Victorian Certificate of Education – Free Trial Examinations

STUDENT NUMBER									Letter

MATHEMATICAL METHODS

Free Trial Written Examination 2

Reading time: 15 minutes
Writing time: 2 hours

QUESTION AND ANSWER BOOK

Structure of book

Section	*Number of questions*	*Number of questions to be answered*	*Number of marks*
A	20	20	20
B	4	4	60
			Total: 80

- Students are permitted to bring into the examination room: pens, pencils, highlighters, erasers, sharpeners, rulers, a protractor, set squares, aids for curve sketching, one bound reference, one approved technology (calculator or software) and, if desired, one scientific calculator. Calculator memory DOES NOT need to be cleared. For approved computer-based CAS, full functionality may be used.
- Students are NOT permitted to bring into the examination room: blank sheets of paper and/or correction fluid/tape.

Materials supplied
- Question and answer book of 21 pages
- Formula sheet
- Answer sheet for multiple-choice questions

Instructions
- Write your **student number** in the space provided above on this page.
- Check that your **name** and **student number** as printed on your answer sheet for multiple-choice questions are correct, **and** sign your name in the space provided to verify this.
- Unless otherwise indicated, the diagrams in this book are **not** drawn to scale.
- All written responses must be in English.

At the end of the examination
- Place the answer sheet for multiple-choice questions inside the front cover of this book.
- You may keep the formula sheet.

Students are NOT permitted to bring mobile phones and/or any other unauthorised electronic devices into the examination room.

Version 2 (April 2019)

SECTION A – Multiple-choice questions

Instructions for Section A

Answer **all** questions in pencil on the answer sheet provided for multiple-choice questions.

Choose the response that is **correct** for the question.

A correct answer scores 1; an incorrect answer scores 0.

Marks will **not** be deducted for incorrect answers.

No marks will be given if more than one answer is completed for any question.

Unless otherwise indicated, the diagrams in this book are **not** drawn to scale.

Question 1

The gradient of a line that is perpendicular to the line segment joining the points $(-1,1)$ and $\left(\frac{3}{4},0\right)$ is

A. $\frac{1}{7}$

B. $\frac{4}{7}$

C. $-\frac{4}{7}$

D. $\frac{7}{4}$

E. $-\frac{7}{4}$

Question 2

The period and maximal domain of the function $g(x)=3\tan\left(\frac{\pi x}{2}\right)$ are respectively

A. π and $[-3,3]$

B. 2 and $\mathbb{R}\setminus\{2k-1 \mid k\in\mathbb{Z}\}$

C. 2 and $\mathbb{R}\setminus\{4k-1 \mid k\in\mathbb{Z}\}$

D. 4 and $\mathbb{R}\setminus\{4k-1 \mid k\in\mathbb{Z}\}$

E. 4 and $[-3,3]$

Question 3

Let g be a differentiable function for all $x\in\mathbb{R}$ such that $g'(2)=0$ and $g'(x)<0$ for all $x\in\mathbb{R}\setminus\{2\}$.

Which one of the following statements about the graph of g is true?

A. There is a local minimum at $(2, g(2))$.

B. There is a local maximum at $(2, g(2))$.

C. There is a stationary point of inflection at $(2, g(2))$.

D. There is a non-stationary point of inflection at $(2, g(2))$.

E. There is more than one stationary point.

Question 4

The average rate of change of the function $f(x)=\log_e\left(\sqrt{x}+1\right)+2\sqrt{x}$ between $x=0$ and $x=4$ is equal to

A. $\log_e(3)+4$

B. $\frac{1}{4}\log_e(3)+1$

C. $\frac{1}{4}\log_e(3)+4$

D. $\frac{3}{4}\log_e(3)+\frac{8}{3}$

E. $9\log_e(3)+32$

Question 5

Consider the graph of the function $w(x)=x^{-3}$.

