Quality Assessment Tasks

NAME: ___________________________

UNITS 3 & 4 Practice Examination

VCE®Mathematical Methods

Written examination 2

Reading time: 15 minutes **Writing time: 2 hours**

QUESTION AND ANSWER BOOK
Structure of book

Section	*Number of questions*	*Number of questions to be answered*	*Number of marks*
A	20	20	20
B	4	4	60
			80

- Students are permitted to bring into the examination room: pens, pencils, highlighters, erasers, sharpeners, rulers, a protractor, set-squares, aids for curve sketching, one bound reference, one approved CAS calculator or CAS software and, if desired, one scientific calculator. Calculator memory DOES NOT need to be cleared.
- Students are NOT permitted to bring into the examination room: blank sheets of paper and/or white out liquid/tape.

Materials supplied
- A question and answer book of 27 pages.
- A double-sided page of formulas.
- An answer sheet for multiple-choice questions.

Instructions
- Write your name in the space provided above on this page.
- Write your name on the multiple-choice answer sheet.
- Unless otherwise indicated the diagrams in this book are **not** drawn to scale.
- All written responses must be in English.

At end of Examination
- Place the answer sheet for multiple-choice questions inside the front cover of this book.

Students are NOT permitted to bring mobile phones and/or any other unauthorised electronic devices into the examination room.

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SECTION A – Multiple-choice questions

Instructions for Section A

Answer **all** questions in pencil on the answer sheet provided for multiple-choice questions.
Choose the response that is **correct** for the question.
A correct answer scores 1; an incorrect answer scores 0.
Marks will **not** be deducted for incorrect answers.
No marks will be given if more than one answer is completed for any question.
Unless otherwise indicated, the diagrams in this book are **not** drawn to scale.

Question 1

A special die numbered 1, 2, 3, 4, 5 and 5 is thrown. The random variable P is the number of 5s in twelve throws of both dice. Pr ($P \geq 3$) to four decimal places is

A. 0.1811

B. 0.3931

C. 0.6069

D. 0.6775

E. 0.8189

Question 2

A discrete random variable X has the following probability distribution:

X	-1	0	1	2	3
$\Pr(X = x)$	$2p$	0.3	$p - q$	$2q$	0.1

What is the largest possible value of the mean of X?

A. 0.1

B. 0.4

C. 0.6

D. 0.9

E. 2.1

Question 3
A continuous random variable has a probability distribution given by

$f(x) = \frac{3\sqrt{4-x}}{16}; 0 \leq x \leq 4$

$f(x) = 0$ elsewhere.

Pr (X < a) = 0.875

The value of a is

A. 1

B. 2

C. $\sqrt[3]{2}$

D. 3

E. $3\sqrt[3]{2}$

Question 4
The mass of Ezy-Crème donuts is normally distributed with mean 60*g* and variance 25*g*. Given that a donut chosen at random is over 55*g*, what is the probability that the mass is 70*g* or less?

A. 0.9729

B. 0.9772

C. 0.8185

D. 0.1314

E. 0.0761

Question 5

An eight-sided (octahedral) die has the numbers 1 to 8 on its faces. The die is thrown 12 times. The random variable P represents the occurrences of a prime number. (Students should note that 1 is **not** a prime number.) The mean and standard deviation of P is

A. $\mu = 6, \sigma = 3$

B. $\mu = 6, \sigma = \sqrt{3}$

C. $\mu = 3, \sigma = 3$

D. $\mu = 3, \sigma = \sqrt{3}$

E. $\mu = 4, \sigma = 3$

Question 6

The sum of all the solutions to $\{x : -\pi \le x \le \frac{\pi}{2}; 4\cos^2 2x = 3\}$ is

A. 0

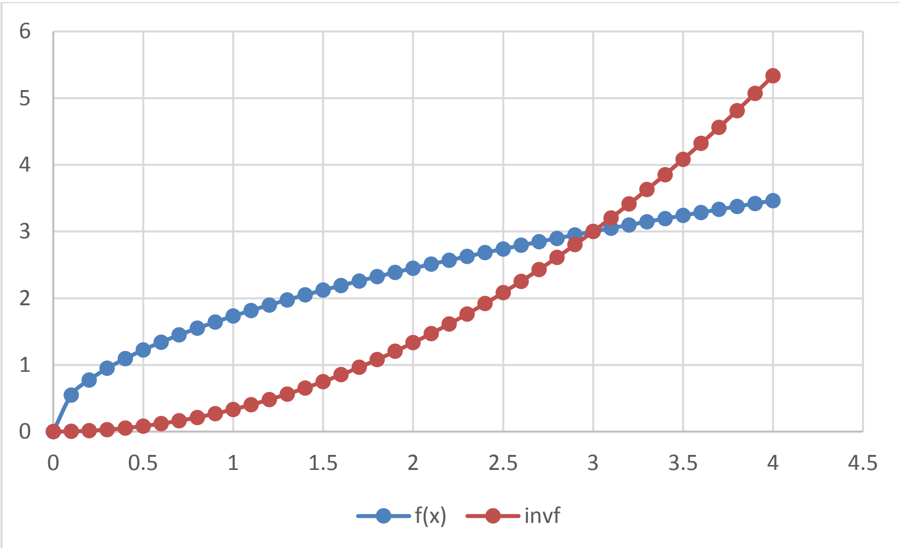
B. $\frac{\pi}{12}$

C. $\frac{-\pi}{2}$

D. $\frac{-11\pi}{12}$

E. $\frac{-3\pi}{2}$

Question 7



$f(x)=\sqrt{3x}$. The area enclosed by $f(x)$ and $f^{-1}(x)$ is

A. 6

B. 2.25

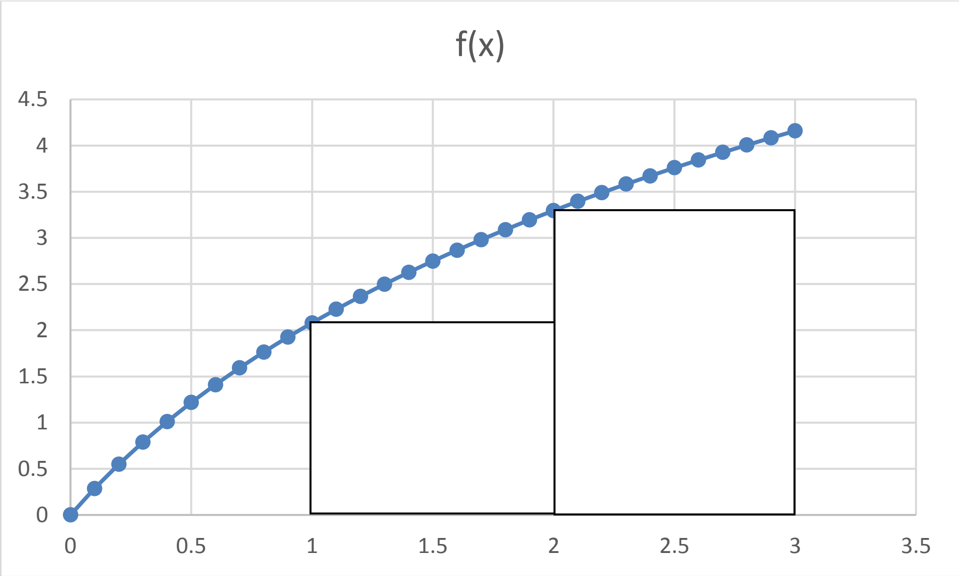
C. 3

D. $6-3\sqrt{3}$

E. $9-3\sqrt{3}$

Question 8

A student uses a rectangle approximation to estimate the area under the graph of f as shown:
$f:[0,3]\to R,\ f(x)=3\log_e(x+1)$. You may use technology, or else $\int \log_e x\,dx = x(\log_e x - 1)+C$).



