210} = \frac{19}{42} \quad \textbf{A1}$$

h. Given that at least 3 locals were chosen to approach the desk, find the probability that no internationals were included in the selected 4 travellers.

3 marks

We know that 3 locals were chosen so this is the conditional element of the question which will be included. No internationals implies 4 locals.

$$\Pr(X = 4 \mid X \geq 3) = \frac{\Pr(X = 4 \cap X \geq 3)}{\Pr(X \geq 3)} \quad \textbf{M1}$$

$$\Pr(X = 4 \mid X \geq 3) = \frac{\Pr(X = 4)}{\Pr(X \geq 3)}$$

$$\Pr(X = 4 \mid X \geq 3) = \frac{\binom{6}{4}\binom{4}{0}}{\Pr(X \geq 3)}$$

$$\Pr(X = 4 \mid X \geq 3) = \frac{15/210}{19/42} \quad \textbf{M1}$$

$$\therefore \Pr(X = 4 \mid X \geq 3) = \frac{3}{19} \quad \textbf{A1}$$

Question 3 (15 marks)

Temperature variation for a wintery town can be given by

$$T(t) = -3\cos\left(\frac{\pi}{12}(t-b)\right)+1, \quad t \in [0,24]$$

where t in given in hours, starting at 9pm on a Sunday, and T is measured in $°C$.

a. Find the amplitude and period of $T(t)$.

1 mark

Amplitude is 3° and period is $\frac{2\pi}{\pi/12} = 24$ hours. **A1**

A scientist takes a temperature reading at 10pm on the same day and records it as $\frac{-3\sqrt{2}}{2}+1$

b. Show that the smallest positive value of $b = 4$, if $b > 0$.

2 marks

$$Let \ \ t = 1$$

$$T(1) = -3\cos\left(\frac{\pi}{12}(1-b)\right)+1 = \frac{-3\sqrt{2}}{2}+1$$

$$-3\cos\left(\frac{\pi}{12}(1-b)\right) = \frac{-3\sqrt{2}}{2}$$ **M1**

$$\cos\left(\frac{\pi}{12}(1-b)\right) = \frac{\sqrt{2}}{2}$$

$$\frac{\pi}{12}(1-b) = \frac{-\pi}{4}$$

$$1-b = -3$$ **A1**

$$\therefore b = 4$$

c. State the initial temperature for the town.

1 mark

Initial temperature occurs when $t = 0$.

$$T(0) = -3\cos\left(\frac{\pi}{12}(0-4)\right)+1$$

$$T(0) = -3\cos\left(\frac{-\pi}{3}\right)+1$$

$$T(0) = -3\left(\frac{1}{2}\right)+1 = \frac{-1}{2}$$

Hence, initial temperature is $-0.5°C$ **A1**

d. If the temperature is taken from the initial time for one period, find the t intercepts, correct to 3 decimal places, hence state the time to the nearest minute.

3 marks

Find the t – intercepts when the temperature is zero.

$$-3\cos\left(\frac{\pi}{12}(t-4)\right)+1=0 \qquad 0 \le t \le 24$$

$$\cos\left(\frac{\pi}{12}(t-4)\right)=\frac{1}{3} \qquad -4 \le t-4 \le 20$$

$$(Basic\ \ Angle)\,\theta=\cos^{-1}\left(\frac{1}{3}\right),\ 1st\,\&\,4th\ \ quads \qquad \frac{-\pi}{3} \le \frac{\pi}{12}(t-4) \le \frac{5\pi}{3}$$ **M1**

$$\frac{\pi}{12}(t-4)=\theta, 2\pi-\theta$$

$$t=\frac{12}{\pi}\theta+4, \frac{12}{\pi}(2\pi-BA)+4$$

$$\therefore t=8.702, 23.298$$ **M1**

This corresponds to 8 hours 42 minutes and 23 hours 18 minutes, hence the time will be Monday 5.42am and 8.18pm respectively.

A1

e. Find the average value of the temperature over the first 12 hours.

3 marks

Use the integral to show the average value over the first 12 hours

$$Average\ \ value=\frac{1}{12-0}\int_0^{12}-3\cos\left(\frac{\pi}{12}(t-4)\right)+1\,dx$$

$$Avg\,Val=\frac{1}{12}\left[\frac{-36}{\pi}\sin\left(\frac{\pi}{12}(t-4)\right)+t\right]_0^{12}$$ **M1**

$$Avg\,Val=\frac{1}{12}\left[\left(\frac{-36}{\pi}\sin\left(\frac{\pi}{12}(8)\right)+12\right)-\left(\frac{-36}{\pi}\sin\left(\frac{\pi}{12}(-4)\right)+0\right)\right]$$

$$Avg\,Val=\frac{1}{12}\left[\left(\frac{-36}{\pi}\sin\left(\frac{2\pi}{3}\right)+12\right)-\left(\frac{-36}{\pi}\sin\left(\frac{-\pi}{3}\right)\right)\right]$$

