Instructions

Answer **all** questions in the spaces provided.
In all questions where a numerical answer is required, an exact value must be given, unless otherwise specified.
In questions where more than one mark is available, appropriate working **must** be shown.
Unless otherwise indicated, the diagrams in this book are **not** drawn to scale.

Question 1 (3 marks)

a. Let $y = x^2 \sin(x)$.

Find $\frac{dy}{dx}$. 1 mark

$$\frac{dy}{dx} = \frac{d}{dx}(x^2)\cdot \sin(x) + \frac{d(\sin(x))}{dx}\cdot x^2$$
$$= 2x\sin(x) + x^2\cos(x) \quad \left(= x(2\sin(x) + x\cos(x))\right)$$
$$\left(= 2x\left(\sin(x) + \cos(x)\left(\frac{x}{2}\right)\right)\right)$$

equivalent forms not necessary

b. Evaluate $f'(1)$, where $f: R \to R$, $f(x) = e^{x^2 - x + 3}$. 2 marks

$$f'(x) = (2x-1)e^{x^2-x+3}$$
$$f'(1) = (2-1)e^{1-1+3}$$
$$= e^3$$

Question 2 (3 marks)

A car manufacturer is reviewing the performance of its car model X. It is known that at any given six-month service, the probability of model X requiring an oil change is $\frac{17}{20}$, the probability of model X requiring an air filter change is $\frac{3}{20}$ and the probability of model X requiring both is $\frac{1}{20}$.

Oil

Filter

a. State the probability that at any given six-month service model X will require an air filter change without an oil change. 1 mark

	Oil	No Oil	
Filter	$\frac{1}{20}$	$\frac{2}{20}$	$\frac{3}{20}$
No Filter	$\frac{16}{20}$	$\frac{1}{20}$	$\frac{17}{20}$
	$\frac{17}{20}$	$\frac{3}{20}$	1

$\Pr(\text{Air filter} \cap \text{No Oil})$

$= \frac{3}{20} - \frac{1}{20} = \frac{2}{20} = \underline{\frac{1}{10}}$

b. The car manufacturer is developing a new model, Y. The production goals are that the probability of model Y requiring an oil change at any given six-month service will be $\frac{m}{m+n}$, the probability of model Y requiring an air filter change will be $\frac{n}{m+n}$ and the probability of model Y requiring both will be $\frac{1}{m+n}$, where $m, n \in Z^+$.

Determine m in terms of n if the probability of model Y requiring an air filter change without an oil change at any given six-month service is 0.05 2 marks

	Oil	No Oil	
Filter	$\frac{1}{m+n}$	$\frac{n-1}{m+n}$	$\frac{n}{m+n}$
No filter	$\frac{m-1}{m+n}$	$\frac{1}{m+n}$	$\frac{m}{m+n}$
	$\frac{m}{m+n}$	$\frac{n}{m+n}$	1

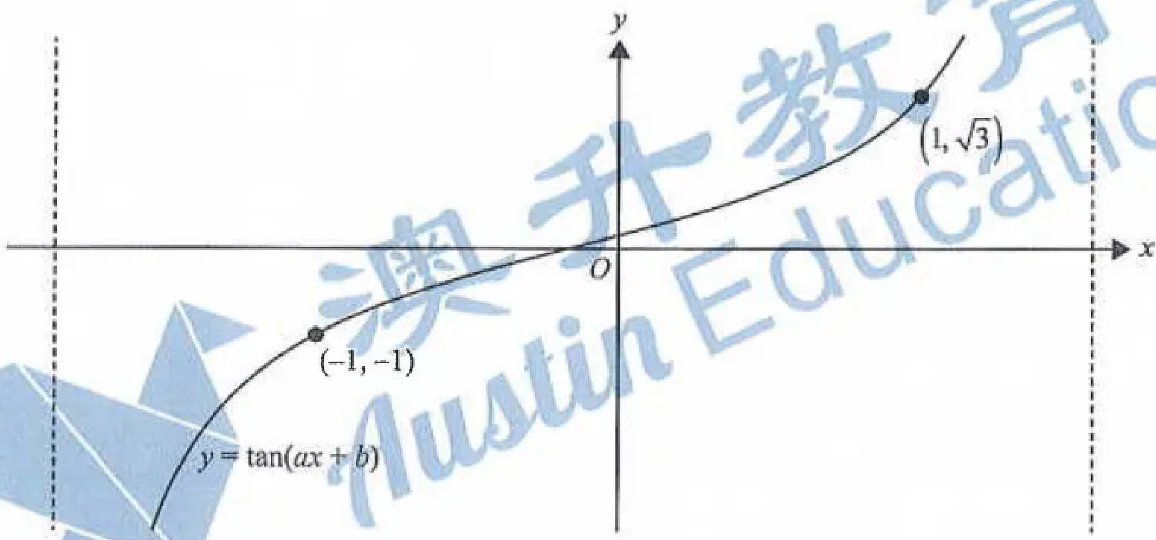
$\Pr(\text{Air filter change} \cap \text{No oil change})$

