TRIUMPH PUBLICATIONS

Fortify Sample Exam 2B

MATHEMATICAL METHODS

Written examination 2

Reading time: 15 minutes
Writing time: 2 hours

QUESTION AND ANSWER BOOK

Structure of book

Section	*Number of questions*	*Number of questions to be answered*	*Number of marks*
A	20	20	20
B	5	5	60
			Total 80

- Students are permitted to bring into the examination room: pens, pencils, highlighters, erasers, sharpeners, rulers, a protractor, set squares, aids for curve sketching, one bound reference, one approved technology (calculator or software) and, if desired, one scientific calculator. Calculator memory DOES NOT need to be cleared. For approved computer-based CAS, full functionality may be used.
- Students are NOT permitted to bring into the examination room: blank sheets of paper and/or correction fluid/tape.

Materials supplied
- Question and answer booklet of 22 pages.
- Formula sheet.
- Working space is provided throughout the book.

Instructions
- Write your **student number** in the space provided above on this page.
- Unless otherwise indicated, the diagrams in this book are **not** drawn to scale.
- All written responses must be in English.

At the end of the examination
- You may keep the formula sheet.

Students are NOT permitted to bring mobile phones and/or any other unauthorised electronic devices into the examination room.

SECTION A - Multiple-choice questions

Instructions for Section A

Answer **all** questions in pencil on the answer sheet provided for multiple-choice questions.
Choose the response that is **correct** for the question.
A correct answer scores 1; an incorrect answer scores 0.
Marks will **not** be deducted for incorrect answers.
No marks will be given if more than one answer is completed for any question.
Unless otherwise indicated, the diagrams in this book are **not** drawn to scale.

Question 1

The quadratic function $f : D \to R,\ f(x) = -(x-3)^2 + 4$ has range $(-5,\ 4]$. A possible domain D of the function is

A. $[3,\ 6)$

B. R

C. $(3,\ 6]$

D. $(-5,\ 4]$

E. $(-6,\ 3]$

Question 2

John is taking a maths exam. The multiple choice section on the exam has 22 questions which each have 5 potential answers. John is a very poor maths student and guesses every answer; that is, he has a 20% chance to answer any given question correctly.
Given that he got at least 8 correct answers, the probability that he got exactly 11 correct answers is closest to

A. 0.0481

B. 0.0360

C. 0.0561

D. 0.0012

E. 0.0221

Question 3

The simultaneous linear equations

$$2x + (m-3)y = \frac{4}{3}m$$
$$(m-4)x + 3y = 8$$

where m is a real number, have **infinitely many solutions** for

A. $m \in R\backslash\{1,\ 0\}$

B. $m \in R\backslash\{0\}$

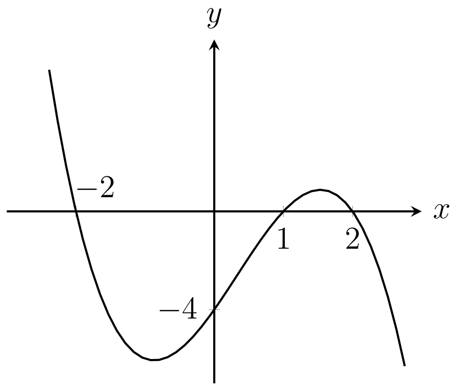
C. $m \in R\backslash\{1\}$

D. $m = 6$

E. $m = 1$

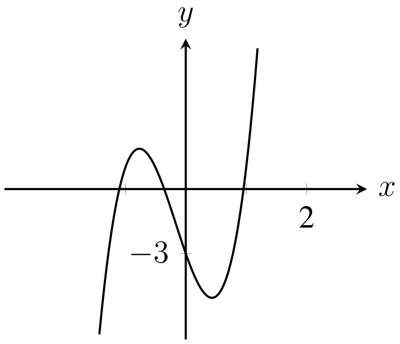
Question 4

The graph of a function f, with domain R, is shown below.

