VICTORIAN CURRICULUM AND ASSESSMENT AUTHORITY

Victorian Certificate of Education 2019

SUPERVISOR TO ATTACH PROCESSING LABEL HERE

STUDENT NUMBER | | | | | | | | | | Letter |

MATHEMATICAL METHODS

Written examination 2

Thursday 7 November 2019

Reading time: 3.00 pm to 3.15 pm (15 minutes)
Writing time: 3.15 pm to 5.15 pm (2 hours)

QUESTION AND ANSWER BOOK

Structure of book

Section	*Number of questions*	*Number of questions to be answered*	*Number of marks*
A	20	20	20
B	5	5	60
			Total 80

- Students are permitted to bring into the examination room: pens, pencils, highlighters, erasers, sharpeners, rulers, a protractor, set squares, aids for curve sketching, one bound reference, one approved technology (calculator or software) and, if desired, one scientific calculator. Calculator memory DOES NOT need to be cleared. For approved computer-based CAS, full functionality may be used.
- Students are NOT permitted to bring into the examination room: blank sheets of paper and/or correction fluid/tape.

Materials supplied
- Question and answer book of 25 pages
- Formula sheet
- Answer sheet for multiple-choice questions

Instructions
- Write your **student number** in the space provided above on this page.
- Check that your **name** and **student number** as printed on your answer sheet for multiple-choice questions are correct, **and** sign your name in the space provided to verify this.
- Unless otherwise indicated, the diagrams in this book are **not** drawn to scale.
- All written responses must be in English.

At the end of the examination
- Place the answer sheet for multiple-choice questions inside the front cover of this book.
- You may keep the formula sheet.

Students are NOT permitted to bring mobile phones and/or any other unauthorised electronic devices into the examination room.

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SECTION A – Multiple-choice questions

Instructions for Section A

Answer **all** questions in pencil on the answer sheet provided for multiple-choice questions.
Choose the response that is **correct** for the question.
A correct answer scores 1; an incorrect answer scores 0.
Marks will **not** be deducted for incorrect answers.
No marks will be given if more than one answer is completed for any question.
Unless otherwise indicated, the diagrams in this book are **not** drawn to scale.

Question 1

Let $f : R \to R,\ f(x) = 3\sin\left(\frac{2x}{5}\right) - 2.$

The period and range of f are respectively

A. 5π and $[-3, 3]$

B. 5π and $[-5, 1]$

C. 5π and $[-1, 5]$

D. $\frac{5\pi}{2}$ and $[-5, 1]$

E. $\frac{5\pi}{2}$ and $[-3, 3]$

Question 2

The set of values of k for which $x^2 + 2x - k = 0$ has two real solutions is

A. $\{-1, 1\}$

B. $(-1, \infty)$

C. $(-\infty, -1)$

D. $\{-1\}$

E. $[-1, \infty)$

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Question 3

Let $f: R\backslash\{4\} \to R$, $f(x) = \dfrac{a}{x-4}$, where $a > 0$.

The average rate of change of f from $x = 6$ to $x = 8$ is

A. $a \log_e(2)$

B. $\dfrac{a}{2}\log_e(2)$

C. $2a$

D. $-\dfrac{a}{4}$

E. $-\dfrac{a}{8}$

Question 4

$\displaystyle\int_0^{\frac{\pi}{6}} \left(a\sin(x) + b\cos(x)\right)dx$ is equal to

A. $\dfrac{\left(2-\sqrt{3}\right)a - b}{2}$

B. $\dfrac{b - \left(2-\sqrt{3}\right)a}{2}$

C. $\dfrac{\left(2-\sqrt{3}\right)a + b}{2}$

D. $\dfrac{\left(2-\sqrt{3}\right)b - a}{2}$

E. $\dfrac{\left(2-\sqrt{3}\right)b + a}{2}$

Question 5

Let $f'(x) = 3x^2 - 2x$ such that $f(4) = 0$.