$= \frac{n-1}{m+n} = 0.05 = \frac{1}{20}$

$20n - 20 = m + n, \quad \boxed{m = 19n - 20}$

Question 3 (3 marks)

Shown below is part of the graph of a period of the function of the form $y = \tan(ax + b)$.



The graph is continuous for $x \in [-1, 1]$.

Find the value of a and the value of b, where $a > 0$ and $0 < b < 1$.

Using $(-1,-1)$: $-1 = \tan(-a+b)$

Using $(1,\sqrt{3})$; $\sqrt{3} = \tan(a+b)$

By inspection: $-a+b = -\frac{\pi}{4}$ ①

$a+b = \frac{\pi}{3}$ ②

① + ② $\Rightarrow$ $2b = \frac{\pi}{12}$, $b = \frac{\pi}{24}$

also $a + \frac{\pi}{24} = \frac{8\pi}{24}$

$a = \frac{7\pi}{24}$

$\therefore \begin{cases} a = \frac{7\pi}{24} \\ b = \frac{\pi}{24} \end{cases}$

Question 4 (3 marks)

Solve the equation $2\log_2(x+5) - \log_2(x+9) = 1$.

Simplifying: $\log_2\left(\frac{(x+5)^2}{(x+9)}\right) = 1$

$2^1 = \frac{(x+5)^2}{(x+9)} \Rightarrow x^2+10x+25 = 2x+18$

$\Rightarrow x^2+8x+7=0$

$\Rightarrow (x+1)(x+7)=0$

$\Rightarrow x=-1,\ x=-7$

check validity:

$x+5>0 \Rightarrow x>-5$

$x+9>0 \Rightarrow x>-9$

$\Rightarrow x>-5$

reject $x=-7$

Hence take 1st: $x=-1$ only.

Question 5 (4 marks)

For a certain population the probability of a person being born with the specific gene SPGE1 is $\frac{3}{5}$.

The probability of a person having this gene is independent of any other person in the population having this gene.

a. In a randomly selected group of four people, what is the probability that three or more people have the SPGE1 gene? 2 marks

Let $X \sim Bi(n=4, p=\frac{3}{5})$ check formula sheet

$$\Pr(X \geq 3) = \Pr(X=3) + \Pr(X=4)$$
$$= {}^4C_3 \times \left(\tfrac{3}{5}\right)^3\left(\tfrac{2}{5}\right)^1 + {}^4C_4\left(\tfrac{3}{5}\right)^4\left(\tfrac{2}{5}\right)^0$$
$$= 4 \times \frac{2}{5} \times \left(\frac{27}{125}\right) + \frac{81}{625}$$
$$= \frac{8 \times 27 + 81}{625} = \frac{297}{625} \text{ (accept 0.4752)}$$

b. In a randomly selected group of four people, what is the probability that exactly two people have the SPGE1 gene, given that at least one of those people has the SPGE1 gene? Express your answer in the form $\frac{a^3}{b^4 - c^4}$, where $a, b, c \in Z^+$. $\Pr(X=2 \mid X \geq 1) = \frac{\Pr(X=2 \cap X \geq 1)}{\Pr(X \geq 1)}$ 2 marks