The ratio of real area to approximation is

A. $4\log_e 4:\log_e 6$

B. $4\log_e 4-3:\log_e 6$

C. $\log_e \frac{256}{6}:1$

D. $4\log_e 4:\log_e 24$

E. $4\log_e 4-3:\log_e 24$

Question 9

The average value of the function $f:R\to R;\ f(x)=ae^{x/a}$ over the domain [0, a] is

A. a

B. $\frac{1}{a}$

C. $e^{a+1/a}$

D. ae

E. $a(e-1)$

Question 10

$f: R \to R;\ f(x) = a\tan\frac{\pi x}{6}$. The chord connecting (1, f(1)) and (2, f(2)) has a length of

A. $\sqrt{3}a$

B. $a(\sqrt{3} - \frac{1}{\sqrt{3}})$

C. $\sqrt{1 + \frac{4a^2}{3}}$

D. $\sqrt{\frac{7a^2}{3}}$

E. $\sqrt{1 + \frac{4a}{3}}$

Question 11

The graph of the function $f: R^+ \to R;\ f(x) = \sqrt{ax + b}$ is subject to the following transformations: (a) translation of -b units along the x axis (b) reflection in the x axis (c) dilation by a factor of a away from the x axis. The resultant function f_T is given by what rule?

A. $f_T = -a\sqrt{ax}$

B. $f_T = -\sqrt{\frac{x}{a}}$

C. $f_T = -a\sqrt{ax + b}$

D. $f_T = a\sqrt{ax - b}$

E. $f_T = -a\sqrt{ax + b(a+1)}$

Question 12

The function $f: R^{+} \to R;\ f(x) = x^2 - 2x$ is subject to the following transformation T:

$T: R^2 \to R^2; \begin{pmatrix} 0 & -2 \\ 1 & 0 \end{pmatrix}\begin{bmatrix} x \\ y \end{bmatrix} + \begin{bmatrix} 1 \\ -1 \end{bmatrix} = \begin{bmatrix} x' \\ y' \end{bmatrix}$ The transformed function is

A. $f_T(x) = \frac{1}{2}\sqrt{2(1-x)}$

B. $f_T(x) = \frac{1}{2}\sqrt{2(3-x)}$

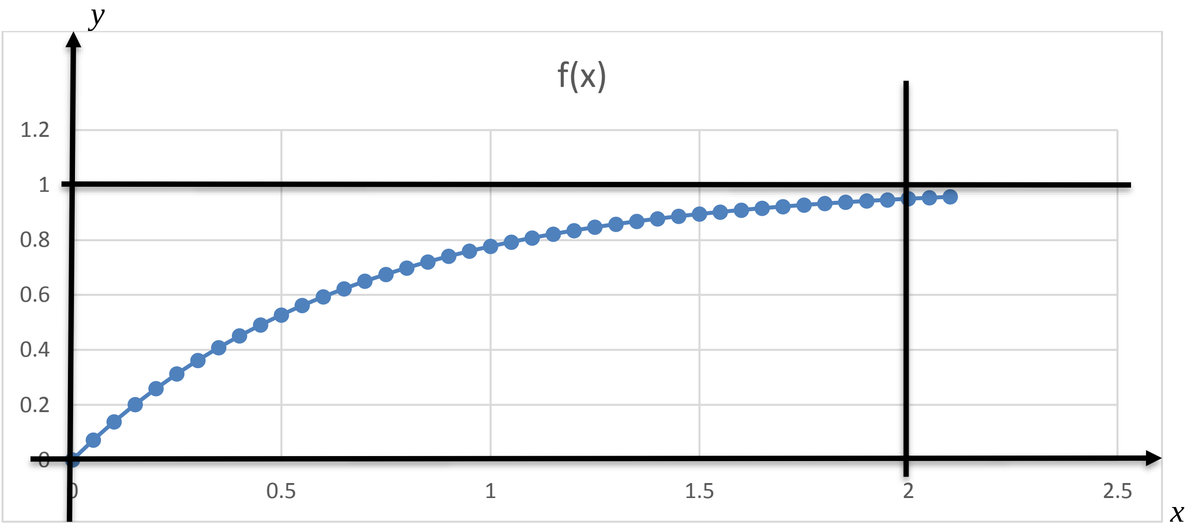
C. $f_T(x) = \pm\frac{1}{2}\sqrt{2(1-x)}$

D. $f_T(x) = \pm\frac{1}{2}\sqrt{2(3-x)}$

E. $f_T(x) = \frac{1}{2}\sqrt{(3-x)}$

Question 13

The area enclosed by $f: [0, \infty) \to R;\ f(x) = 1 - e^{-kx}$, the y axis, and the lines $y = 1$ and $x = 2$ is



A. $\frac{1}{k}(1-e^{-2k})$

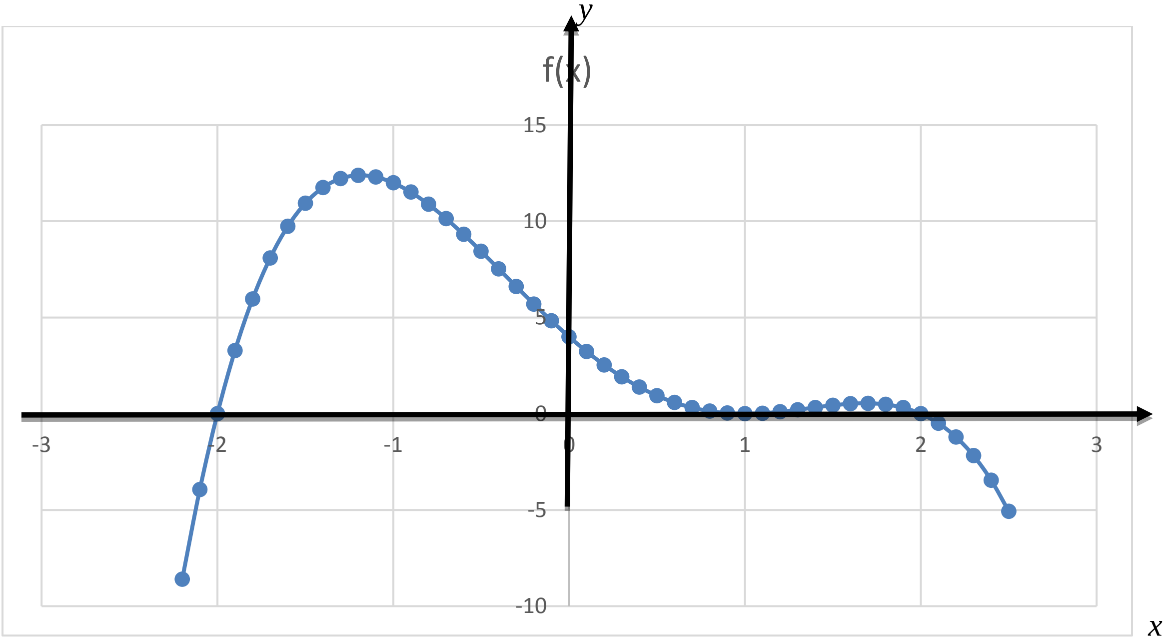
B. $2 - \frac{1}{k}(1-e^{-2k})$

C. $(1-e^{-2k})$

D. $2 + \frac{1}{k}(1-e^{-2k})$

E. $\frac{1}{k}(1+e^{-2k})$

Question 14



The above graph could represent the function f

A. $f(x) = (x-1)(x-2)(x+2)$

B. $f(x) = (x+1)(x+2)(x-2)$

C. $f(x) = (x-1)^2(x-2)(x+2)$

D. $f(x) = (1-x)(x-2)(x+2)$

E. $f(x) = (1-x)^2(2-x)(x+2)$

Question 15

$f:[2,4) \to R: f(x) = \log_3(x-1)^2$. The inverse function f^{-1} is

A. $f^{-1}: R \to R, f^{-1}(x) = \pm\sqrt{e^x + 1}$

B. $f^{-1}: R \to R, f^{-1}(x) = \pm\sqrt{3^x + 1}$

C. $f^{-1}:[0,2] \to R, f^{-1}(x) = 3^{0.5x} + 1$

D. $f^{-1}:[0,2) \to R, f^{-1}(x) = 3^{0.5x} + 1$

E. $f^{-1}:[2,4) \to R, f^{-1}(x) = 3^{0.5x} + 1$

Question 16

Given that $\frac{d}{dx}e^{-ax}\cos bx = e^{-ax}(-a\cos bx - b\sin bx)$, calculate $\int_0^1 e^{-2x}(2\cos \pi x + \pi \sin \pi x)dx$.