The rule of f is

A. $f(x) = x^3 - x^2$

B. $f(x) = x^3 - x^2 + 48$

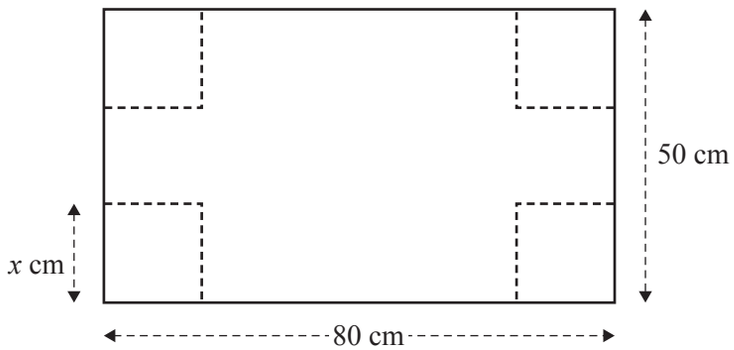
C. $f(x) = x^3 - x^2 - 48$

D. $f(x) = 6x - 2$

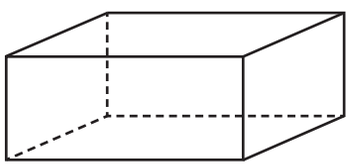
E. $f(x) = 6x - 24$

Question 6

A rectangular sheet of cardboard has a length of 80 cm and a width of 50 cm. Squares, of side length x centimetres, are cut from each of the corners, as shown in the diagram below.



A rectangular box with an open top is then constructed, as shown in the diagram below.



The volume of the box is a maximum when x is equal to

A. 10

B. 20

C. 25

D. $\frac{100}{3}$

E. $\frac{200}{3}$

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Question 7

The discrete random variable X has the following probability distribution.

x	0	1	2	3
$\Pr(X = x)$	a	$3a$	$5a$	$7a$

The mean of X is

A. $\frac{1}{16}$

B. 1

C. $\frac{35}{16}$

D. $\frac{17}{8}$

E. 2

Question 8

An archer can successfully hit a target with a probability of 0.9. The archer attempts to hit the target 80 times. The outcome of each attempt is independent of any other attempt.

Given that the archer successfully hits the target at least 70 times, the probability that the archer successfully hits the target exactly 74 times, correct to four decimal places, is

A. 0.3635

B. 0.8266

C. 0.1494

D. 0.3005

E. 0.1701

Question 9

The point (a, b) is transformed by

$$T\left(\begin{bmatrix} x \\ y \end{bmatrix}\right) = \begin{bmatrix} \frac{1}{2} & 0 \\ 0 & -2 \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix} + \begin{bmatrix} -\frac{1}{2} \\ -2 \end{bmatrix}$$

If the image of (a, b) is $(0, 0)$, then (a, b) is

A. $(1, 1)$

B. $(-1, 1)$

C. $(-1, 0)$

D. $(0, 1)$

E. $(1, -1)$

Question 10

Which one of the following statements is true for $f: R \to R, f(x) = x + \sin(x)$?

A. The graph of f has a horizontal asymptote

B. There are infinitely many solutions to $f(x) = 4$

C. f has a period of 2π

D. $f'(x) \geq 0$ for $x \in R$

E. $f'(x) = \cos(x)$

Question 11

A and B are events from a sample space such that $\Pr(A) = p$, where $p > 0$, $\Pr(B|A) = m$ and $\Pr(B|A') = n$.

A and B are independent events when

A. $m = n$

B. $m = 1 - p$

C. $m + n = 1$

D. $m = p$

E. $m + n = 1 - p$

Question 12

If $\int_1^4 f(x)\,dx = 4$ and $\int_2^4 f(x)\,dx = -2$, then $\int_1^2 \big(f(x) + x\big)\,dx$ is equal to

A. 2

B. 6

C. 8

D. $\frac{7}{2}$

E. $\frac{15}{2}$

Question 13

The graph of the function f passes through the point $(-2, 7)$.

If $h(x) = f\left(\frac{x}{2}\right) + 5$, then the graph of the function h must pass through the point

A. $(-1, -12)$

B. $(-1, 19)$

C. $(-4, 12)$

D. $(-4, -14)$

E. $(3, 3.5)$

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Question 14

The weights of packets of lollies are normally distributed with a mean of 200 g.

If 97% of these packets of lollies have a weight of more than 190 g, then the standard deviation of the distribution, correct to one decimal place, is

A. 3.3 g

B. 5.3 g

C. 6.1 g

D. 9.4 g

E. 12.1 g

Question 15

Let $f: [2, \infty) \rightarrow R$, $f(x) = x^2 - 4x + 2$ and $f(5) = 7$. The function g is the inverse function of f.