$$\Pr(X=2 \mid X \geq 1) = \frac{\Pr(X=2)}{1 - \Pr(X=0)} = \frac{\frac{216}{625}}{\frac{625-16}{625}} = \frac{216}{625-16} = \boxed{\frac{(6)^3}{(5)^4 - (2)^4}}$$

$$\Pr(X=2) = {}^4C_2 \times \left(\tfrac{3}{5}\right)^2\left(\tfrac{2}{5}\right)^2 = 6 \times \frac{9}{25} \times \frac{4}{25} = \frac{216}{625}$$

$$1 - \Pr(X=0) = 1 - {}^4C_0 \times \left(\tfrac{3}{5}\right)^0\left(\tfrac{2}{5}\right)^4 = 1 - \frac{16}{625} = \frac{625-16}{625}$$

where $a=6$, $b=5$ & $c=2$

DO NOT WRITE IN THIS AREA

Question 6 (8 marks)

Let $f: [0, 2] \to R$, where $f(x) = \frac{1}{\sqrt{2}}\sqrt{x}$.

a. Find the domain and the rule for f^{-1}, the inverse function of f. 2 marks

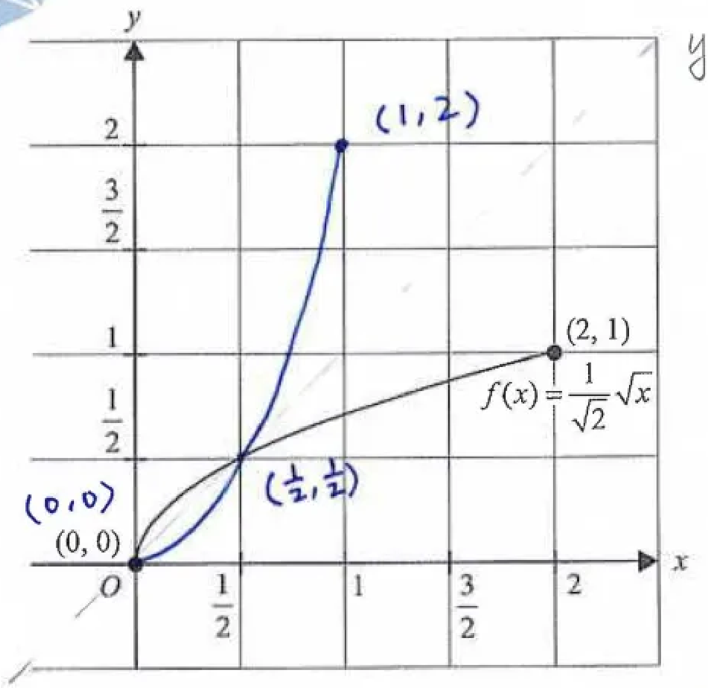
$\text{Dom}(f) = [0, 2]$, $f(x)$ is strictly increasing

$f(0) = 0$, $f(2) = 1 \Rightarrow \text{Range}(f) = [0, 1] = \text{dom}(f^{-1})$

let $y = f(x)$: $y = \sqrt{\frac{x}{2}}$; swap x & y for inverse : $x = \sqrt{\frac{y}{2}}$; $x^2 = \frac{y}{2}$

$\therefore f^{-1}: [0, 1] \to \mathbb{R}, f^{-1}(x) = 2x^2$

The graph of $y = f(x)$, where $x \in [0, 2]$, is shown on the axes below.