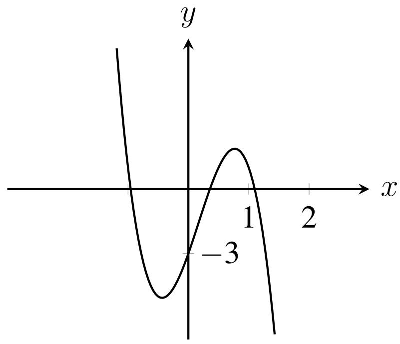


The graph which best represents $f(-2x)+1$ is

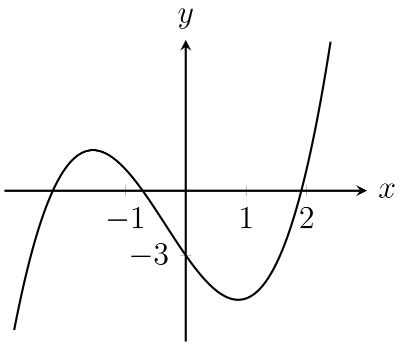
A.



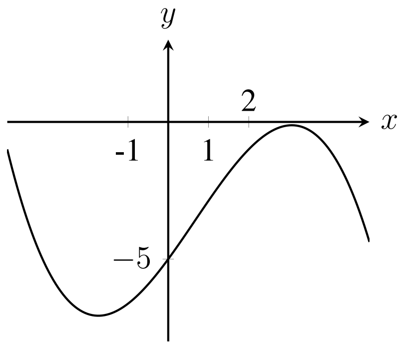
B.



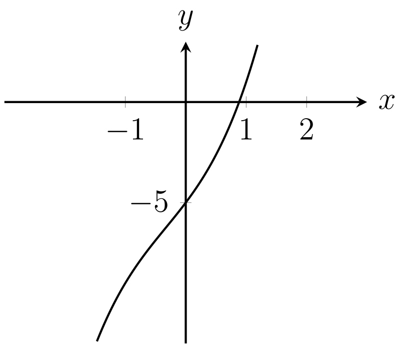
C.



D.



E.



Question 5

The gradient of a line perpendicular to the line which passes through $(0,\ 3)$ and $(-9,\ 0)$ is

A. 3

B. -3

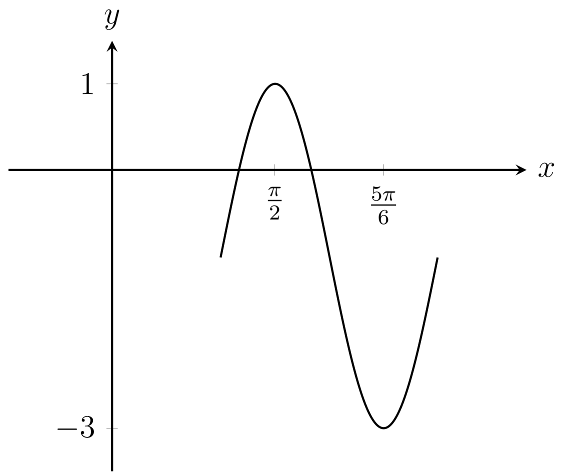
C. 6

D. $\dfrac{1}{3}$

E. $-\dfrac{1}{3}$

Question 6

Consider the graph below.



The graph shown could have equation

A. $y = 2\sin\left(3x - \dfrac{\pi}{3}\right) - 1$

B. $y = 2\cos(3x - \pi) + 1$

C. $y = 2\sin\left(x + \dfrac{\pi}{3}\right) - 1$

D. $y = 4\cos 2\left(x - \dfrac{\pi}{6}\right) + 1$

E. $y = 2\sin 3\left(x - \dfrac{\pi}{3}\right) - 1$

Question 7

Let $f : R^+ \to R$ be a differentiable function. Then, for all $x \in R^+$, the derivative of $f(\log_e(x))$ with respect to x is equal to

A. $\dfrac{f'(\log_e(x))}{x}$

B. $\dfrac{f'(x)}{x}$

C. $\log_e(x)f'(x)$

D. $\mathrm{e}f'(\log_e(x))$

E. $f'(\log_e(x))$

Question 8

The transformation $T : R^2 \to R^2$ that maps the curve with equation $y = \cos(x)$ onto the curve with equation $y = \frac{1}{2}\cos\left(\frac{x-\pi}{2}\right) - 3$ is given by