$g'(7)$ is equal to

A. $\frac{1}{6}$

B. 5

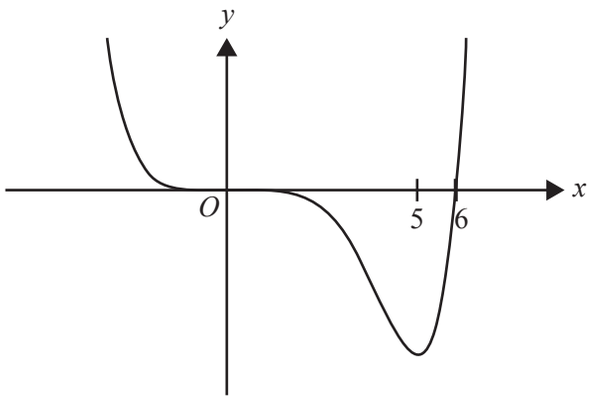
C. $\frac{\sqrt{7}}{14}$

D. 6

E. $\frac{1}{7}$

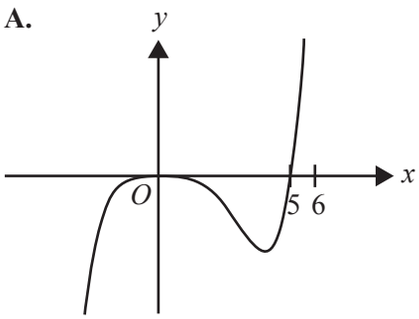
Question 16

Part of the graph of $y = f(x)$ is shown below.

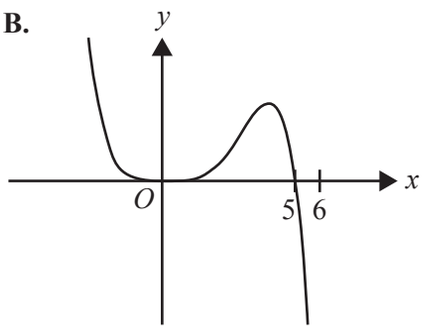


The corresponding part of the graph of $y = f'(x)$ is best represented by

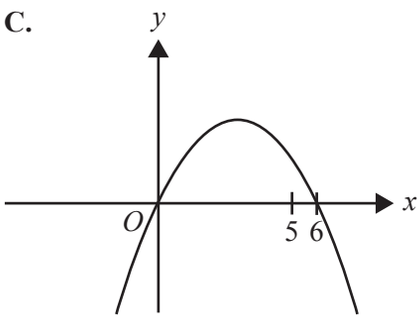
A.



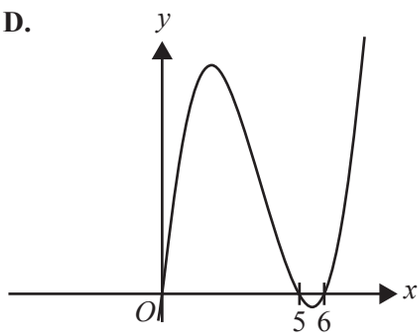
B.



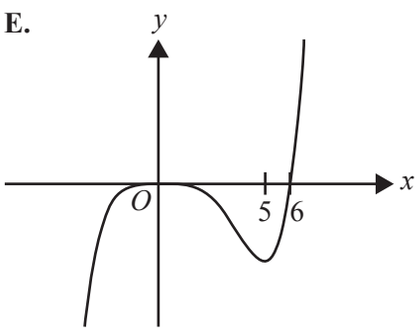
C.



D.



E.



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Question 17

A box contains n marbles that are identical in every way except colour, of which k marbles are coloured red and the remainder of the marbles are coloured green. Two marbles are drawn randomly from the box.

If the first marble is **not** replaced into the box before the second marble is drawn, then the probability that the two marbles drawn are the same colour is