The following transformations are applied, in the given order, to the graph of w:

- Dilation by a factor of 2 from the y-axis
- Translation of 3 units in the positive x-direction
- Reflection about the line $y=x$

Which one of the following is the rule of the transformed graph?

A. $y=2x^{\frac{-1}{3}}+3$

B. $y=2x^{\frac{-1}{3}}-3$

C. $y=2x^{\frac{-1}{3}}+6$

D. $y=\frac{1}{2}x^{\frac{-1}{3}}+3$

E. $y=\frac{1}{2}x^{\frac{-1}{3}}-3$

Question 6

Consider the graph of the function $f:D\to\mathbb{R},\ f(x)=\frac{kx}{x+1}$, where D is the maximal domain of f and $k\neq 0$.

The equations of the asymptotes of the graph of f^{-1}, the inverse function of f, are

A. $x=-1$ and $y=0$

B. $x=-1$ and $y=k$

C. $x=0$ and $y=-1$

D. $x=k$ and $y=-1$

E. $x=-k$ and $y=-1$

Question 7

Let $p:\mathbb{R}\to\mathbb{R},\ p(x)=x^3+kx^2+3x+1$, where $k\in\mathbb{R}$.

The graph of p will have no stationary points if

A. $k=9$

B. $k=-3$ or $k=3$

C. $k<-3$ or $k>3$

D. $-3\le k\le 3$

E. $-3<k<3$

Question 8

Let $h(x)=\begin{cases}2x & -8\le x<0\\ ax(4-x) & 0\le x\le 4\end{cases}$.

Given that the average value of h over its domain is 0, the value of a is

A. 1

B. 6

C. 12

D. -6

E. $\frac{4}{3}$

Question 9

The simultaneous linear equations $mx-4y=m$ and $2x-my=1$ will have a unique solution only for

A. $m>0$

B. $m\in\{-2\sqrt{2},\ 2\sqrt{2}\}$

C. $m\in\mathbb{R}\setminus\{-2\sqrt{2},\ 2\sqrt{2}\}$

D. $m\in\{-2, 2\}$

E. $m\in\mathbb{R}\setminus\{-2, 2\}$

Question 10

For which one of the following functions is the functional relation $f(x+y)=f(x)+f(y)+f(x)f(y)$ true for all $x, y\in\mathbb{R}$?

A. $f(x)=x-1$

B. $f(x)=1-x$

C. $f(x)=e^x-1$

D. $f(x)=1-e^x$

E. $f(x)=\log_e(x-1)$

Question 11

The binomial variable S is such that $\mathrm{E}(S)=6$ and $\mathrm{Var}(S)=5$.

$\Pr(S>6)$ is closest to

A. 0.3926
B. 0.3932
C. 0.3933
D. 0.5691
E. 0.5778

Question 12

Where f is a differentiable function for all $x \in \mathbb{R}$, the derivative of $\cos(2f(x))$ with respect to x is equal to

A. $\sin(2f(x))$
B. $-\sin(2f(x))$
C. $2f'(x)\sin(2f(x))$
D. $-2f'(x)\sin(2f(x))$
E. $-2f'(x)\cos(2f(x))$

Question 13

The continuous random variable T has a probability density function f, where

$$f(t)=\begin{cases}\dfrac{e^{t}+e^{-t}}{4} & -a \le t \le a \\ 0 & \text{elsewhere}\end{cases},$$

and $a>0$.

The value of a is

A. $\log_e\left(\sqrt{2}+1\right)$
B. $\log_e\left(\sqrt{3}+2\right)$
C. $\log_e\left(\sqrt{5}+2\right)$
D. $\log_e\left(\dfrac{\sqrt{17}+1}{4}\right)$
E. 1

Question 14

If $\int_0^2 g(x)\,dx=4$ and $\int_0^4 \left(4g\left(\dfrac{x}{2}\right)+ax\right)dx=64$, then the value of a is

A. 3
B. 4
C. 6
D. 7
E. 8

Question 15

Let X be a discrete random variable such that $\Pr(X = x) = ax^2(4 - x)$, where $X \in \{1, 2, 3\}$ and $a \neq 0$.