A. $T\left(\begin{bmatrix} x \\ y \end{bmatrix}\right) = \begin{bmatrix} \frac{1}{2} & 0 \\ 0 & 1 \end{bmatrix}\begin{bmatrix} x \\ y \end{bmatrix} + \begin{bmatrix} -\pi \\ -6 \end{bmatrix}$

B. $T\left(\begin{bmatrix} x \\ y \end{bmatrix}\right) = \begin{bmatrix} 0 & 1 \\ 2 & 0 \end{bmatrix}\begin{bmatrix} x \\ y \end{bmatrix} + \begin{bmatrix} \frac{\pi}{2} \\ 6 \end{bmatrix}$

C. $T\left(\begin{bmatrix} x \\ y \end{bmatrix}\right) = \begin{bmatrix} -2 & 0 \\ 0 & -\frac{1}{2} \end{bmatrix}\begin{bmatrix} x \\ y \end{bmatrix} + \begin{bmatrix} \pi \\ 3 \end{bmatrix}$

D. $T\left(\begin{bmatrix} x \\ y \end{bmatrix}\right) = \begin{bmatrix} \frac{1}{2} & 0 \\ 0 & 2 \end{bmatrix}\begin{bmatrix} x \\ y \end{bmatrix} + \begin{bmatrix} \pi \\ -3 \end{bmatrix}$

E. $T\left(\begin{bmatrix} x \\ y \end{bmatrix}\right) = \begin{bmatrix} 2 & 0 \\ 0 & \frac{1}{2} \end{bmatrix}\begin{bmatrix} x \\ y \end{bmatrix} + \begin{bmatrix} \pi \\ -3 \end{bmatrix}$

Question 9

The upper limit of a 95% confidence interval of a particular survey is 47%. If 1,000 people participated in this survey, the sample proportion, $\hat{p}$, is closest to

A. 0.42

B. 0.43

C. 0.44

D. 0.45

E. 0.46

Question 10

In a particular town, there are 20,000 citizens. From census data, it is known that 1 in 8 citizens are retired. There are 20 people in a cafe on a Friday morning. For this group of 20 people, $\hat{p}$ is the random variable of the distribution of sample proportions of retired citizens.

The value of $\Pr(\hat{p} > 0.1)$ is closest to

A. 0.5353

B. 0.4647

C. 0.7331

D. 0.7653

E. 0.6417

Question 11

Consider the functions

- $f : [-1, \infty) \to R,\ f(x) = \sqrt{x+1}$
- $g : [-a, a] \to R,\ g(x) = 2\cos(x)$

What is the maximum value of a for which $f(g(x))$ exists?

A. $\frac{\pi}{2}$

B. 1

C. $\frac{2\pi}{3}$

D. 2

E. $\frac{\pi}{3}$

Question 12

Charlie loves to go horse riding. If he goes riding one week, the probability that he goes riding the next week is 0.9. If he doesn't go riding one week, the probability that he doesn't go riding the next is 0.2. Charlie didn't go riding this week. The probability that he goes riding exactly once in the next two weeks is

A. 0.04

B. 0.16

C. 0.24

D. 0.36

E. 0.96

Question 13

If $\int_0^2 f(x)\,dx = 5$ and $\int_0^2 bx^2 + 5f(x)\,dx = 41$, the value of b is

A. -1

B. 2

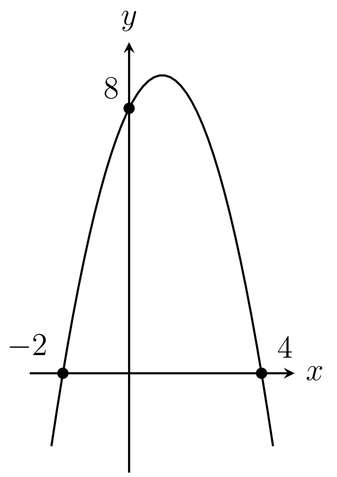
C. 4

D. 6

E. 8

Question 14

The derivative function $g'(x)$ is shown below.