A. -2

B. $-e^{-2}-1$

C. $e^{-2}+1$

D. $2\pi e^{-2}$

E. $-2\pi e^{-2}$

Question 17

State the period and range of the function $g: R \to R, g(x) = 3 - 2\sin(\frac{\pi x}{a} + \frac{\pi}{2})$.

A. $\frac{2}{a}$ and [-2, 2]

B. $2a$ and [1, 5]

C. a and [-2, 2]

D. $2a$ and [-2, 2]

E. $\frac{2}{a}+\frac{1}{2}$ and [1, 5]

Question 18

Solve the equation $f(x) = 0$ where $f:[0,\pi] \to R, f(x) = (\sin\frac{x}{2} - \sqrt{3}\cos\frac{x}{2})(\sin\frac{x}{2} + \sqrt{3}\cos\frac{x}{2})$.

A. $x = \frac{2\pi}{3}$

B. $x = \frac{\pi}{3}$

C. $x = \frac{\pi}{6}, \frac{2\pi}{3}$

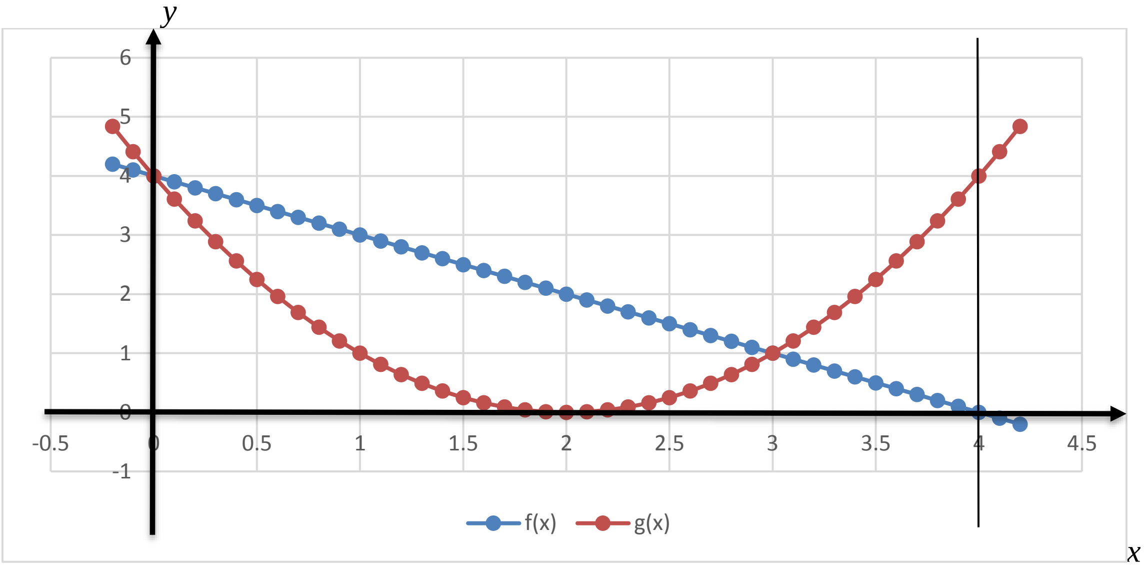
D. $x = \pm\frac{2\pi}{3}$

E. There are no solutions

Question 19

$f:[0,4]\to R,\ f(x)=4-x$

$g:[0,4]\to R,\ g(x)=(x-2)^2$



The two areas A (in the domain [0, 3]) and B (in the domain [3, 4]) are enclosed by the graphs of $f(x)$, $g(x)$ and the line $x = 4$ are in the ratio

A. 3 : 2

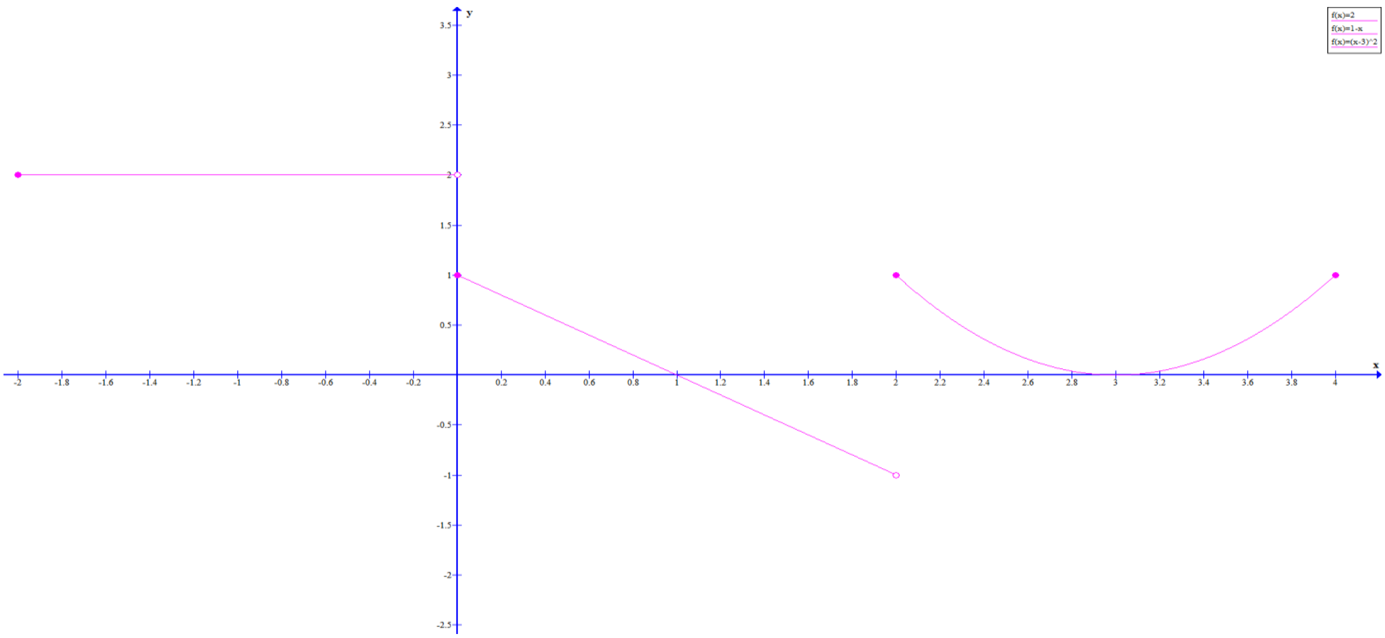
B. 9 : 11

C. 11 : 8

D. 18 : 11

E. 27 : 11

Question 20



The function *f* as shown could be

A. $f : R \to R,\ f(x) = (x-3)^2$

B. $f : [-2,0] \to R,\ f(x) = 2; [0,2] \to R,\ f(x) = 1-x; [2,4] \to R,\ f(x) = (x-3)^2$

C. $f : (-2,0] \to R,\ f(x) = 2; (0,2] \to R,\ f(x) = 1-x; (2,4] \to R,\ f(x) = (x-3)^2$

D. $f : [-2,0) \to R,\ f(x) = 2; [0,2) \to R,\ f(x) = 1-x; [2,4] \to R,\ f(x) = (x-3)^2$

E. *f* is not a function.

SECTION B

Instructions for Section B

Answer **all** questions in the spaces provided.

In all questions where a numerical answer is required, an exact value must be given unless otherwise specified.

In questions where more than one mark is available, appropriate working **must** be shown.

Unless otherwise indicated, the diagrams in this book are **not** drawn to scale.