The value of a is

A. $\frac{3}{64}$

B. $\frac{1}{20}$

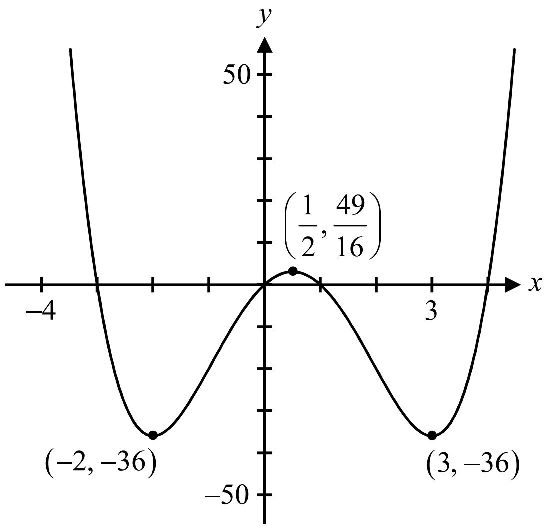
C. $\frac{3}{44}$

D. $\frac{1}{10}$

E. $\frac{27}{256}$

Question 16

Part of the graph of a quartic polynomial function f and the coordinates of its stationary points are shown below.



The graph of $y = -f\left(\sqrt{x}\right)$, defined over its maximal domain, has

A. a local minimum at $(-4, -36)$, a local maximum at $\left(\frac{1}{4}, \frac{49}{16}\right)$ and a local minimum at $(9, -36)$.

B. a local maximum at $(4, 36)$, a local minimum at $\left(\frac{1}{4}, -\frac{49}{16}\right)$ and a local maximum at $(9, 36)$.

C. a local maximum at $(-2, 6)$, a local minimum at $\left(\frac{1}{2}, -\frac{7}{4}\right)$ and a local maximum at $(3, 6)$.

D. a local minimum at $\left(\frac{1}{4}, -\frac{49}{16}\right)$ and a local maximum at $(9, 36)$ only.

E. a local minimum at $\left(\frac{1}{2}, -\frac{7}{4}\right)$ only.

Question 17

A 95% confidence interval for the proportion of train that are late in a rural town is calculated from a large sample to be $(0.035434, 0.064566)$, correct to six decimal places.

The sample size from which this interval was constructed is

A. 156
B. 215
C. 512
D. 860
E. 1050

Question 18

Let $f:\mathbb{R}\to\mathbb{R},\ f(x)=ae^{-mx}$ and let $g:\mathbb{R}\to\mathbb{R},\ g(x)=be^{-nx}$, where $a,b,m,n\in\mathbb{N}$.

Given that $\int f(x)dx=g'(x)$, which one of the following is **not necessarily** true?

A. $\frac{m}{n}\in\mathbb{N}$

B. $\frac{n}{m}\in\mathbb{N}$

C. $\frac{m^2}{n^2}\in\mathbb{N}$

D. $\frac{a}{b}\in\mathbb{N}$

E. $\frac{b}{a}\in\mathbb{N}$

Question 19

A bag contains an equal number of black and red marbles. Let $\hat{P}$ be the distribution of sample proportions of red marbles when samples of size four are taken from the bag **without replacement**.

Given that $\Pr\left(\hat{P}=0\right)=\frac{1}{33}$, the total number of marbles in the bag is

A. 6
B. 12
C. 16
D. 24
E. 32

Question 20

Consider the graph of $f:\mathbb{R}\to\mathbb{R},\ f(x)=2x$ and $g:\left(-\frac{\pi}{2},\frac{\pi}{2}\right)\to\mathbb{R},\ g(x)=a\tan(x)$, where $a\in\mathbb{R}\setminus\{0\}$.