$$Avg\,Val=\frac{1}{12}\left[\left(\frac{-36}{\pi}\times\frac{\sqrt{3}}{2}+12\right)-\left(\frac{-36}{\pi}\times\frac{-\sqrt{3}}{2}\right)\right]$$ **M1**

$$Avg\,Val=\frac{1}{12}\left(\frac{-18\sqrt{3}}{\pi}+12-\frac{-18\sqrt{3}}{\pi}\right)$$

$$\therefore Avg\,Val=1-\frac{3\sqrt{3}}{\pi}$$ **A1**

f. Find the stationary points of $T(t)$ over the whole day. 2 marks

Find when the derivative is equal to zero

$$T'(t) = \frac{\pi}{4}\sin\left(\frac{\pi}{12}(t-4)\right) = 0 \qquad 0 \le t \le 24$$

$$\sin\left(\frac{\pi}{12}(t-4)\right) = 0 \qquad -4 \le t-4 \le 20$$

$$(Basic\ \ Angle)\,\theta = 0, \pi\ \ 1st\,\&\,2nd\ \ quads \qquad \frac{-\pi}{3} \le \frac{\pi}{12}(t-4) \le \frac{5\pi}{3}$$ **M1**

$$\frac{\pi}{12}(t-4) = 0, \pi$$

$$t-4 = 0, 12$$

$$\therefore t = 4, 16$$

Hence, the stationary points occur at $(0,4)$ *and* $(16,0)$. **A1**

g. Find the equation of the normal at $t = 6$. 3 marks

Coordinate at $t = 6$

$$T(6) = -3\cos\left(\frac{\pi}{12}(2)\right) + 1$$

$$T(6) = -3\cos\left(\frac{\pi}{6}\right) + 1$$

$$T(6) = -3 \times \frac{\sqrt{3}}{2} + 1$$

$$\therefore T(6) = 1 - \frac{3\sqrt{3}}{2} \qquad \left(6, 1 - \frac{3\sqrt{3}}{2}\right)$$ **M1**

Gradient of the tangent at $t = 6$

$$T'(6) = \frac{\pi}{4}\sin\left(\frac{\pi}{12}(2)\right)$$

$$T'(6) = \frac{\pi}{4}\sin\left(\frac{\pi}{6}\right)$$

$$\therefore T'(6) = \frac{\pi}{4} \times \frac{1}{2} = \frac{\pi}{8}$$ **M1**

Use perpendicular gradient for equation of the normal

$$T - \left(1 - \frac{3\sqrt{3}}{2}\right) = \frac{-8}{\pi}(t-6)$$

$$\therefore T(t) = \frac{-8}{\pi}t + \frac{48}{\pi} + 1 - \frac{3\sqrt{3}}{2}$$ **A1**

Question 4 (14 marks)

Let $f:(-\infty, a) \to R,\ f(x) = \dfrac{3}{(x-5)^2} - 1$.

a. Find the largest possible value of a for $f^{-1}(x)$ to exist.

1 mark

For an inverse to exist, the original function must be one-to-one.

The maximum x value is the vertical asymptote, at $x = 5$, hence $a = 5$. **A1**

b. Find the inverse function, $f^{-1}(x)$, using full notation.

2 marks

For the original function, domain $f(x) \in (-\infty, 5)$ and range $f(x) \in (-1, \infty)$.

$$\textit{Let } y = f(x),\ \textit{swap } x \,\&\, y$$

$$x = \frac{3}{(y-5)^2} - 1$$

$$x + 1 = \frac{3}{(y-5)^2}$$

$$(y-5)^2 = \frac{3}{x+1} \quad \textbf{M1}$$

$$y - 5 = \pm\sqrt{\frac{3}{x+1}} \quad \textit{but dom } f(x) \in (-\infty, 5)$$

$$y = 5 - \sqrt{\frac{3}{x+1}}$$

$$\therefore f^{-1}:(-1, \infty) \to R,\ f^{-1}(x) = 5 - \sqrt{\frac{3}{x+1}} \quad \textbf{A1}$$

c. Find the coordinates of the point(s) of intersection between $f(x) = f^{-1}(x)$, correct to 2 decimal places.

2 marks

Equate either functions given in $f(x) = f^{-1}(x) = x$

$$\textit{Let } \frac{3}{(x-5)^2} - 1 = x \quad \textit{OR} \quad 5 - \sqrt{\frac{3}{x+1}} = x \quad \textbf{M1}$$

On CAS

$$\therefore x = -0.91, 4.24 \quad \textit{hence } (-0.91, 0) \textit{ and } (4.24, 0) \quad \textbf{A1}$$

d. The function $h(x)$ is obtained by applying the transformation T to the function $f(x)$, where

$$T\left(\begin{bmatrix} x \\ y \end{bmatrix}\right) = \begin{bmatrix} -1 & 0 \\ 0 & 3 \end{bmatrix}\begin{bmatrix} x \\ y \end{bmatrix} + \begin{bmatrix} 6 \\ -2 \end{bmatrix}$$

i. Find the equation $h(x)$.