Question 1 (16 marks)

Two sheep farms are owned by Narelle and Trevor. The thickness of the wool in microns is normally distributed. The continuous variables N and T describe the wool thickness of their respective flocks.

a. i. $\mu_N = 23$. Pr ($N < 21 = 0.1151$) Find σ_N to three decimal places. 1 mark

ii. $\mu_T = m$; $\sigma_T = 1.0$. Pr ($T > 25 = 0.8413$) Find m. 1 mark

b. i. If the maximum width for superfine wool is a, and the probability that any of Narelle's fleeces will qualify as superfine wool is 0.960, find a to three decimal places. 1 mark

ii. What is the probability that a fleece from Trevor's farm will qualify as superfine to three decimal places? 1 mark

iii. Narelle has 500 sheep and Trevor 300. Find the total probability that any fleece chosen at random is superfine. 2 marks

iv. Given that a fleece chosen at random is superfine, what is the probability it came from Narelle's farm? 1 mark

c. **i.** A sample of five fleeces is chosen from Trevor's farm. Find the probability that at least four of the fleeces will qualify as superfine. (Answer to three decimal places.) 1 mark

ii. The random variable $\hat{P}$ is the proportion of superfine fleeces from Trevor's farm. Find the 95% confidence limits of $\hat{P}$ to two decimal places. 2 marks

d. The probability density function which describes the market price for superfine wool per kilogram is:

$f(x) = Ax^2; 0 < x \le 0.5$

$f(x) = Be^{0.5-x}; 0.5 < x \le l$

$f(x) = 0$ otherwise. Note that x represents units of \$1000.

i. The median price is $x = 0.4500$. Calculate the constant A. 1 mark

ii. The function is continuous at $x = 0.5$. Calculate the constant B. 1 mark

iii. Calculate the constant l to four decimal places. 2 marks

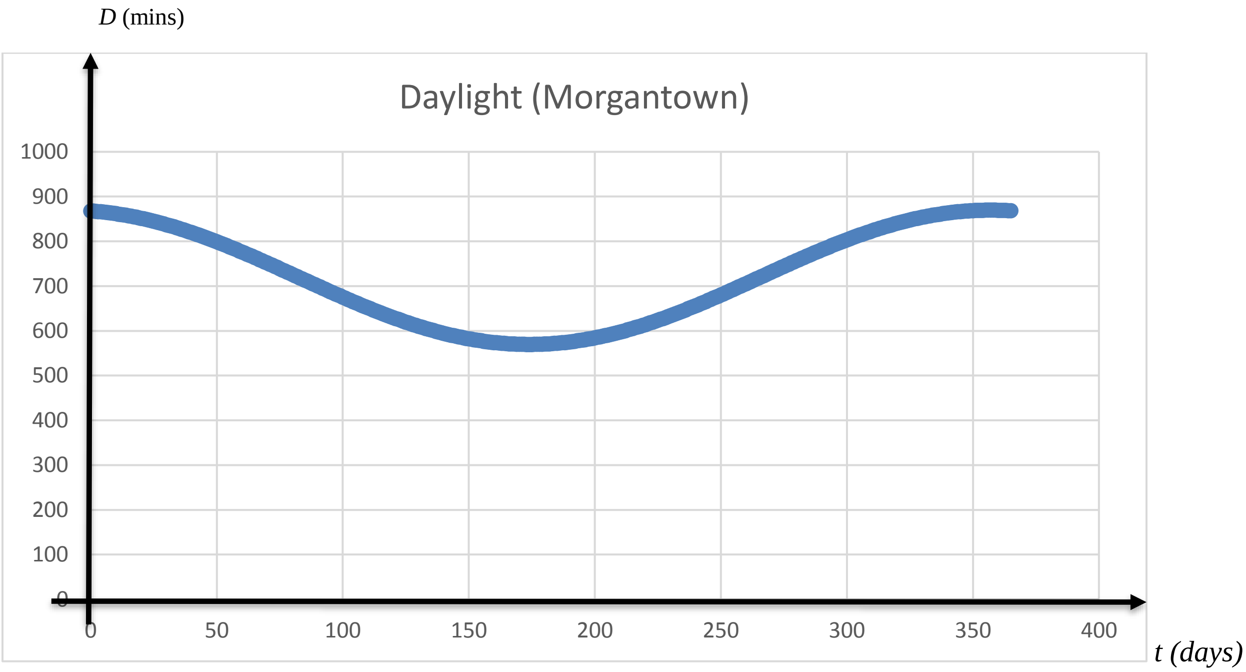
e. The time T needed to shear a sheep is normally distributed with mean 5.2 minutes and variance 2.25 min^2. If a sheep needs more than 5 minutes to shear, find the probability (3 decimal places) that it takes less than 6 minutes. 2 marks

Question 2 (11 marks)

The times of daylight during the year in Morgantown can be modelled by the function

$$D(t) = 150\cos\left(\frac{\pi}{183}(t+9)\right) + 720$$

where t is the number of days after Jan 1st, 2020 and D is measured in minutes. Round your answers **up** to the next whole number when it comes out as a fraction or decimal.



a. i. Find the coordinates of the equinoxes (i.e. the points where the length of daytime and night-time are equal). 2 marks

ii. Find the coordinates of the solstices (i.e. the points where the hours of daylight are at their maximum and minimum). 2 marks

b. **i.** Give the coordinates of the point where *D(t)* is **decreasing** at its maximum rate.

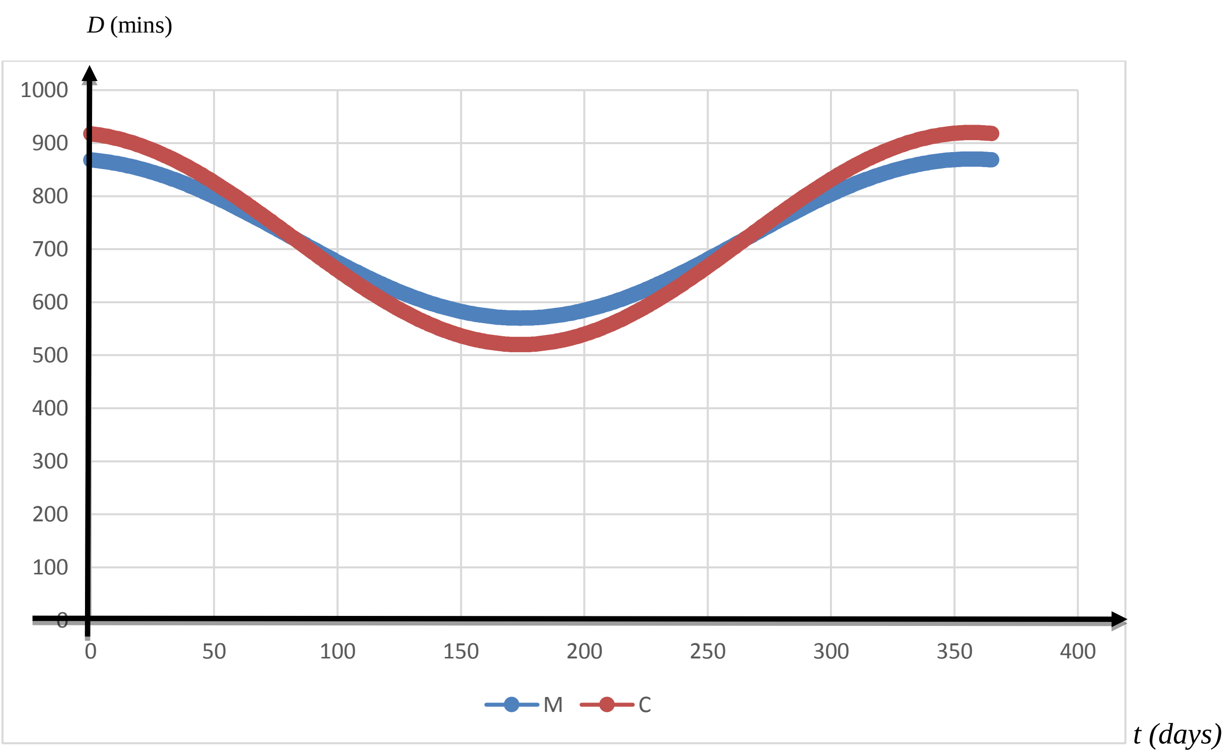
2 marks

ii. Give the coordinates of the point where *D(t)* is **increasing** at its maximum rate.