The integral of $g'(x)$ is equal to $g(x) + c$. The equation $g(x) + c = 0$ has three solutions. The possible range of values of c is

A. $-2 < c < 4$

B. $c < -\frac{80}{3} \cup c > \frac{28}{3}$

C. $-\frac{80}{3} < c < \frac{28}{3}$

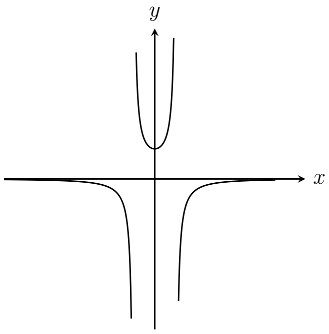
D. $c < -\frac{28}{3} \cup c > \frac{80}{3}$

E. $-\frac{28}{3} < c < \frac{80}{3}$

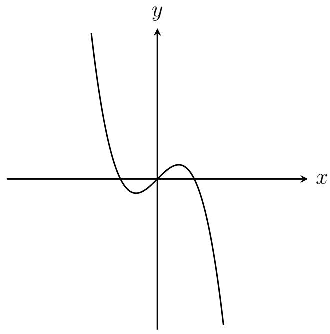
Question 15

If $f'(x) = -2x^2 + 1$, which one of the following graphs could represent the graph of $y = f(x)$?

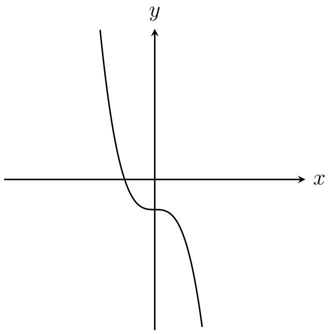
A.



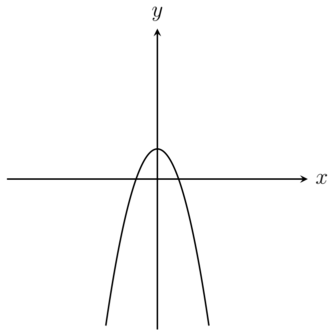
B.



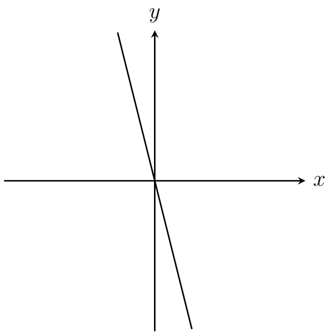
C.



D.



E.



Question 16

If the tangent to the graph of $y = \log_e(ax)$, $a \neq 0$, at $x = b$ passes through the origin, b is equal to

A. $-\dfrac{e}{a}$

B. 1

C. $\dfrac{e}{a}$

D. 0

E. e

Question 17

The inverse function of $f : \left(\frac{3}{2}, \infty\right) \to R,\ f(x) = \frac{1}{\sqrt{2x-3}}$ is

A. $f^{-1} : \left(-\frac{3}{2}, \infty\right) \to R,\ f^{-1}(x) = \frac{1}{(2x-3)^2}$

B. $f^{-1} : \left(\frac{3}{2}, \infty\right) \to R,\ f^{-1}(x) = 2x^2 + \frac{3}{2}$

C. $f^{-1} : R\backslash\{0\} \to R,\ f^{-1}(x) = \frac{1}{2x^2} + \frac{3}{2}$

D. $f^{-1} : R^+ \to R,\ f^{-1}(x) = \frac{1}{2x^2} - \frac{3}{2}$

E. $f^{-1} : R^+ \to R,\ f^{-1}(x) = \frac{1+3x^2}{2x^2}$

Question 18

If $f(x+2) = x(x+4)$, then $f(x)$ is equal to

A. $x^2 + 4x + 2$

B. $x^2 + 8x + 12$

C. $x^2 + 4x + 4$

D. $x^2 + 4x$

E. $x^2 - 4$

Question 19

The height of any given tree in a particular forest area is given by X metres, where X is a normally distributed random variable with mean 8.4 metres and standard deviation k metres.