y = x
(Not required)
(Symmetry axis)

b. On the axes above, sketch the graph of f^{-1} over its domain. Label the endpoints and point(s) of intersection with the function f, giving their coordinates. 2 marks

c. Find the total area of the two regions: one region bounded by the functions f and f^{-1}, and the other region bounded by f, f^{-1} and the line $x = 1$. Give your answer in the form $\frac{a - b\sqrt{b}}{6}$, where $a, b \in Z^{+}$. 4 marks

$$\text{Area} = \int_0^{\frac{1}{2}} \left(f(x) - f^{-1}(x)\right) dx + \int_{\frac{1}{2}}^{1} \left(f^{-1}(x) - f(x)\right) dx$$

$$= 2\int_0^{\frac{1}{2}} \left(x - f^{-1}(x)\right) dx + \int_{\frac{1}{2}}^{1} \left(2x^2 - \frac{1}{\sqrt{2}} \cdot \sqrt{x}\right) dx$$

$$= 2\int_0^{\frac{1}{2}} \left(x - 2x^2\right) dx + \int_{\frac{1}{2}}^{1} \left(2x^2 - \frac{\sqrt{x}}{\sqrt{2}}\right) dx$$

$$= 2\left[\frac{x^2}{2} - \frac{2x^3}{3}\right]_0^{\frac{1}{2}} + \left[\frac{2x^3}{3} - \frac{1}{\sqrt{2}} \cdot \frac{2}{3}\sqrt{x^3}\right]_{\frac{1}{2}}^{1}$$

$$= 2\left(\frac{1}{2} \times \frac{1}{2} \times \frac{1}{2} - \frac{2}{3} \times \frac{1}{8} - 0\right) + \left(\left(\frac{2}{3} - \frac{2}{3\sqrt{2}}\right) - \left(\frac{2}{3} \times \frac{1}{8} - \frac{2}{3\sqrt{2}}\sqrt{\frac{1}{8}}\right)\right)$$

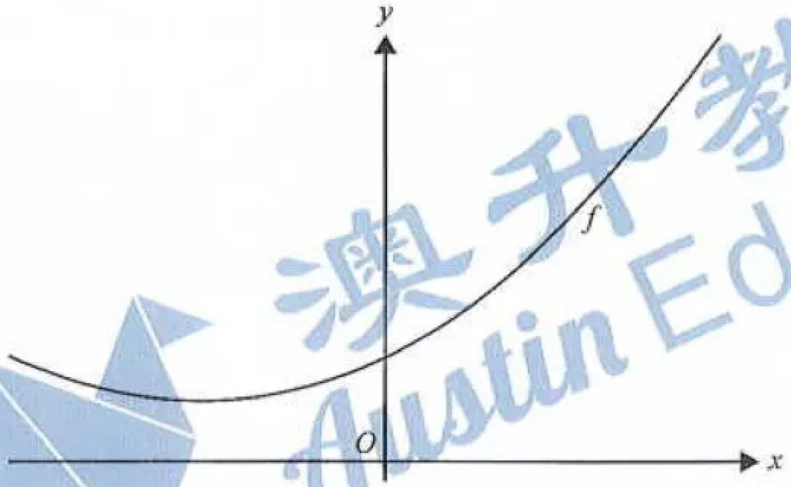
$$= \frac{1}{4} - \frac{2}{3} \times \frac{1}{4} + \frac{2}{3} - \frac{2}{3\sqrt{2}} - \frac{1}{3} \times \frac{1}{4} + \frac{\cancel{2}}{3\sqrt{2}} \times \frac{1}{\cancel{2}\sqrt{2}}$$

$$= \frac{11}{12} - \frac{3}{12} - \frac{2\sqrt{2}}{6} + \frac{1}{6}$$

$$= \frac{8}{12} + \frac{2}{12} - \frac{2\sqrt{2}}{6} = \frac{5 - 2\sqrt{2}}{6} \text{ units}^2.$$

Question 7 (8 marks)

Consider the function $f(x) = x^2 + 3x + 5$ and the point $P(1, 0)$. Part of the graph of $y = f(x)$ is shown below.



a. Show that point P is not on the graph of $y = f(x)$. 1 mark

$f(1) = 1 + 3 + 5$ OR $f(x) = x^2 + 3x + 5$

$= 9$ $\Delta = 3^2 - 4 \times 1 \times 5 = -11 < 0 \Rightarrow$ Never touches x axis

$f(1) \neq 0 \Rightarrow (1, 0)$ is not on the graph of $y = f(x)$

b. Consider a point $Q(a, f(a))$ to be a point on the graph of f.

i. Find the slope of the line connecting points P and Q in terms of a. 1 mark

$m_{PQ} = \frac{0 - f(a)}{1 - a} = \frac{f(a)}{a - 1}$

$\therefore m_{PQ} = \frac{a^2 + 3a + 5}{a - 1}$

ii. Find the slope of the tangent to the graph of f at point Q in terms of a. 1 mark

$f'(x) = 2x + 3$

$\therefore m_Q = f'(a) = 2a + 3$

DO NOT WRITE IN THIS AREA

iii. Let the tangent to the graph of f at $x = a$ pass through point P.