1 mark

c. The times of daylight during the year in Castleford can be modelled by the function

$D(t) = 200\cos(\frac{\pi}{183}(t+9)) + 720$:

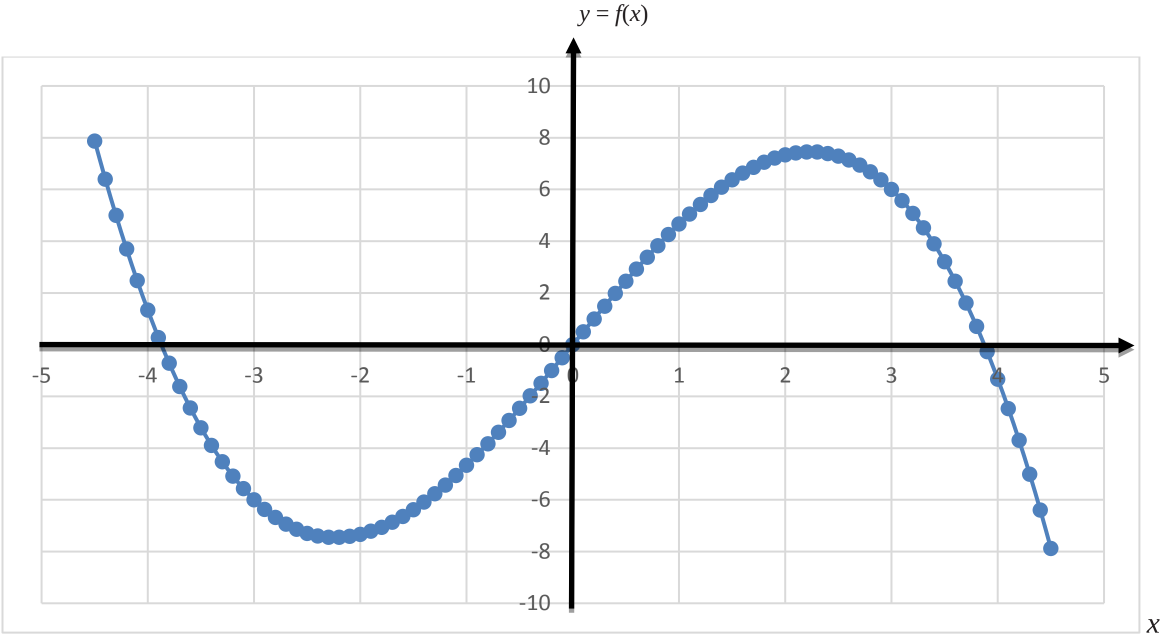


Find all the values of t for which Castleford has more daylight than Morgantown. 2 marks

d. Which of Castleford and Morgantown could be expected to experience more daylight over the whole year? Justify your answer. 2 marks

Question 3 (18 marks)

$$f: R \to R,\ f(x) = 5x - \frac{x^3}{3}$$



a. Find the axis intercepts of $f(x)$. 3 marks

b. Find the coordinates of the turning points of $f(x)$. 3 marks

c. Consider the function $g: R \to R; g(x) = 5x - \frac{x^3}{3} + a$

i. Find the values of a for which $g(x)$ has exactly two intercepts with the x axis. 2 marks

ii. Find all values of a for which $g(x)$ has only one intercept with the x axis. 1 mark

d. Find the area enclosed by $f(x)$ and the x axis. 2 marks

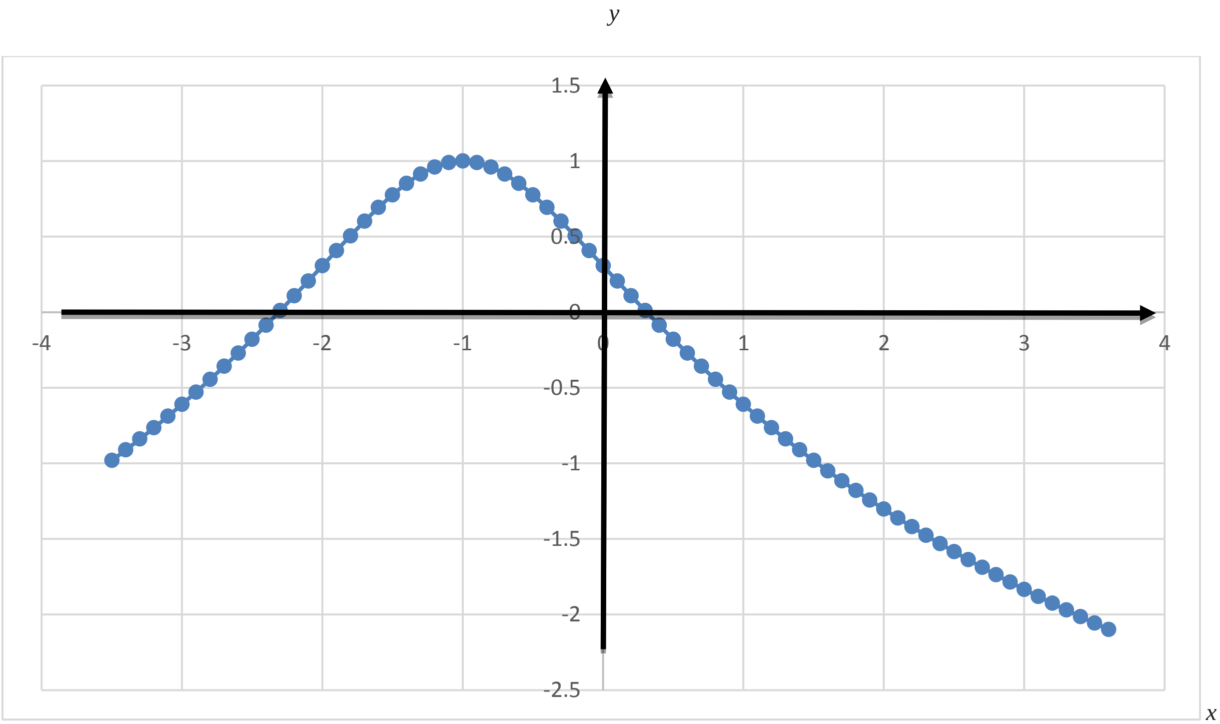
e. i. Find the equation of the tangent to $f(x)$ at $x = 3$. 2 marks

ii. At what point does the normal at $x = 3$ cut the y axis? 2 marks

iii. Find the area of the quadrilateral bounded by the tangent, the normal and the axes. 3 marks

Question 4 (15 marks)

$f: R \to R,\ f(x) = 1 - \log_e(x^2 + 2x + 2)$



a. Find the axis intercepts of $f(x)$. 3 marks

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b. Find the coordinates of the turning point of $f(x)$. 2 marks

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c. Find the inverse function f^{-1} and state its maximal domain. 3 marks

d. Consider the functions *g* and *h* and the transformation matrix *T* where T maps *g*(*x*) onto *h*(*x*):

$g: R^+ \to R; g(x) = \log_e x$; $h: R^+ \to R; h(x) = \log_k x$

$T: R^2 \to R^2; \begin{pmatrix} a & 0 \\ 0 & b \end{pmatrix} \begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} x' \\ y' \end{bmatrix}$

i. Find the constants *a* and *b*. 2 marks

ii. Describe the transformation *T*. 1 mark

e. i. Consider the function $f_k: R^+ \to R, f_k(x) = k - \log_k x$.

Find the equation of the tangent to this curve at $x = k$. 2 marks

ii. Find the inverse function $f_k^{-1}(x)$ and state its domain. 2 marks

END OF QUESTION AND ANSWER BOOK

Mathematical Methods Formulas

Mathematical Methods Formulas

Mensuration:

Area of a triangle: ½ bc sin A

Differential Calculus:

$$\frac{d}{dx}x^n = nx^{n-1}$$

$$\frac{d}{dx}(ax+b)^n = an(ax+b)^{n-1}$$

$$\frac{d}{dx}e^{ax} = ae^{ax}$$

$$\frac{d}{dx}\sin ax = a\cos ax$$

$$\frac{d}{dx}\cos ax = -a\sin ax$$

$$\frac{d}{dx}\tan ax = \frac{a}{\cos^2 ax}$$

$$\frac{d}{dx}(uv) = u\frac{dv}{dx} + v\frac{du}{dx}$$

$$\frac{d}{dx}\left(\frac{u}{v}\right) = \frac{v\frac{du}{dx} - u\frac{dv}{dx}}{v^2}$$

Integral Calculus:

$$\int x^n dx = \frac{x^{n+1}}{n+1} + C; n \neq -1$$

$$\int (ax+b)^n dx = \frac{(ax+b)^{n+1}}{a(n+1)} + C; n \neq -1$$

$$\int \frac{1}{x} dx = \log_e x + C$$

$$\int e^{ax} dx = \frac{e^{ax}}{a} + C$$

$$\int \sin ax dx = \frac{-1}{a}\cos ax + C$$

Multiple Choice Answer Sheet

Student Name:

Shade the letter that corresponds to each correct answer.