3 marks

Applying this transformation to this original equation

$$\begin{bmatrix} x' \\ y' \end{bmatrix} = \begin{bmatrix} -1 & 0 \\ 0 & 3 \end{bmatrix}\begin{bmatrix} x \\ y \end{bmatrix} + \begin{bmatrix} 6 \\ -2 \end{bmatrix}$$

$$\begin{bmatrix} x' \\ y' \end{bmatrix} = \begin{bmatrix} -x \\ 3y \end{bmatrix} + \begin{bmatrix} 6 \\ -2 \end{bmatrix}$$

$$x' = -x + 6 \quad \therefore x = 6 - x'$$

$$y' = 3y - 2 \quad \therefore y = \frac{y'+2}{3}$$ **M1**

Substitute x and y into the original equation

$$\frac{y'+2}{3} = \frac{3}{(6-x'-5)^2} - 1$$

$$\frac{y+2}{3} = \frac{3}{(1-x)^2} - 1$$ **M1**

$$y + 2 = \frac{9}{(1-x)^2} - 3$$

$$y = \frac{9}{(1-x)^2} - 5$$

$$\therefore h(x) = \frac{9}{(1-x)^2} - 5$$ **A1**

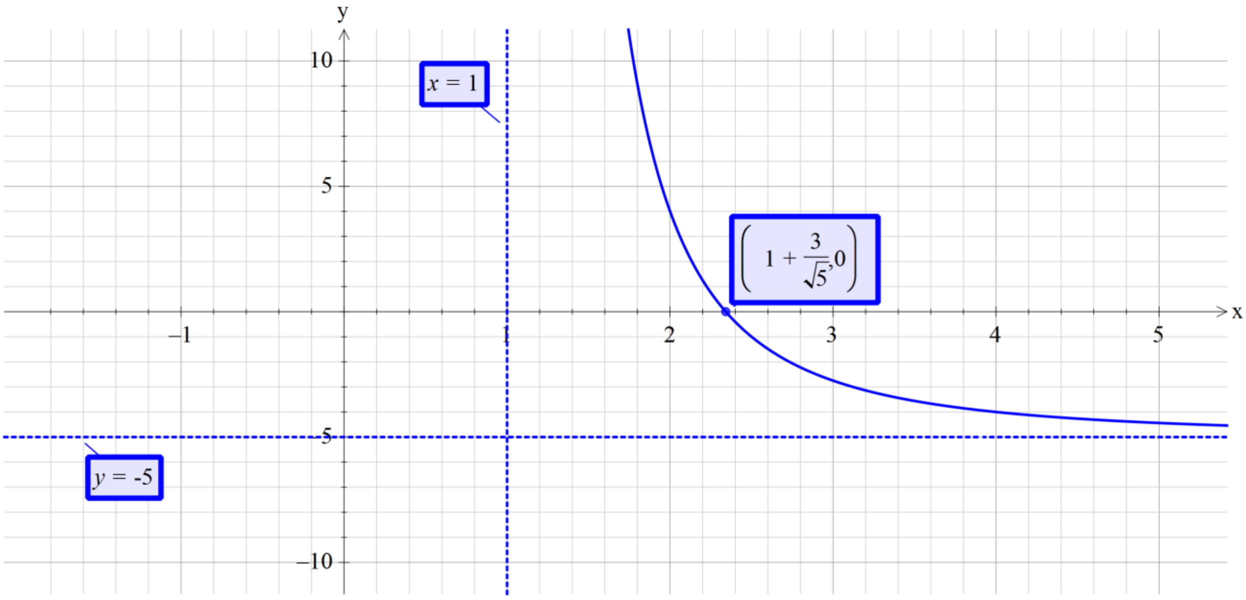
ii. State the equations of the asymptotes for $h(x)$.

1 mark

From the previous question, asymptotes occur at $x = 1$ *and* $y = -5$. **A1**

e. Sketch the graph of $h(x)$ within the relevant domain.

3 marks



1 mark – Shape **1 mark** – x-intercept **1 mark** – asymptotes

f. Find the area for $h(x)$ enclosed between $x = 2.5, 5$ and the x-axis.

2 marks

The area in question is negative and so must be made positive

$$Area = \int_{5}^{5/2} 9(1-x)^{-2} - 5 \ dx$$

$$A = \left[\frac{9(1-x)^{-1}}{-1 \times -1} - 5x \right]_{5}^{5/2}$$

$$A = \left[\frac{9}{1-x} - 5x \right]_{5}^{5/2}$$

$$A = \left[\left(\frac{9}{1-5/2} - 5\left(\frac{5}{2}\right) \right) - \left(\frac{9}{1-5} - 5(5) \right) \right]$$ **M1**

$$A = \left[\left(\frac{9}{-3/2} - \frac{25}{2} \right) - \left(\frac{-9}{4} - 25 \right) \right]$$

$$A = \left(-6 - \frac{25}{2} \right) - \left(\frac{-9}{4} - 25 \right)$$

$$\therefore A = 19 - \frac{41}{4} = \frac{35}{4} \ units^2$$ **A1**