Find the values of a. 2 marks

Equating gradients: $\dfrac{a^2+3a+5}{a-1} = 2a+3$

$a^2+3a+5 = (2a+3)(a-1) = 2a^2+a-3$

$a^2-2a-8=0$, $(a-4)(a+2)=0$

$\therefore a=4$ or $a=-2$

iv. Give the equation of one of the lines passing through point P that is tangent to the graph of f. 1 mark

If $a=4$, $\underline{y=11x-11}$, passing through $(1,0)$.

If $a=-2$, $\underline{y=1-x}$, passing through $(1,0)$ too.

c. Find the value, k, that gives the shortest possible distance between the graph of the function of $y = f(x-k)$ and point P. 2 marks

$y = x^2+3x+5$

$= x^2+3x+\left(\frac{3}{2}\right)^2-\left(\frac{3}{2}\right)^2+5$

$= \left(x+\frac{3}{2}\right)^2 - \frac{9}{4}+\frac{20}{4}$

$= \left(x+\frac{3}{2}\right)^2+\frac{11}{4}$

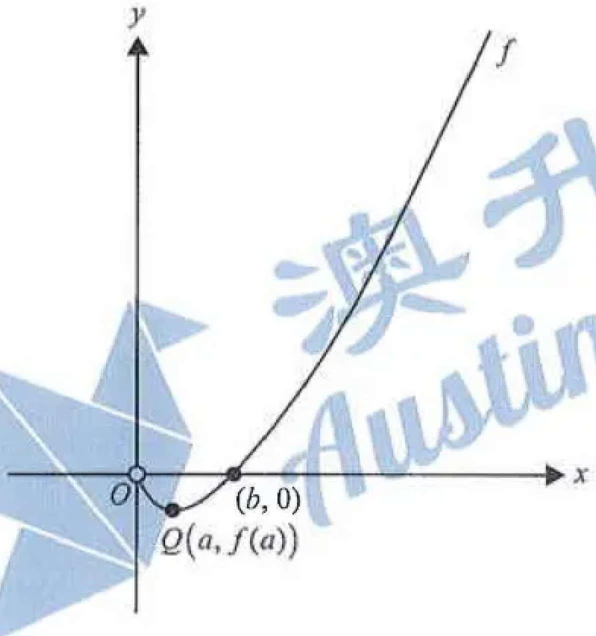
$\boxed{\therefore k = \frac{5}{2}.}$

Compare the t.p $\left(-\frac{3}{2}, \frac{11}{4}\right)$ and $(1,0)$

Need to translate $y=f(x)$ to the right by $\frac{5}{2}$ units for shortest distance $\underline{\frac{11}{4}}$

Question 8 (8 marks)

Part of the graph of $y = f(x)$, where $f: (0, \infty) \to R, f(x) = x \log_e(x)$, is shown below.



The graph of f has a minimum at the point $Q(a, f(a))$, as shown above.

a. Find the coordinates of the point Q. 2 marks

$f'(x) = 1 \cdot \ln(x) + x \cdot \frac{1}{x} = \ln(x) + 1 = 0$

$\ln(x) = -1 \quad , \quad x = e^{-1}$

$f(e^{-1}) = e^{-1} \cdot \ln(e^{-1}) = \frac{1}{e} \cdot (-1) = -\frac{1}{e}$

$\therefore Q: (e^{-1}, -e^{-1})$ (or $(\frac{1}{e}, -\frac{1}{e})$)