Question	A	B	C	D	E
Question 1	[ ]	[ ]	[ ]	[ ]	[ ]
Question 2	[ ]	[ ]	[ ]	[ ]	[ ]
Question 3	[ ]	[ ]	[ ]	[ ]	[ ]
Question 4	[ ]	[ ]	[ ]	[ ]	[ ]
Question 5	[ ]	[ ]	[ ]	[ ]	[ ]
Question 6	[ ]	[ ]	[ ]	[ ]	[ ]
Question 7	[ ]	[ ]	[ ]	[ ]	[ ]
Question 8	[ ]	[ ]	[ ]	[ ]	[ ]
Question 9	[ ]	[ ]	[ ]	[ ]	[ ]
Question 10	[ ]	[ ]	[ ]	[ ]	[ ]
Question 11	[ ]	[ ]	[ ]	[ ]	[ ]
Question 12	[ ]	[ ]	[ ]	[ ]	[ ]
Question 13	[ ]	[ ]	[ ]	[ ]	[ ]
Question 14	[ ]	[ ]	[ ]	[ ]	[ ]
Question 15	[ ]	[ ]	[ ]	[ ]	[ ]
Question 16	[ ]	[ ]	[ ]	[ ]	[ ]
Question 17	[ ]	[ ]	[ ]	[ ]	[ ]
Question 18	[ ]	[ ]	[ ]	[ ]	[ ]
Question 19	[ ]	[ ]	[ ]	[ ]	[ ]
Question 20	[ ]	[ ]	[ ]	[ ]	[ ]

QATs
Quality Assessment Tasks

Solution Pathway

NOTE: This task is sold on condition that it is NOT placed on any school network or social media site (such as Facebook, Wikispaces, etc.) at any time.

NOT FOR PRIVATE TUTOR USE.

Below are sample answers. Please consider the merit of alternative responses.

Section A: Multiple-choice answers

1.	**E**	**6.**	**E**	**11.**	**E**	**16.**	**C**
2.	**C**	**7.**	**C**	**12.**	**B**	**17.**	**B**
3.	**D**	**8.**	**B**	**13.**	**A**	**18.**	**A**
4.	**A**	**9.**	**E**	**14.**	**E**	**19.**	**E**
5.	**B**	**10.**	**C**	**15.**	**D**	**20.**	**D**

Section A: Multiple-choice solutions

1. $P \sim \text{Bi}(12, \tfrac{1}{3})$. $\Pr(P \geq 3) = 1 - \Pr(P \leq 2) = 0.8189$

2. From the sum of probabilities, $3p + q = 0.6$, hence $q = 0.6 - 3p$ and $p \leq 0.2$

The mean is $3q - p + 0.3 = 3(0.6 - 3p) - p + 0.3 = 2.1 - 10p$.

However, $p \geq q$ from $\Pr(X = 1)$, so the minimum value of p equals q, in which case:

$p = 0.6 - 3p$, therefore $4p = 0.6$ and $p = 0.15$

Hence p lies between [0.15, 0.2].

If $p = 0.15$, $\mu = 0.6$. If $p = 0.2$, $\mu = 0.1$. Hence max value of $\mu = 0.6$.

3. $\Pr(X < a) = 0.875$ $\int_0^a \frac{3}{16}\sqrt{4-x}\,dx = -\frac{1}{8}[(4-x)^{3/2}]_0^a = 1 - \frac{1}{8}[(4-a)^{3/2}] = 0.875$

$(4-a)^{3/2} = 1 \therefore a = 3$

4. $\sigma^2 = 25 \Rightarrow \sigma = 5g$; $Y \sim N(60, 5)$, so $Pr\left(Y \leq 70 \mid Y > 55\right) = Pr\left(Z \leq 2 \mid Z > -1\right)$

$= \frac{\Pr(-1 < Z \leq 2)}{\Pr(Z > -1)} = \frac{0.9772 - 0.1587}{0.8413} = 0.9729$

5. Primes from 1 to 8 are 2, 3, 5 and 7, hence $p = 0.5$. $P \sim \text{Bi}(12, \tfrac{1}{2})$.

$\mu_P = np = 6$; $\sigma_P = \sqrt{np(1-p)} = \sqrt{3}$

6. Since the cosine graph is symmetrical about the *y* axis, most of the solutions will cancel each other out. They are: $x = \pm\frac{\pi}{12}; \pm\frac{5\pi}{12}; \pm\frac{7\pi}{12}; \pm\frac{11\pi}{12}$ etc.

But $\frac{7\pi}{12}$ and $\frac{11\pi}{12}$ lie outside the domain. Hence the solution sum is $\frac{-7\pi - 11\pi}{12} = \frac{-3\pi}{2}$.

7. $f(x) = \sqrt{3x}$. $f^{-1}(x) = \frac{x^2}{3}$; $f(x) = f^{-1}(x)$ at (0, 0) and (3, 3).

$$A = \int_0^3 \sqrt{3x} - \frac{x^2}{3}\,dx = [\frac{2\sqrt{3}}{3}x^{3/2} - \frac{x^3}{9}]_0^3 = 3$$

8. Actual area:

$$\int_0^3 3\log_e(x+1)dx = 3\int_1^4 \log_e u\,du = 3[u(\log_e u - 1]_1^4 = 12\log_e 4 - 9$$

Estimated area: Two rectangles of width 1 unit and height $f(1)$ and $f(2)$:

$$A \approx 3\log_e 2 + 3\log_e 3 = 3\log_e 6$$

9. The average value of the function $f: R \to R; f(x) = ae^{x/a}$ over the domain [0, *a*] is:

$$\frac{1}{a-0}\int_0^a ae^{x/a}dx = \int_0^a e^{x/a}dx = [ae^{x/a}]_0^a = a(e-1)$$

10. $f: R \to R; f(x) = a\tan\frac{\pi x}{6}$.

The chord connecting the points on the curve at *x* = 1 and *x* = 2 has a length of:

$$f(1) = a\tan\frac{\pi}{6} = \frac{a}{\sqrt{3}};\ f(2) = a\tan\frac{\pi}{3} = \sqrt{3}a$$

Rise = 2 – 1 = 1; Run = $a(\sqrt{3} - \frac{1}{\sqrt{3}}) = \frac{2a}{\sqrt{3}}$

$$L = \sqrt{1 + \frac{4a^2}{3}}$$

11. $f: R^+ \to R; f(x) = \sqrt{ax+b}$ is subject to the following transformations: (a) translation of -*b* parallel to the *y* axis (b) reflection in the *x* axis (c) dilation by a factor of *a* away from the *x* axis. The resultant function f_T is given by what rule?

$$f(x) = \sqrt{ax+b}$$

$$f_1(x) = \sqrt{a(x+b)+b} = \sqrt{ax + b(a+1)}$$

$$f_2(x) = -\sqrt{ax + b(a+1)}$$

$$f_T(x) = -a\sqrt{ax + b(a+1)}$$

12. The function $f: R^+ \to R;\ f(x) = x^2 - 2x$ is subject to the following transformation T:

$$T: R^2 \to R^2; \begin{pmatrix} 0 & -2 \\ 1 & 0 \end{pmatrix}\begin{bmatrix} x \\ y \end{bmatrix} + \begin{bmatrix} 1 \\ -1 \end{bmatrix} = \begin{bmatrix} x' \\ y' \end{bmatrix}$$

$$TX + C = X'$$

$$X = T^{-1}(X' - [^{1}_{-1}])$$

$$\begin{bmatrix} x \\ y \end{bmatrix} = \begin{pmatrix} 0 & 1 \\ -0.5 & 0 \end{pmatrix}\begin{bmatrix} x'-1 \\ y'+1 \end{bmatrix} = \begin{bmatrix} y'+1 \\ 0.5(1-x') \end{bmatrix}$$

$$\therefore 0.5(1-x') = (y'+1)^2 - 2(y'+1)$$

$$\therefore 0.5(1-x') = y'^2 - 1$$

$$\therefore y'^2 = 0.5(3-x')$$

$$\therefore y' = \pm\frac{1}{2}\sqrt{2(3-x')}$$

Positive root because of the domain restriction, therefore $f_T(x) = \frac{1}{2}\sqrt{2(3-x)}$

13. Area under the graph = $\int_0^2 (1-e^{-kx})dx = [x + \frac{1}{k}e^{-kx}]_0^2 = 2 + \frac{1}{k}e^{-2k} - \frac{1}{k} = 2 - \frac{1}{k}(1-e^{-2k})$

Area of rectangle = 2, so area enclosed = 2 – (2 – $\frac{1}{k}(1-e^{-2k})$)= $\frac{1}{k}(1-e^{-2k})$

14. The graph is an inverted quartic with a repeated root at $x = 1$ and the other two roots at -2 and 2. Hence $f(x) = -A(x-1)^2(x+2)(x-2)$. Students should check $f(0) = 4$ hence A = 1.