A. $\dfrac{k^2+(n-k)^2}{n^2}$

B. $\dfrac{k^2+(n-k-1)^2}{n^2}$

C. $\dfrac{2k(n-k-1)}{n(n-1)}$

D. $\dfrac{k(k-1)+(n-k)(n-k-1)}{n(n-1)}$

E. ${}^{n}C_{2}\left(\dfrac{k}{n}\right)^{2}\left(1-\dfrac{k}{n}\right)^{n-2}$

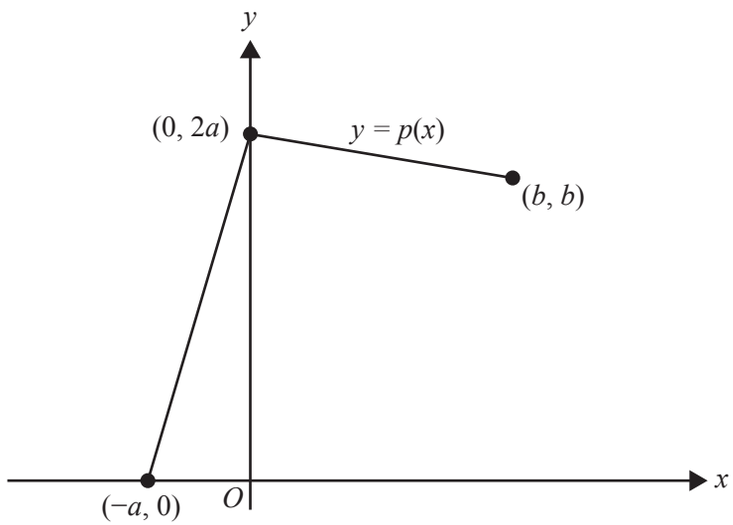
Question 18

The distribution of a continuous random variable, X, is defined by the probability density function f, where

$$f(x) = \begin{cases} p(x) & -a \le x \le b \\ 0 & \text{otherwise} \end{cases}$$

and $a, b \in R^+$.

The graph of the function p is shown below.



It is known that the average value of p over the interval $[-a, b]$ is $\frac{3}{4}$.

$\Pr(X > 0)$ is

A. $\frac{2}{3}$

B. $\frac{3}{4}$

C. $\frac{4}{5}$

D. $\frac{7}{9}$

E. $\frac{5}{6}$

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Question 19

Given that $\tan(\alpha) = d$, where $d > 0$ and $0 < \alpha < \frac{\pi}{2}$, the sum of the solutions to $\tan(2x) = d$, where $0 < x < \frac{5\pi}{4}$, in terms of α, is

A. 0

B. 2α

C. $\pi + 2\alpha$

D. $\frac{\pi}{2} + \alpha$

E. $\frac{3(\pi + \alpha)}{2}$

Question 20

The expression $\log_x(y) + \log_y(z)$, where x, y and z are all real numbers greater than 1, is equal to

A. $-\frac{1}{\log_y(x)} - \frac{1}{\log_z(y)}$

B. $\frac{1}{\log_x(y)} + \frac{1}{\log_y(z)}$

C. $-\frac{1}{\log_x(y)} - \frac{1}{\log_y(z)}$

D. $\frac{1}{\log_y(x)} + \frac{1}{\log_z(y)}$

E. $\log_y(x) + \log_z(y)$

END OF SECTION A
TURN OVER

SECTION B

Instructions for Section B

Answer **all** questions in the spaces provided. Write using blue or black pen.
In all questions where a numerical answer is required, an exact value must be given unless otherwise specified.
In questions where more than one mark is available, appropriate working **must** be shown.
Unless otherwise indicated, the diagrams in this book are **not** drawn to scale.

Question 1 (11 marks)
Let $f : R \to R, f(x) = x^2 e^{-x^2}$.

a. Find $f'(x)$. 1 mark

b. **i.** State the nature of the stationary point on the graph of f at the origin. 1 mark

ii. Find the maximum value of the function f and the values of x for which the maximum occurs. 2 marks

iii. Find the values of $d \in R$ for which $f(x) + d$ is always negative. 1 mark

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c. **i.** Find the equation of the tangent to the graph of f at $x = -1$. 1 mark

ii. Find the area enclosed by the graph of f and the tangent to the graph of f at $x = -1$, correct to four decimal places. 2 marks