A sample of 300 trees is researched and 11 of these trees are shorter than 7.3 metres tall. The value of k is

A. 1.79

B. 1.10

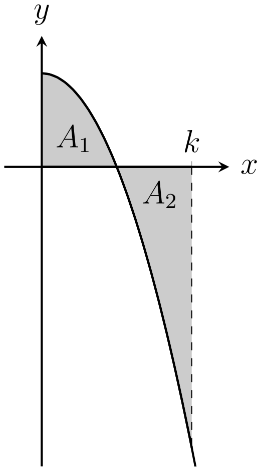
C. 0.61

D. 0.04

E. 0.73

Question 20

Part of the graph of $f : R \to R,\ f(x) = 9 - x^2$ is shown below.



The area of the region A_1 is equal to the area of the region A_2. The value of k is

A. 0

B. 18

C. $2\sqrt{3}$

D. 6

E. $3\sqrt{3}$

END OF SECTION A
TURN OVER

SECTION B

Instructions for Section B

Answer **all** questions in the spaces provided.

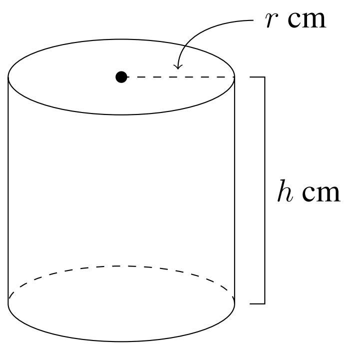
In all questions where a numerical answer is required, an exact value must be given unless otherwise specified.

In questions where more than one mark is available, appropriate working **must** be shown.

Unless otherwise indicated, the diagrams in this book are **not** drawn to scale.

Question 1 (7 marks)

A tin can with a closed lid is made in the shape of a cylinder as shown below, with the radius of the lid r cm and height h cm.



The volume of the can is 192π cm^3.

a. Find an expression for h in terms of r.

2 marks

b. Show that the total surface area of the can, A cm^2, is given by $A = 2\pi\left(r^2 + \frac{192}{r}\right)$.

2 marks

c. Find the value of h for which the tin has the minimum possible total surface area.

3 marks

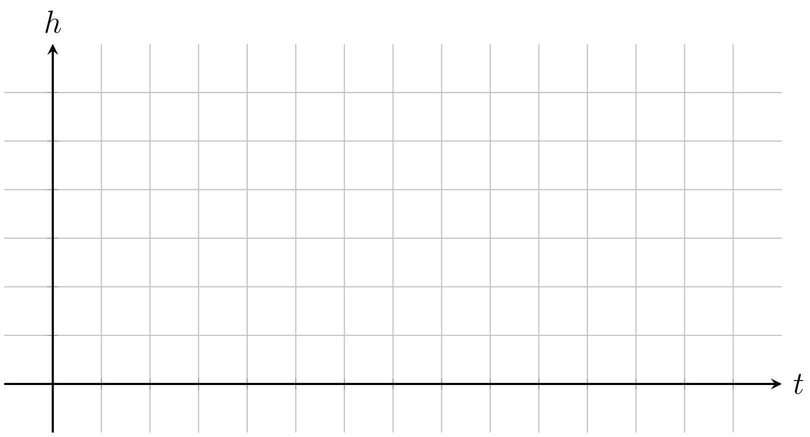
Question 2 (12 marks)

While Zoe and Jessica are riding a Ferris Wheel, they realise that as the wheel spins the height of their carriage above ground level, h metres, t minutes after the ride starts, can be modelled by the function $h(t) = a - b\cos(ct)$. Once the ride starts, it takes the girls 2 minutes to reach the top.

a. If the bottom of the wheel is 3 metres off the ground, and the diameter of the wheel is 16 metres, find the values of a, b and c. The wheel does not stop during a regular ride.

3 marks

b. Sketch the graph of h against t for the interval $0 \le t \le 8$ (a regular ride length) on the axes below. Label all endpoints and axis intercepts. 3 marks



c. How many minutes have the girls been on the ride when they first reach a height of 8 metres? Give your answer correct to two decimal places.