equivalent answer

DO NOT WRITE IN THIS AREA

b. Using $\frac{d\left(x^2 \log_e(x)\right)}{dx} = 2x\log_e(x) + x$, show that $x\log_e(x)$ has an antiderivative $\frac{x^2 \log_e(x)}{2} - \frac{x^2}{4}$. 1 mark

$$\int \frac{d}{dx}\left(x^2 \log_e(x)\right) dx = 2\int x \cdot \log_e(x)\,dx + \int x\,dx$$

$$x^2 \log_e(x) = 2\int x \cdot \log_e(x)\,dx + \frac{1}{2}x^2$$

$$\frac{1}{2}x^2 \log_e(x) = \int x \cdot \log_e(x)\,dx + \frac{1}{4}x^2$$

$$\int x \log_e(x)\,dx = \frac{1}{2}x^2 \log_e(x) - \frac{1}{4}x^2 + C$$

when $C=0$, an antiderv. is $\frac{1}{2}x^2 \log_e(x) - \frac{1}{4}x^2$

c. Find the area of the region that is bounded by f, the line $x = a$ and the horizontal axis for $x \in [a, b]$, where b is the x-intercept of f. 2 marks

$a = \frac{1}{e}$, $b = 1$

$\ln(1) = 0$

$\therefore x = 1$

$b = 1$

$$\text{Area} = -\int_{\frac{1}{e}}^{1} x \log_e(x)\,dx$$

$$= \int_{1}^{\frac{1}{e}} x \cdot \log_e(x)\,dx$$

$$= \left[\frac{1}{2}x^2 \log_e(x) - \frac{1}{4}x^2\right]_1^{\frac{1}{e}}$$

$$= \left(\frac{1}{2} \times \left(\frac{1}{e}\right)^2 \log_e\left(\frac{1}{e}\right) - \frac{1}{4} \times \left(\frac{1}{e}\right)^2\right) - \left(\frac{1}{2} \times \ln(1) - \frac{1}{4}\right)$$

$$= \frac{1}{2e^2}(-1) - \frac{1}{4e^2} + \frac{1}{4}$$

$$= \frac{1}{4} - \frac{3}{4e^2}$$ or other equivalent forms

d. Let $g: (a, \infty) \to R$, $g(x) = f(x) + k$ for $k \in R$. Strictly $\uparrow$

i. Find the value of k for which $y = 2x$ is a tangent to the graph of g. 1 mark

$\boxed{k = e}$

If $f'(x) = g'(x) = 2$ $\quad \therefore (e, 2e)$ will lie on $y = g(x)$

$\ln(x) + 1 = 2$ $\quad \therefore x \cdot \ln(x) + k = y$

$\ln(x) = 1,\ x = e$ $\quad e\ln(e) + k = 2e,\ k = e$

ii. Find all values of k for which the graphs of g and g^{-1} do not intersect. 2 marks

$g(x) = x \cdot \ln(x) + k = x$

$k = x - x\ln(x)$

$k = \underbrace{x(1 - \ln(x))}$

let $h(x) = x \cdot (1 - \ln(x))$

$h'(x) = 1 \cdot (1 - \ln(x)) + x \cdot -\frac{1}{x}$

$= 1 - \ln(x) - 1$

$= -\ln(x) = 0$ for t.ps

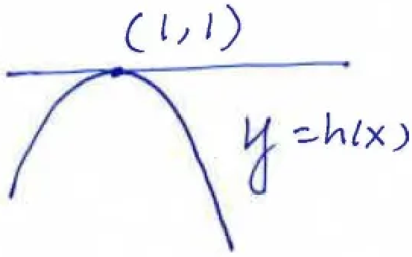
$x = 1$ $\quad h(1) = 1 \times (1 - \ln(1))$

$= 1.$

If $x = \frac{1}{2}$, $h'(\frac{1}{2}) = -\ln(\frac{1}{2}) = \ln(2) > 0$

If $x = \frac{3}{2}$, $h'(\frac{3}{2}) = -\ln(\frac{3}{2}) < 0$

Hence $k > 1$ for the g & g^{-1} not to intersect with each other



END OF QUESTION AND ANSWER BOOK