The maximal set of values of a for which the graphs of f and g only intersect or touch once is

A. $a\in(0,2)$
B. $a\in(0,2\pi]$
C. $a\in[2,\infty)$
D. $a\in(-\infty,2)$
E. $a\in(-\infty,0)\cup[2,\infty)$

END OF SECTION A

SECTION B

Instructions for Section B

Answer **all** questions in the spaces provided.

In all questions where a numerical answer is required, an exact value must be given unless otherwise specified.

In questions where more than one mark is available, appropriate working **must** be shown.

Unless otherwise indicated, the diagrams in this book are **not** drawn to scale.

Question 1 (10 marks)

Let $f:[0, 8\pi] \rightarrow \mathbb{R},\ f(x) = 2 - 4\sin\left(\frac{x}{2}\right)$.

a. State the period and range of *f*. 1 mark

b. **i.** State $f'(x)$. 1 mark

ii. For what values of *x* is $f'(x) > 0$? 1 mark

iii. The rule of f' can be obtained from the rule of *f* under a transformation *T*, where

$$T:\mathbb{R}^2 \rightarrow \mathbb{R}^2,\ T\left(\begin{bmatrix} x \\ y \end{bmatrix}\right) = \begin{bmatrix} 1 & 0 \\ 0 & a \end{bmatrix}\begin{bmatrix} x \\ y \end{bmatrix} + \begin{bmatrix} \pi \\ b \end{bmatrix}.$$

Find the values of *a* and *b*. 2 marks

c. **i.** Find the equations of the tangents to the graph of f at $x = \frac{\pi}{3}$ and $x = \frac{13\pi}{3}$. 2 marks

ii. Find the perpendicular distance between these tangents. 3 marks

Question 2 (13 marks)

A team of biologists are researching the effect of different toxins on a species of bacteria. In their investigation, three agar plates are used. The first plate contains no toxin, the second plate contains toxin X, and the third plate contains toxin Y.

Let $P_n(t)$, where $t \geq 0$ and $0 < P_n(t) < 1$, denote the **proportion** of the nth agar plate that is covered by the bacteria t days after they are added.

The investigation was set up such that 20% of each of the plates were initially covered by the bacteria.

The bacteria on the first agar plate spread according to the function $P_1(t) = 1 - \dfrac{m}{e^{kt} + m}$, where $m, k > 0$.

a. It was found that after four days, 80% of the first plate was covered by bacteria.

Find the values of m and k and hence, show that $P_1(t) = 1 - \dfrac{4}{2^t + 4}$. 2 marks

b. **i.** Find the average rate of change of P_1 over the interval $[0, 4]$. 1 mark

ii. Find the value(s) of t for which the growth rate of the bacteria on the first plate is equal to the average rate of change of P_1 over the interval $[0, 4]$, correct to one decimal place. 2 marks

The bacteria on the second agar plate spread according to the function $P_2(t)=1-\frac{4}{2^t+4}-\frac{3}{50}t$.

c. Due to toxin X, the bacteria on the second plate spread to a maximum coverage, but eventually begin to die.

i. Find the value of t for which the proportion of the second plate that is covered by the bacteria is maximum, correct to one decimal place. 1 mark

ii. Find the maximum **percentage** of the second plate that the bacteria cover, correct to the nearest integer. 1 mark

d. Find the duration of time, in days, for which the bacteria cover more than 50% of the second plate, correct to one decimal place. 2 marks

e. Find the time taken, in days, for the bacteria on the second plate to be eradicated, correct to one decimal place. 1 mark

Due to toxin Y, the bacteria on the third agar plate spread according to the function $P_3(t) = 1 - \frac{4}{2^t + 4} - qt$, where $q > 0$.

f. The maximum percentage of the third plate that is covered by the bacteria is 50%.