15. $f: [2,4) \to R: f(x) = \log_3(x-1)^2 = 2\log_3(x-1)$

$$x = 2\log_3(y-1)$$

$$\therefore f^{-1}(x) = 3^{0.5x} + 1$$

$$f(2) = 0;\ f(4) = 2$$

Range of f = domain of f^{-1} = [0, 2)

16. $\int_0^1 e^{-2x}(2\cos\pi x + \pi\sin\pi x)dx = \int_1^0 e^{-2x}(-2\cos\pi x - \pi\sin\pi x)dx = [e^{-2x}\cos\pi x]_1^0 = 1 + e^{-2}$

17. Period = $\dfrac{2\pi}{\pi/a} = 2a$; range = 3 ± 2 = [1,5]

18. $(\sin\frac{x}{2} - \sqrt{3}\cos\frac{x}{2})(\sin\frac{x}{2} + \sqrt{3}\cos\frac{x}{2}) = 0$

$\therefore \sin^2\frac{x}{2} = 3\cos^2\frac{x}{2}$

$\therefore \tan^2\frac{x}{2} = 3$

$\therefore \tan\frac{x}{2} = \pm\sqrt{3}$

$\therefore \frac{x}{2} = \pm\frac{\pi}{3}, \pm\frac{2\pi}{3}...$

$\therefore x = \pm\frac{2\pi}{3}, \pm\frac{4\pi}{3}...$

The only solution which lies inside the domain [0, π] is $x = \frac{2\pi}{3}$

19. $f:[0,4] \to R, f(x) = 4 - x;\ g:[0,4] \to R, g(x) = (x-2)^2$

A: $\int_0^3 4 - x - (x-2)^2 dx = \frac{9}{2}$; **B:** $\int_3^4 [(x-2)^2 + x - 4]dx = \frac{11}{6}$

A : B = 27 : 11

20. $f:[-2,0) \to R, f(x) = 2; [0,2) \to R, f(x) = 1 - x; [2,4] \to R, f(x) = (x-3)^2$

Section B: Extended Answer Solutions

Question 1 (16 marks)

Two sheep farms are owned by Narelle and Trevor. The thickness of the wool in microns is normally distributed. The continuous variables *N* and *T* describe the wool thickness of their respective flocks.

a. i. $\mu_N = 23$. Pr ($N < 21$) = 0.1151. Find σ_N to three decimal places. 1 mark

Pr ($N < 21$) = 0.1151 therefore Pr ($N < 25$) = 0.8849

$Z = 1.200$

$$\frac{N - \mu_N}{\sigma_N} = 1.2 \therefore \sigma_N = \frac{25 - 23}{1.2} = 1.667$$

ii. $\mu_T = m$; $\sigma_T = 1.0$. Pr ($T > 25 = 0.8413$) Find *m*: 1 mark

Pr ($T > 25 = 0.8413$). Invnorm 0.8413 gives $Z = 1.0000$. Therefore $m = 26$.

b. i. If the maximum width for superfine wool is *a*, and the probability that any of Narelle's fleeces will qualify for superfine wool is 0.960, find *a* to three decimal places. 1 mark

Invnorm 0.960 = 1.750: $\frac{a - 23}{1.667} = 1.750 \therefore a = 25.917$

ii. $Pr(T \le 25.917) = \Pr(Z \le 0.083) = 0.533$ 1 mark

iii. Make a probability table:

Pr	***S***	***S'***	***Total***
T	⅜ * 0.533		0.375
N	⅝ * 0.960		0.625
Total			**1**

Filling in the remaining numbers gives:

Pr	***S***	***S'***	***Total***
T	0.200	0.175	0.375
N	0.600	0.025	0.625
Total	0.800	0.200	**1**

Pr(S) = 0.800 2 marks

iv. $\Pr(N \mid S) = \dfrac{\Pr(N \cap S)}{\Pr(S)} = \dfrac{0.600}{0.800} = 0.750$ 1 mark

c. **i.** S ~ Bi (5, 0.533)

$\Pr(S_T \geq 4) = 5 \times 0.533^4 (1 - 0.533) + 0.533^5 = 0.231$ 1 mark

ii. The random variable $\hat{P}$ is the proportion of superfine fleeces from Trevor's farm. Find the 95% confidence limits of $\hat{P}$ to two decimal places. 2 marks

p = 0.533, n = 300

$$\sigma = \sqrt{\frac{p(1-p)}{n}} = \sqrt{\frac{0.533 \times 0.0.467}{300}} = 0.03$$

Confidence limits are (0.47, 0.59)

d. The probability density function which describes the price at market for superfine wool per kilogram is given by:

$f(x) = Ax^2; 0 < x \leq 0.5$
$f(x) = Be^{0.5-x}; 0.5 < x \leq l$
$f(x)$ = 0 otherwise. x represents units of $1000.

i. The median price is x = 0.4500. Calculate the constant A. 1 mark

$$\int_0^{0.5} Ax^2 dx = 0.45 \therefore \frac{A(0.5)^3}{3} = 0.5 \therefore A = 16.4609$$

ii. The function is continuous at x = 0.5. Calculate the constant B. 1 mark

The two functions must have equal values at x = 0.5.

$A(0.5)^2 = Be^0 \therefore B = 4.1152$

iii. Calculate the constant l to four decimal places. 2 marks

The total probability must equal 1:

$$\int_0^{0.5} Ax^2 dx + \int_{0.5}^{l} Be^{0.5-x} dx = 1$$

$$\therefore 0.45 + 4.1152\int_{0.5}^{l} e^{0.5-x} dx = 1$$

$$\therefore [-e^{0.5-x}]_{0.5}^{l} = \frac{0.55}{4.1152} = 0.13365$$

$$\therefore 1 - e^{0.5-l} = 0.13365$$

$$\therefore 0.5 - l = \log_e 0.86635 = -0.1435$$

$$\therefore l = 0.6435$$

e. The time T needed to shear a sheep is normally distributed with mean 5.2 minutes and variance 2.25 min². If a sheep needs more than 5 minutes to shear, find the probability (3 decimal places) that it takes less than 6 minutes. 2 marks

$T \sim N(5.2, 2.25)$ and $\sigma = \sqrt{2.25} = 1.5$

$$\Pr(T \le 6 \mid T > 5) = \frac{\Pr(5 < T \le 6)}{\Pr(T > 5)} = \frac{\Pr(-0.1333 < Z \le 0.5333)}{\Pr(Z > -0.1333)} = 0.463$$

Question 2 (11 marks)

The times of daylight during the year in Morgantown can be modelled by the function

$$D(t) = 150\cos\left(\frac{\pi}{183}(t+9)\right) + 720$$

where t is the number of days after Jan 1st, 2020 and D is measured in minutes. Throughout question 2, round your answers **up** to the next whole number when it comes out as a fraction or decimal.

a. i. Find the coordinates of the equinoxes (i.e. the points where the length of daytime and night-time are equal). 2 marks

At the equinoxes, $D(t)$ = 720 (or 12 hours):

$$\cos\left(\frac{\pi}{183}(t+9)\right) = 0$$

$$\therefore \frac{\pi}{183}(t+9) = \frac{\pi}{2}; \frac{3\pi}{2}$$

$$\therefore t + 9 = 91.5, 274.5$$

$$\therefore t = 83, 266$$

(83, 720) and (266, 720)

ii. Find the coordinates of the solstices (i.e. the points where the hours of daylight are at their maximum and minimum). 2 marks