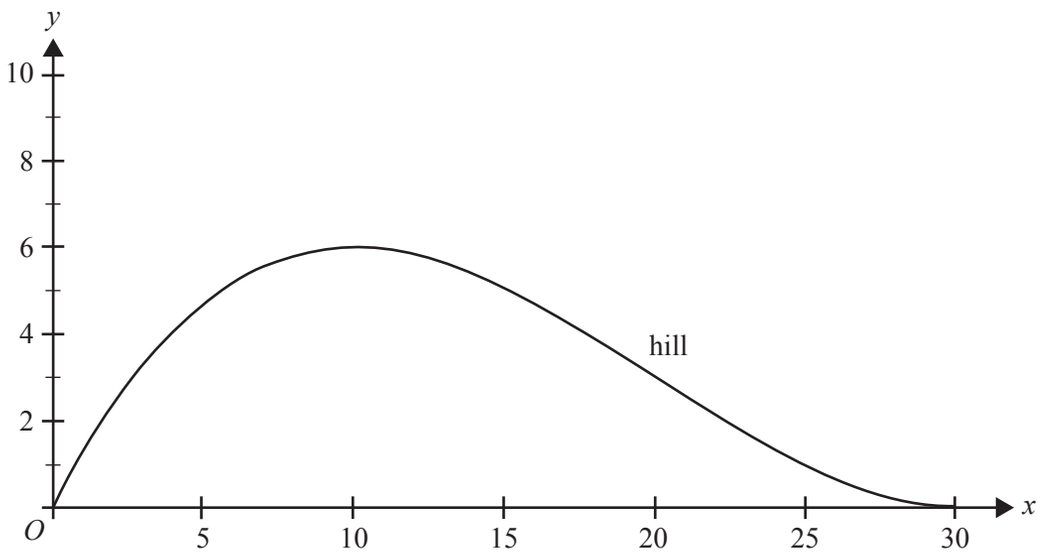
d. Let $M(m, n)$ be a point on the graph of f, where $m \in [0, 1]$.

Find the minimum distance between M and the point $(0, e)$, and the value of m for which this occurs, correct to three decimal places. 3 marks

Question 2 (11 marks)

An amusement park is planning to build a zip-line above a hill on its property.

The hill is modelled by $y = \frac{3x(x-30)^2}{2000}$, $x \in [0, 30]$, where x is the horizontal distance, in metres, from an origin and y is the height, in metres, above this origin, as shown in the graph below.



a. Find $\frac{dy}{dx}$. 1 mark

b. State the set of values for which the gradient of the hill is strictly decreasing. 1 mark

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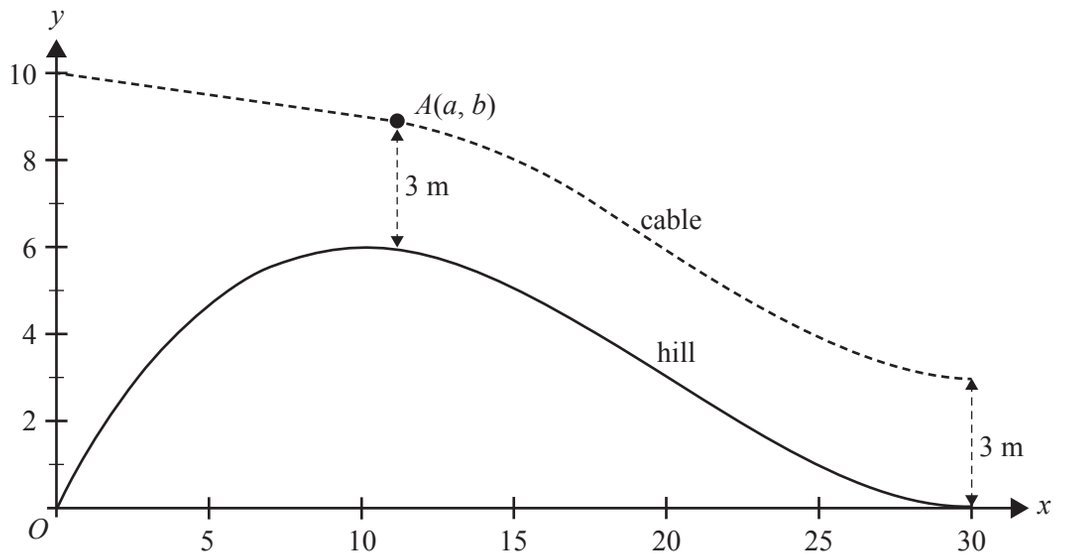
The cable for the zip-line is connected to a pole at the origin at a height of 10 m and is straight for $0 \leq x \leq a$, where $10 \leq a \leq 20$. The straight section joins the curved section at $A(a, b)$. The cable is then exactly 3 m vertically above the hill from $a \leq x \leq 30$, as shown in the graph below.



c. State the rule, in terms of x, for the height of the cable above the horizontal axis for $x \in [a, 30]$. 1 mark

d. Find the values of x for which the gradient of the cable is equal to the average gradient of the hill for $x \in [10, 30]$. 3 marks