1 mark

d. During a regular ride, for how long is Zoe and Jessica's carriage above a height of 14 metres? Give your answer correct to the nearest second.

3 marks

e. How fast are Zoe and Jessica travelling while on the wheel? Give your answer in metres per second.

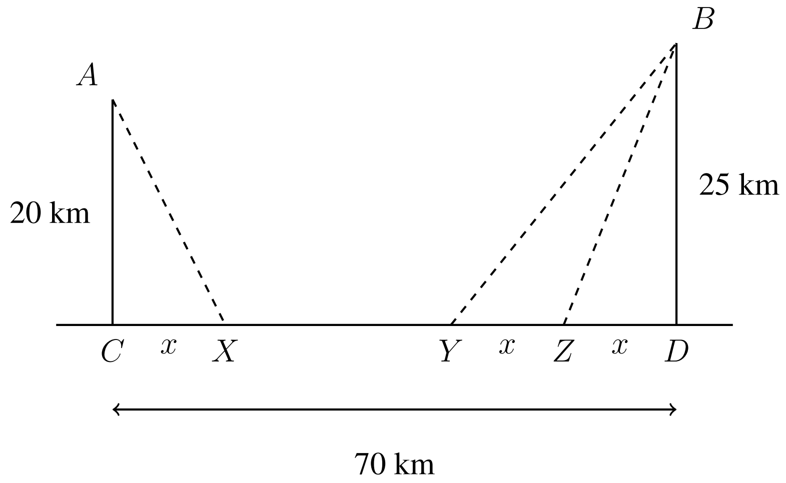
2 marks

Question 3 (18 marks)

Destiny and Tom are a hiking couple. Earlier in the day, they enter Jurassic National Park at point B. They then ride their mountain bikes using sealed roads to the premium hiking location at point A, which is 20 km north of an intersection at point C.

The sealed roads run along $ACDB$. There are off-road 'shortcut roads' along AX, BY, and BZ.

$CD = 70$ km, $CX = YZ = ZD = x$ km, $AC = 20$ km and $BD = 25$ km.



The trek is more difficult than the couple thought it would be, so they return to their bikes at point A later than they had planned. They know that the National Park closes its gates at a certain time, depending on the season. Sensors at the front gate at point B recognise when it gets too dark, and the gate automatically closes so that no hikers can enter the park at night.

The intensity of light in winter decreases over time according to the equation

$$L : [0,\ 12] \to R,\ L(t) = 50(1 + 19e^{-t})$$

where L is the intensity of light (lux) and t is the time in hours after 4:30pm.

The gates shut when the light intensity reaches 80 lux.

a. At what time will the gates shut and trap Destiny and Tom in the park overnight, correct to the nearest minute?

2 marks

To avoid being trapped in the park, the couple must leave point A as soon as possible. They jump on their mountain bikes and start pedalling, but they need to decide which way to go. They can travel on sealed roads at 32 km/hour or they can travel through the bush via the off-road shortcuts at 25 km/hour.

b. Show that Destiny and Tom will remain trapped in the park if they only take sealed roads.

1 mark

In order to reach the gates in time, Destiny and Tom take the off-road shortcut from A to X, ride along the sealed road from X to Z and then ride up another off-road shortcut from Z to B.

c. Show that the time taken to reach the front gates, T hours, is given by

$$T = \frac{1}{25}(\sqrt{400 + x^2} + \sqrt{625 + x^2}) + \frac{1}{16}(35 - x)$$

2 marks

d. Find the value of x which allows the couple to reach the front gates at point B in the minimum time. Give your answer correct to the nearest metre.

2 marks

e. Show that Destiny and Tom can avoid being trapped.

1 mark

f. Could the hikers still make it to the gate before it shuts if they took the route $AXYB$? Give the value for x km that allows for this in the minimum time possible, and state this minimum time.

3 marks

Destiny and Tom decide that it would be best to hike in the summer next time because they would get more daylight. In summer, the equation that represents the intensity of light is slightly different to the equation in winter. However, the brightest and darkest points between 4:30pm and 4:30am are the same in both summer and winter.