At the solstices, *D(t)* = maximum and minimum, hence:

$$\cos(\frac{\pi}{183}(t+9)) = \pm 1 \therefore \frac{\pi}{183}(t+9) = \pi; 2\pi$$
$$\therefore t+9 = 183, 366$$
$$\therefore t = 174, 357$$

$$D(174) = -150 + 720 = 570$$
$$D(357) = 150 + 720 = 870$$

Coordinates are (174, 570) and (357, 870)

b. i. Give the coordinates of the point where *D(t)* is **decreasing** at its maximum rate. 2 marks

$D'(t) = -\frac{150\pi}{183}\sin(\frac{\pi}{183}(t+9))$ This function reaches its minimum value when:

$$-\sin(\frac{\pi}{183}(t+9)) = -1$$
$$\therefore \frac{\pi}{183}(t+9) = \frac{\pi}{2}$$
$$\therefore t+9 = 91.5$$
$$t = 83$$

Coordinates are (83, 720)

Students may use technology, or else take the second derivative and solve it for $D''(t) = 0$

ii. Give the coordinates of the point where *D(t)* is **increasing** at its maximum rate. 1 mark

$$-\sin(\frac{\pi}{183}(t+9)) = 1$$
$$\therefore \frac{\pi}{183}(t+9) = \frac{3\pi}{2}$$
$$\therefore t+9 = 274.5$$
$$t = 266$$

Coordinates: (266, 720)

c. The times of daylight during the year in Castleford can be modelled by the function

$D(t) = 200\cos(\frac{\pi}{183}(t+9)) + 720$ where t = days after Jan 1st, 2020 and D is measured in minutes.

Find all the values of t for which Castleford has more daylight than Morgantown. 2 marks

Students may use technology, but logic alone should indicate that the values of t required are the six months between the spring and autumn equinoxes: $\{t : t \leq 83\} \cup \{t : t \geq 266\}$

d. Which of Castleford and Morgantown could be expected to experience more daylight over the whole year? Justify your answer. 2 marks

Again, logic would indicate that everywhere on earth receives the same average amount of daylight (720 minutes). However, to gain the second mark students should either mention that sine and cosine curves have equal areas above and below the median line, or else give an equation of the general form

$\int_0^{2\pi} \cos t\,dt = 0$ or equivalent.

Question 3 (17 marks)

$$f : R \to R,\ f(x) = 5x - \frac{x^3}{3}$$

a. Find the axis intercepts of *f*(*x*). 3 marks

(0,0); ($-\sqrt{15}$, 0) and ($\sqrt{15}$, 0) (1 mark each)

b. Find the coordinates of the turning points of *f*(*x*). 3 marks

$$f'(x) = 5 - x^2$$
$$f'(x) = 0 \therefore x = \pm\sqrt{5}$$

(1 mark)

$$f(\pm\sqrt{5}) = \pm\frac{10\sqrt{5}}{3}$$

(1 mark)

($-\sqrt{5}$, $\frac{-10\sqrt{5}}{3}$) and ($\sqrt{5}$, $\frac{10\sqrt{5}}{3}$) (1 mark)

c. Consider the function $g : R \to R; g(x) = 5x - \frac{x^3}{3} + a$

i. Find the values of *a* for which *g*(*x*) has exactly two intercepts with the *x* axis. 2 marks

$$a = \frac{\pm 10\sqrt{5}}{3}$$

ii. Find all values of *a* for which *g*(*x*) has only one intercept with the *x* axis. 1 mark

$$\{a : a < \frac{-10\sqrt{5}}{3}\} \cup \{a : a > \frac{10\sqrt{5}}{3}\}$$

d. Find the area enclosed by $f(x)$ and the x axis. 2 marks

$$A = \int_0^{\sqrt{15}} f(x)dx - \int_{-\sqrt{15}}^{0} f(x)dx$$

$$= 2\int_0^{\sqrt{15}} 5x - \frac{x^3}{3}\, dx$$

$$= [5x^2 - \frac{x^4}{6}]_0^{\sqrt{15}}$$

$$= 37.5$$

e. i. Find the equation of the tangent to $f(x)$ at $x = 3$: 2 marks

$f(3) = 6$; gradient = $f'(3) = -4$

$$y - 6 = -4(x-3)$$

$$\therefore y = -4x + 18$$

ii. At what point does the normal at $x = 3$ cut the y axis? 2 marks

Normal has equation $y = \frac{x+21}{4}$ and cuts the y axis at (0, 5.25)

iii. Find the area of the quadrilateral bounded by the tangent, the normal and the axes: 3 marks

Area of trapezium from $x = 0$ to 3: $\frac{1}{2} \times 3\{y(0) + y(3)\} = \frac{3}{2}(\frac{21}{4} + 6) = \frac{135}{8}$ (1 mark)

Area of triangle from $x = 3$ to 4.5: $\frac{1}{2} \times 6 \times \frac{3}{2} = \frac{9}{2}$. (1 mark)

Total area = $\frac{171}{8}$ (1 mark)

Question 4 (15 marks)

$f: R \rightarrow R,\ f(x) = 1 - \log_e(x^2 + 2x + 2)$

a. Find the axis intercepts of $f(x)$. 3 marks

$f(0) = 1 - \log_e 2$ (1 mark)

$f(x) = 1 - \log_e(x^2 + 2x + 2) = 0$

$\therefore x^2 + 2x + 2 = e$ (1 mark)

$\therefore (x+1)^2 = e - 1$

$\therefore x = \pm\sqrt{e-1} - 1$ (1 mark)

b. Find the coordinates of the turning point of $f(x)$. 2 marks

$f'(x) = -\dfrac{2x+2}{x^2+2x+2} = 0$ at $x = -1$ (1 mark)

$f(-1) = 1 - \log_e(1 - 2 + 2) = 1$ hence the turning point is at (-1,1) (1 mark)

c. Find the inverse function f^{-1} and state its maximal domain. 3 marks

$x = 1 - \log_e(y^2 + 2y + 2)$

$\therefore \log_e((y+1)^2 + 1) = 1 - x$

$\therefore (y+1)^2 + 1 = e^{1-x}$ (1 mark)

$\therefore y = \pm\sqrt{e^{1-x} - 1} - 1$

$f^{-1}(1 - \log_e 2) = 0$ so we need the positive root. For the maximal domain we need $e^{1-x} \geq 1 \therefore x \leq 1$

(1 mark)

$f^{-1}: (-\infty, 1] \rightarrow R,\ f^{-1}(x) = \sqrt{e^{1-x} - 1} - 1$ (1 mark)

d. Consider the functions *g* and *h* and the transformation matrix *T* where T maps *g*(*x*) onto *h*(*x*).

$g: R^+ \to R; g(x) = \log_e x$; $h: R^+ \to R; h(x) = \log_k x$

$T: R^2 \to R^2; \begin{pmatrix} a & 0 \\ 0 & b \end{pmatrix} \begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} x' \\ y' \end{bmatrix}$

i Find the constants *a* and *b*: 2 marks

$\log_k x = \dfrac{\log_e x}{\log_e k}$ so $a = 1$ and $b = \dfrac{1}{\log_e k}$ (1 mark each)

ii Describe the transformation *T*: 1 mark

Dilation away from the *x* axis by a factor of $\dfrac{1}{\log_e k}$

e. i. Consider the function $f_k : R^+ \to R,\ f_k(x) = k - \log_k x$.

Find the equation of the tangent to this curve at $x = k$: 2 marks

$f_k'(x) = \dfrac{-1}{x \log_e k} \therefore f_k'(k) = \dfrac{-1}{k \log_e k}$ (1 mark)

$f_k(k) = k - 1$

Equation of tangent:

$$y - k + 1 = \frac{-1}{k \log_e k}(x - k)$$
$$\therefore y = \frac{-1}{k \log_e k} x + k + \frac{1}{\log_e k} - 1$$

(1 mark)

ii. Find the inverse function $f_k^{-1}(x)$ and state its domain: 2 marks

$x = k - \log_k y$

$\therefore \log_k y = k - x$ (1 mark)

$\therefore y = k^{k-x}$

Domain = R (1 mark)