The gradients of the straight and curved sections of the cable approach the same value at $x = a$, so there is a continuous and smooth join at A.

e. **i.** State the gradient of the cable at A, in terms of a. 1 mark

ii. Find the coordinates of A, with each value correct to two decimal places. 3 marks

iii. Find the value of the gradient at A, correct to one decimal place. 1 mark

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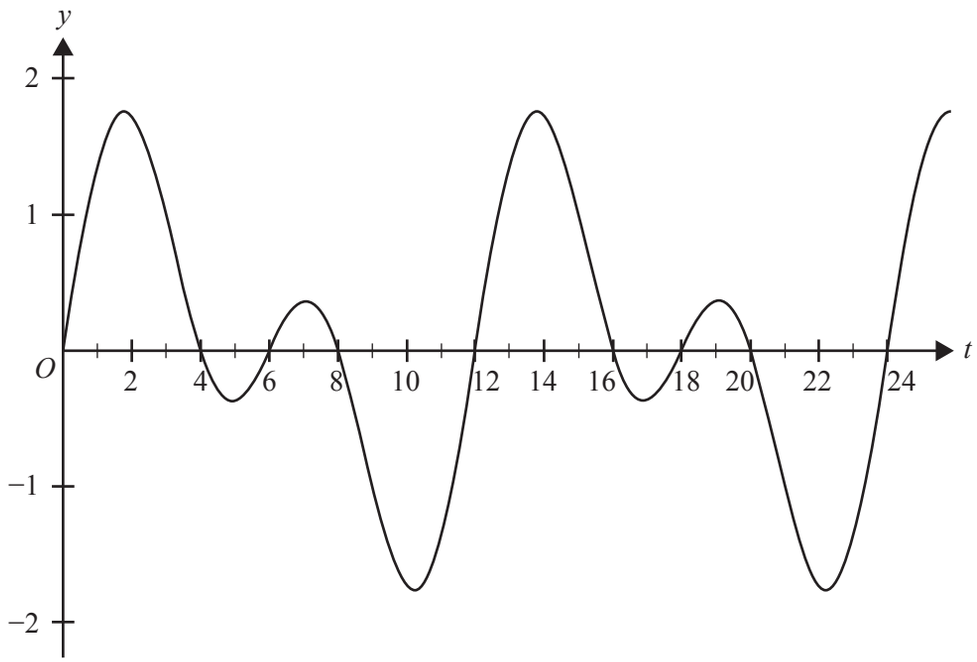
Question 3 (9 marks)

During a telephone call, a phone uses a dual-tone frequency electrical signal to communicate with the telephone exchange.

The strength, f, of a simple dual-tone frequency signal is given by the function $f(t) = \sin\left(\frac{\pi t}{3}\right) + \sin\left(\frac{\pi t}{6}\right)$, where t is a measure of time and $t \geq 0$.

Part of the graph of $y = f(t)$ is shown below.



a. State the period of the function. 1 mark

b. Find the values of t where $f(t) = 0$ for the interval $t \in [0, 6]$. 1 mark

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c. Find the maximum strength of the dual-tone frequency signal, correct to two decimal places. 1 mark

d. Find the area between the graph of f and the horizontal axis for $t \in [0, 6]$. 2 marks

Let g be the function obtained by applying the transformation T to the function f, where

$$T\left(\begin{bmatrix} x \\ y \end{bmatrix}\right) = \begin{bmatrix} a & 0 \\ 0 & b \end{bmatrix}\begin{bmatrix} x \\ y \end{bmatrix} + \begin{bmatrix} c \\ d \end{bmatrix}$$

and a, b, c and d are real numbers.

e. Find the values of a, b, c and d given that $\int_2^0 g(t)dt + \int_2^6 g(t)dt$ has the same area calculated in **part d.** 2 marks

f. The rectangle bounded by the line $y = k$, $k \in R^+$, the horizontal axis, and the lines $x = 0$ and $x = 12$ has the same area as the area between the graph of f and the horizontal axis for one period of the dual-tone frequency signal.

Find the value of k. 2 marks

Question 4 (17 marks)

The Lorenz birdwing is the largest butterfly in Town A.