The light intensity in summer is modelled by the equation $L = a + b\mathrm{e}^{-ct}$, where L is the intensity of light in summer, t is the time after 4:30pm and a, b and c are real numbers.

g. If the graph of the function modelling light intensity in summer passes through the point $\left(2,\ 50\left(1+\frac{19}{\mathrm{e}^{\frac{4}{3}}}\right)\right)$, find the values of a, b and c.

3 marks

h. If the gates still shut when the light intensity reaches 80 lux, at what time will the gates be closed during summer, correct to the nearest minute?

2 marks

i. In summer, the park issues a warning through their PA system advising hikers to start heading back to the front gates. If this warning is issued at 8:45pm, what is the intensity of light in the park at this time, correct to two decimal places?

2 marks

Question 4 (11 marks)

A local hardware store is conducting a survey to find out how many households own a barbecue.

a. The store discovers that any given household has a 40% chance to own a barbecue.

i. If the store surveys 15 people, what is the probability that none of them own a barbecue, correct to four decimal places?

1 mark

ii. What is the probability that less than 8 of them don't own a barbecue?

2 marks

b. In a larger survey of 120 people, $\hat{p}$ is the random variable of the distribution of sample proportions of households that own a barbecue.

i. Find $\Pr\left(\hat{p} \leq \frac{5}{12}\right)$, correct to four decimal places.

2 marks

ii. 45% of households in the larger survey own a barbecue. Find the **96%** confidence interval for the estimate of the proportion of households in this town that own a barbecue, correct to four decimal places.

3 marks

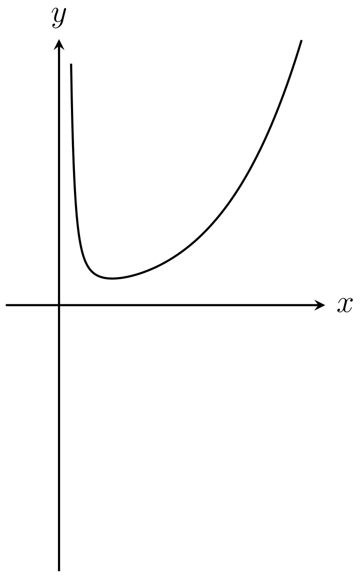
c. The store conducts a different survey to find out the proportion of households in the town that own a fire pit. After conducting the survey, they are a% certain that between 10% and 18% of households in the town own a fire pit.

If 200 people are surveyed, find the value of a, correct to one decimal place.

3 marks

Question 5 (12 marks)

The graph of $f : (0,\ \infty) \to R, f(x) = \dfrac{x^{\log_e(x)}}{2}$ is shown below.



a. Find:

i. the rule for the derivative function $f'(x)$.

2 marks

ii. the gradient of $y = f(x)$ at $x = 2$, correct to two decimal places.

1 mark

The tangent, $y = t(x)$, at $x = k$, passes through the origin.

b. Find the value of k, correct to three decimal places.

2 marks

c. Find the rule for the function $t(x)$, giving any values correct to three decimal places.

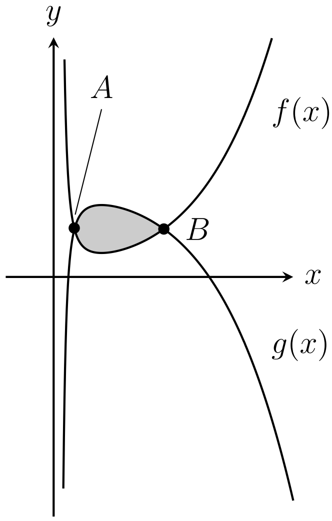
1 mark

Point $P(p, f(p))$ lie on the graph of f such that the distance from origin O to point P is at a minimum.

d. Find the coordinates of point P, correct to three decimal places.

3 marks

The graph of $g : (0, \infty) \to R$, $g(x) = \dfrac{-x^{\log_e(x)}}{2} + 2$ is drawn on the same axes as shown below.



Points A and B are the points of intersection between $y = f(x)$ and $y = g(x)$.

e. Find:

i. the x values of points A and B.