Find the value of q, correct to three decimal places, and hence, find the time taken, in days, for the bacteria on the third dish to be eradicated, correct to one decimal place. 3 marks

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Question 3 (17 marks)

Metapaint is a company that supplies many hardware stores with paint. The company uses machines to dispense many types of paints into cans.

On a particular day, it is found that the machine dispensing white paint is not working correctly, and as a result, often overfills the white paint cans. The probability that the machine overfills a can with white paint is 8%. The amount of white paint the machine dispenses into any given can is independent of the amount dispensed into any other can of white paint.

a. At the end of the production line, 32 cans of white paint are checked to see whether they have been overfilled.

i. Find the probability that at least one can of white paint has been overfilled, correct to four decimal places. 1 mark

ii. Given that at least on can of white paint has been overfilled, find the probability that fewer than four cans have been overfilled, correct to four decimal places. 2 marks

The volumes of blue paint filled by a different machine are known to vary normally with a mean of 300 mL and a standard deviation of 8 mL.

The cans of blue paint have a maximum capacity of 315 mL.

b. Find the probability that a randomly selected can of blue paint has been overfilled, correct to four decimal places. 1 mark

Metapaint decides to take a sample of 200 filled cans of blue paint. For samples of size 200, let $\hat{P}$ be the random variable of the distribution of sample proportions of cans of blue paint that have been overfilled.

c. Find $\Pr\left(\hat{P} < 0.05 \,\middle|\, \hat{P} > 0.01\right)$, correct to three decimal places. Do not use a normal approximation in your calculations. 3 marks

d. Using a normal approximation of $\hat{P}$, find the value of b such that $\Pr\left(\hat{P} < b\right) = 0.9$, correct to four decimal places. 2 marks

It is known that the machine that fills the green paint cans also dispenses volumes of paint which vary normally with a mean of 300 mL.

The cans of green paint have a maximum capacity of 315 mL.

e. It is found that 5% of green paint cans are being overfilled.

Find the standard deviation for the normal distribution that applies to the machine that dispenses volumes of green paint, correct to four decimal places. 2 marks

A sample of 50 randomly selected green paint cans is taken from the end of the production line to investigate the proportion of cans that are filled with less than 300 mL of green paint.

f. Suppose the cans are inspected one at a time.

Find the probability that the first can found to have been filled with less than 300 mL of green paint is the fourth one. 1 mark

g. After checking all 50 cans, it is found that 36 cans contain less than 300 mL of green paint.

Determine the approximate 95% confidence interval for the estimate of the proportion of interest. Give values correct to four decimal places. 1 mark

The volumes of red paint, dispensed by a different machine, are known to vary according to a probability density function f, where

$$f(r)=\begin{cases}\dfrac{r-310}{900}\log_e\left(\dfrac{310-r}{60}\right) & 250\le r\le 310\\[2ex] 0 & \text{elsewhere}\end{cases},$$

and r given in millilitres.

h. **i.** Determine the expected volume of red pain dispensed by this machine. 1 mark

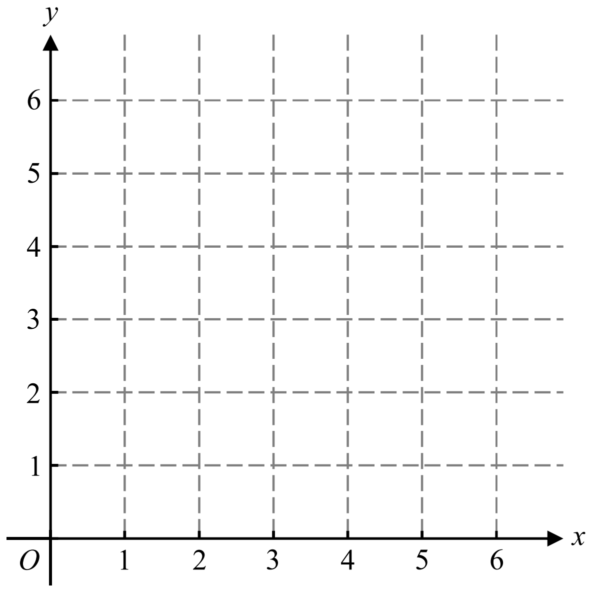
ii. Determine the standard deviation of the volumes of red paint dispensed by this machine. Express your answer in the form $\dfrac{a\sqrt{b}}{c}$, where $a,b,c\in\mathbb{N}$. 2 marks