The probability density function that describes its life span, X, in weeks, is given by

$$f(x) = \begin{cases} \frac{4}{625}(5x^3 - x^4) & 0 \le x \le 5 \\ 0 & \text{elsewhere} \end{cases}$$

a. Find the mean life span of the Lorenz birdwing butterfly. 2 marks

b. In a sample of 80 Lorenz birdwing butterflies, how many butterflies are expected to live longer than two weeks, correct to the nearest integer? 2 marks

c. What is the probability that a Lorenz birdwing butterfly lives for at least four weeks, given that it lives for at least two weeks, correct to four decimal places? 2 marks

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The wingspans of Lorenz birdwing butterflies in Town A are normally distributed with a mean of 14.1 cm and a standard deviation of 2.1 cm.

d. Find the probability that a randomly selected Lorenz birdwing butterfly in Town A has a wingspan between 16 cm and 18 cm, correct to four decimal places. 1 mark

e. A Lorenz birdwing butterfly is considered to be **very small** if its wingspan is in the smallest 5% of all the Lorenz birdwing butterflies in Town A.

Find the greatest possible wingspan, in centimetres, for a **very small** Lorenz birdwing butterfly in Town A, correct to one decimal place. 1 mark

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Each year, a detailed study is conducted on a random sample of 36 Lorenz birdwing butterflies in Town A. A Lorenz birdwing butterfly is considered to be **very large** if its wingspan is greater than 17.5 cm. The probability that the wingspan of any Lorenz birdwing butterfly in Town A is greater than 17.5 cm is 0.0527, correct to four decimal places.

f. **i.** Find the probability that three or more of the butterflies, in a random sample of 36 Lorenz birdwing butterflies from Town A, are **very large**, correct to four decimal places. 1 mark

ii. The probability that n or more butterflies, in a random sample of 36 Lorenz birdwing butterflies from Town A, are **very large** is less than 1%.

Find the smallest value of n, where n is an integer. 2 marks

iii. For random samples of 36 Lorenz birdwing butterflies in Town A, $\hat{P}$ is the random variable that represents the proportion of butterflies that are **very large**.

Find the expected value and the standard deviation of $\hat{P}$, correct to four decimal places. 2 marks

iv. What is the probability that a sample proportion of butterflies that are **very large** lies within one standard deviation of 0.0527, correct to four decimal places? Do not use a normal approximation. 2 marks

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g. The Lorenz birdwing butterfly also lives in Town B.

In a particular sample of Lorenz birdwing butterflies from Town B, an approximate 95% confidence interval for the proportion of butterflies that are **very large** was calculated to be (0.0234, 0.0866), correct to four decimal places.

Determine the sample size used in the calculation of this confidence interval. 2 marks

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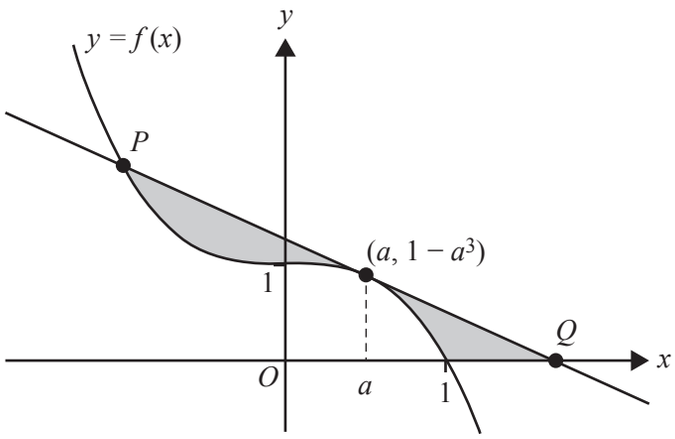
Question 5 (12 marks)

Let $f: R \to R$, $f(x) = 1 - x^3$. The tangent to the graph of f at $x = a$, where $0 < a < 1$, intersects the graph of f again at P and intersects the horizontal axis at Q. The shaded regions shown in the diagram below are bounded by the graph of f, its tangent at $x = a$ and the horizontal axis.



a. Find the equation of the tangent to the graph of f at $x = a$, in terms of a. 1 mark

b. Find the x-coordinate of Q, in terms of a. 1 mark

c. Find the x-coordinate of P, in terms of a. 2 marks

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Let A be the function that determines the total area of the shaded regions.

d. Find the rule of A, in terms of a. 3 marks

e. Find the value of a for which A is a minimum. 2 marks

Consider the regions bounded by the graph of f^{-1}, the tangent to the graph of f^{-1} at $x = b$, where $0 < b < 1$, and the vertical axis.

f. Find the value of b for which the total area of these regions is a minimum. 2 marks

g. Find the value of the acute angle between the tangent to the graph of f and the tangent to the graph of f^{-1} at $x = 1$. 1 mark

END OF QUESTION AND ANSWER BOOK

VICTORIAN CURRICULUM AND ASSESSMENT AUTHORITY

Victorian Certificate of Education
2019

MATHEMATICAL METHODS

Written examination 2

FORMULA SHEET

Instructions

This formula sheet is provided for your reference.
A question and answer book is provided with this formula sheet.