2 marks

ii. the area of the shaded region; the area bound by the graphs of $y = f(x)$ and $y = g(x)$.

1 mark

END OF QUESTION AND ANSWER BOOK

TRIUMPH
PUBLICATIONS

MATHEMATICAL METHODS

Written examination 2

FORMULA SHEET

Instructions

This formula sheet is provided for your reference.
A questions and answer book is provided with this formula sheet.

Students are NOT permitted to bring mobile phones and/or any other unauthorised electronic devices into the examination room.

Formula Sheet

Mensuration

area of a trapezium	$\frac{1}{2}(a+b)h$	volume of a pyramid	$\frac{1}{3}Ah$
curved surface area of a cylinder	$2\pi rh$	volume of a sphere	$\frac{4}{3}\pi r^3$
volume of a cylinder	$\pi r^2 h$	area of a triangle	$\frac{1}{2}bc\sin(A)$
volume of a cone	$\frac{1}{3}\pi r^2 h$		

Calculus

$\frac{d}{dx}\left(x^n\right) = nx^{n-1}$		$\int x^n\,dx = \frac{1}{n+1}x^{n+1} + c,\ n \neq -1$	
$\frac{d}{dx}\left((ax+b)^n\right) = an(ax+b)^{n-1}$		$\int (ax+b)^n\,dx = \frac{1}{a(n+1)}(ax+b)^{n+1} + c,\ n \neq -1$	
$\frac{d}{dx}\left(e^{ax}\right) = ae^{ax}$		$\int e^{ax}\,dx = \frac{1}{a}e^{ax} + c$	
$\frac{d}{dx}(\log_e(x)) = \frac{1}{x}$		$\int \frac{1}{x}\,dx = \log_e(x) + c,\ x > 0$	
$\frac{d}{dx}(\sin(ax)) = a\cos(ax)$		$\int \sin(ax)\,dx = -\frac{1}{a}\cos(ax) + c$	
$\frac{d}{dx}(\cos(ax)) = -a\sin(ax)$		$\int \cos(ax)\,dx = \frac{1}{a}\sin(ax) + c$	
$\frac{d}{dx}(\tan(ax)) = \frac{a}{\cos^2(ax)} = a\sec^2(ax)$			
product rule	$\frac{d}{dx}(uv) = u\frac{dv}{dx} + v\frac{du}{dx}$	quotient rule	$\frac{d}{dx}\left(\frac{u}{v}\right) = \frac{v\frac{du}{dx} - u\frac{dv}{dx}}{v^2}$
chain rule	$\frac{dy}{dx} = \frac{dy}{du} \times \frac{du}{dx}$		

Formula Sheet

Probability

$\Pr(A) = 1 - \Pr(A')$		$\Pr(A \cup B) = \Pr(A) + \Pr(B) - \Pr(A \cap B)$	
$\Pr(A \mid B) = \dfrac{\Pr(A \cap B)}{\Pr(B)}$			
mean	$\mu = \mathrm{E}(X)$	variance	$\mathrm{var}(X) = \sigma^2 = \mathrm{E}((X-\mu)^2) = \mathrm{E}(X^2) - \mu^2$

Probability distribution		**Mean**	**Variance**
discrete	$\Pr(X = x) = p(x)$	$\mu = \Sigma\, x\, p(x)$	$\sigma^2 = \Sigma\, (x-\mu)^2\, p(x)$
continuous	$\Pr(a < X < b) = \int_a^b f(x)\, dx$	$\mu = \int_{-\infty}^{\infty} x\, f(x)\, dx$	$\sigma^2 = \int_{-\infty}^{\infty} (x-\mu)^2\, f(x)\, dx$

Sample proportions

$\hat{P} = \dfrac{X}{n}$		mean	$\mathrm{E}(\hat{P}) = p$
standard deviation	$\mathrm{sd}(\hat{P}) = \sqrt{\dfrac{p(1-p)}{n}}$	approximate confidence interval	$\left(\hat{p} - z\sqrt{\dfrac{p(1-p)}{n}},\ \hat{p} + z\sqrt{\dfrac{p(1-p)}{n}}\right)$

END OF FORMULA SHEET