i. Given that the maximum capacity of a red paint can is 300 mL, find the proportion of red paint cans that will be overfilled, correct to four decimal places. 1 mark

Question 4 (20 marks)

Let $f:[0,4]\to\mathbb{R},\ f(x)=4-\frac{1}{4}x^2$.

a. Sketch that graph of *f* on the axes provided below. Label endpoints. 1 mark



b. **i.** Find the rule and domain of f^{-1}, the inverse function of *f*. 2 marks

ii. Find the area, *B*, of the region enclosed by the graphs of *f* and f^{-1}. 3 marks

c. A rectangle *OKMN* is formed with vertices at $O(0,0)$, $K(0, f(m))$, $M(m, f(m))$ and $N(m, 0)$, where $0 \le m \le 4$.

i. Find an expression for the area of the rectangle, in terms of m. 1 mark

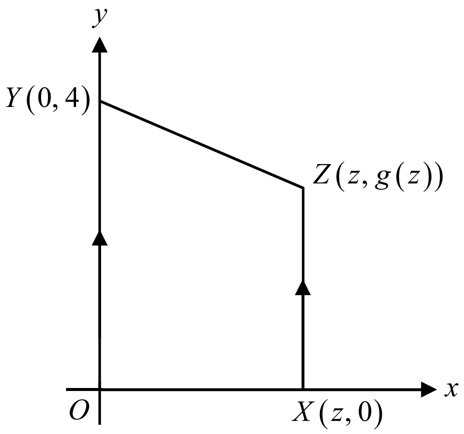
ii. Find the value of m for which the rectangle has maximum area A_R, and find this maximum area. 2 marks

iii. Let S be the area of the region enclosed by the graph of f and the coordinate axes. Find $\frac{A_R}{S}$, expressing your answer in the form $\frac{1}{\sqrt{a}}$, where $a \in \mathbb{N}$. 2 marks

Let $g:[0,4]\to\mathbb{R},\ g(x)=4-4\left(\frac{x}{4}\right)^k$, where $k>1$.

d. Verify that the points $(0, 4)$ and $(4, 0)$ lie on the graph of g for all $k>1$. 1 mark

A trapezium *OYZX* is formed with vertices at $O(0,0)$, $Y(0,4)$, $Z(z, g(z))$, and $X(z,0)$, as shown in the where the line segments *OY* and *ZX* are parallel, and $0<z<4$, as shown in the diagram below.



e. Let $T(z)$ be the rule that gives the area of the trapezium.

Show that $T(z)=4z-2z\left(\frac{z}{4}\right)^k$. 1 mark

Let $A_T(k)$ be the rule of the function $A_T : (1, \infty) \to \mathbb{R}$ that gives the maximum possible area of a trapezium for a given value of *k*.

f. **i.** Show that $A_T(k) = \dfrac{16k}{k+1}\left(\dfrac{2}{k+1}\right)^{\frac{1}{k}}$. 3 marks

ii. Find the smallest value of *p* such that $A_T(k) < p$ for all $k > 1$. 1 mark

g. **i.** Find the area, *A*, of the region enclosed by the graph of *g* and the coordinate axes, in terms of *k*. 1 mark

ii. Find the value of *k* for which the ratio $\dfrac{A_T}{A}$ is a minimum, and hence, find the area of the maximum sized trapezium for this value of *k*, correct to four decimal places. 2 marks

END OF QUESTION AND ANSWER BOOK