Students are NOT permitted to bring mobile phones and/or any other unauthorised electronic devices into the examination room.

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Mathematical Methods formulas

Mensuration

area of a trapezium	$\frac{1}{2}(a+b)h$	volume of a pyramid	$\frac{1}{3}Ah$
curved surface area of a cylinder	$2\pi rh$	volume of a sphere	$\frac{4}{3}\pi r^3$
volume of a cylinder	$\pi r^2 h$	area of a triangle	$\frac{1}{2}bc\sin(A)$
volume of a cone	$\frac{1}{3}\pi r^2 h$		

Calculus

$\frac{d}{dx}\left(x^n\right) = nx^{n-1}$	$\int x^n dx = \frac{1}{n+1}x^{n+1} + c, \ n \neq -1$
$\frac{d}{dx}\left((ax+b)^n\right) = an(ax+b)^{n-1}$	$\int (ax+b)^n dx = \frac{1}{a(n+1)}(ax+b)^{n+1} + c, n \neq -1$
$\frac{d}{dx}\left(e^{ax}\right) = ae^{ax}$	$\int e^{ax} dx = \frac{1}{a}e^{ax} + c$
$\frac{d}{dx}\left(\log_e(x)\right) = \frac{1}{x}$	$\int \frac{1}{x} dx = \log_e(x) + c, \ x > 0$
$\frac{d}{dx}\left(\sin(ax)\right) = a\ \cos(ax)$	$\int \sin(ax) dx = -\frac{1}{a}\cos(ax) + c$
$\frac{d}{dx}\left(\cos(ax)\right) = -a\ \sin(ax)$	$\int \cos(ax) dx = \frac{1}{a}\sin(ax) + c$
$\frac{d}{dx}\left(\tan(ax)\right) = \frac{a}{\cos^2(ax)} = a\ \sec^2(ax)$	

product rule	$\frac{d}{dx}(uv) = u\frac{dv}{dx} + v\frac{du}{dx}$	quotient rule	$\frac{d}{dx}\left(\frac{u}{v}\right) = \frac{v\frac{du}{dx} - u\frac{dv}{dx}}{v^2}$
chain rule	$\frac{dy}{dx} = \frac{dy}{du}\frac{du}{dx}$		

Probability

$\Pr(A) = 1 - \Pr(A')$		$\Pr(A \cup B) = \Pr(A) + \Pr(B) - \Pr(A \cap B)$	
$\Pr(A\|B) = \dfrac{\Pr(A \cap B)}{\Pr(B)}$			
mean	$\mu = \mathrm{E}(X)$	variance	$\mathrm{var}(X) = \sigma^2 = \mathrm{E}((X-\mu)^2) = \mathrm{E}(X^2) - \mu^2$

Probability distribution		Mean	Variance
discrete	$\Pr(X = x) = p(x)$	$\mu = \sum x\, p(x)$	$\sigma^2 = \sum (x-\mu)^2\, p(x)$
continuous	$\Pr(a < X < b) = \int_a^b f(x)dx$	$\mu = \int_{-\infty}^{\infty} x\, f(x)dx$	$\sigma^2 = \int_{-\infty}^{\infty} (x-\mu)^2 f(x)dx$

Sample proportions

$\hat{P} = \dfrac{X}{n}$		mean	$\mathrm{E}(\hat{P}) = p$
standard deviation	$\mathrm{sd}(\hat{P}) = \sqrt{\dfrac{p(1-p)}{n}}$	approximate confidence interval	$\left(\hat{p} - z\sqrt{\dfrac{\hat{p}(1-\hat{p})}{n}},\ \hat{p} + z\sqrt{\dfrac{\hat{p}(1-\hat{p})}{n}}\right)$

END OF FORMULA SHEET