The Mathematical Association of Victoria
Trial Examination 2019

MATHEMATICAL METHODS

Written Examination 2

STUDENT NAME ____

Reading time: 15 minutes
Writing time: 2 hours

QUESTION AND ANSWER BOOK

Structure of examination

Section	*Number of questions*	*Number of questions to be answered*	*Number of marks*
A	20	20	20
B	4	4	60
			Total 80

- Students are permitted to bring into the examination room: pens, pencils, highlighters, erasers, sharpeners, rulers, a protractor, set-squares, aids for curve sketching, one bound reference, one approved technology (calculator or software) and, if desired, one scientific calculator. Calculator memory DOES NOT need to be cleared. For approved computer-based CAS, full functionality may be used.
- Students are NOT permitted to bring into the examination room: blank sheets of paper and/or correction fluid/tape.

Materials supplied
- Question and answer book of 21 pages.
- Formula sheet.
- Answer sheet for multiple-choice questions.

Instructions
- Write your **name** in the space provided above on this page.
- Write your **name** on the multiple-choice answer sheet.
- Unless otherwise indicated, the diagrams in this book are **not** drawn to scale.
- All written responses must be in English.

At the end of the examination
- Place the answer sheet for multiple-choice questions inside the front cover of this book.

Students are NOT permitted to bring mobile phones and/or any other unauthorised electronic devices into the examination room.

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SECTION A- Muliple-choice questions

Instructions for Section A

Answer **all** questions in pencil on the answer sheet provided for multiple – choice questions.
Choose the response that is **correct** for the question.
A correct answer scores1, an incorrect answer scores 0.
Marks will **not** be deducted for incorrect answers.
No marks will be given if more than one answer is completed for any question.
Unless otherwise indicated, the diagrams in this book are **not** drawn to scale.

Question 1

The period and amplitude of the graph of $y=-3\cos\left(2\pi x-\frac{\pi}{2}\right)$ respectively are

A. 3 and 2π
B. -3 and 2π
C. 2π and 3
D. 1 and -3
E. 1 and 3

Question 2

Consider $f(x)=\sin(x)$ for $x\in[0,2\pi]$ and $g(x)=(x+1)^2$ over its maximal domain. The domain of $g(f(x))$ is

A. $[-1,\ 1]$
B. $[-2,\ 2]$
C. $[0,\ \infty)$
D. $[0,\ 2\pi]$
E. $[0,\ \pi]$

Question 3

The general solution of the equation $2\sin(2x)=1$ can be written as

A. $x=\frac{\pi}{12}+\pi n$ or $x=\frac{5\pi}{12}+\pi n,\ n\in R$
B. $x=\frac{\pi}{6}+2\pi n$ or $x=\frac{5\pi}{6}+2\pi n,\ n\in Z$
C. $x=\frac{\pi}{12}+\pi n$ or $x=\frac{5\pi}{12}+\pi n,\ n\in Z$
D. $x=\frac{\pi}{12}+2\pi n$ or $x=\frac{5\pi}{12}+2\pi n,\ n\in Z$
E. $x=\pm\frac{\pi}{6}+2\pi n,\ n\in Z$

Question 4

The number of distinct x-intercepts for the graph of $f(x)=3x^4-60x^3+450x^2-1500x+k+1875$, where k is a real constant could be

A. 0, 1 or 2
B. 1, 2 or 3
C. 0, 1, 2 or 3
D. 0, 2, 3 or 4
E. 0, 2 or 4

Question 5

The inverse of the function $f:\left[\frac{3}{2},\infty\right)\to R, f(x)=-\sqrt{2x-3}+4$ can be expressed as

A. $f^{-1}:[4,\infty)\to R, f^{-1}(x)=\frac{1}{2}(x-4)^2+3$

B. $f^{-1}:(-\infty,4]\to R, f^{-1}(x)=\frac{1}{2}(x-4)^2+3$

C. $f^{-1}:(-\infty,4]\to R, f^{-1}(x)=\frac{1}{2}(x-4)^2+\frac{3}{2}$

D. $f^{-1}:[4,\infty)\to R, f^{-1}(x)=\frac{1}{2}(x-4)^2+\frac{3}{2}$

E. $f^{-1}:\left[\frac{3}{2},\infty\right)\to R, f^{-1}(x)=\frac{x^2}{2}-4x+\frac{19}{2}$

Question 6

Under a sequence of transformations the image of $y=e^x$ is $y_T=2e^{-2x}+3$. The transformations applied, in order, to the graph of $y=e^x$ are

A. A dilation from the x-axis by a factor of 2, followed by a dilation from the y-axis by a factor of 2, then a reflection in the x-axis and a translation in the positive direction of the y-axis by 3 units.

B. A dilation from the x-axis by a factor of 2, followed by a dilation from the y-axis by a factor of 2, then a reflection in the y-axis and a translation in the negative direction of the y-axis by 3 units.

C. A dilation from the x-axis by a factor of 2, followed by a dilation from the y-axis by a factor of 0.5, then a reflection in the x-axis and a translation in the negative direction of the y-axis by 3 units.

D. A dilation from the x-axis by a factor of 2, followed by a dilation from the y-axis by a factor of 0.5, then a reflection in the x-axis and a translation in the positive direction of the y-axis by 3 units.

E. A dilation from the x-axis by a factor of 2, followed by a dilation from the y-axis by a factor of 0.5, then a reflection in the y-axis and a translation in the positive direction of the y-axis by 3 units.

Question 7

The graph of $y=-\log_e\left(2x^2-x\right)$ has only

A. two vertical asymptotes and one horizontal asymptote

B. two vertical asymptotes

C. one vertical asymptote

D. one vertical asymptote and one horizontal asymptote

E. one horizontal asymptote

Question 8

Which one of the following functions does not have an inverse which is a function?

A. $f:R\to R, f(x)=x+2$

B. $g:[-1,\infty)\to R,\ g(x)=x^2+2x+1$

C. $h:R\setminus\{-2\}\to R,\ h(x)=\dfrac{1}{x+2}$

D. $k:(3,\infty)\to R,\ k(x)=\dfrac{1}{(x-3)^2}$

E. $l:(-\infty,1)\to R,\ l(x)=(x+1)(x-2)^2$

Question 9

A matrix transformation is described by $T:R^2\to R^2, T\left(\begin{bmatrix}x\\y\end{bmatrix}\right)=\begin{bmatrix}-1 & 0\\0 & 0.5\end{bmatrix}\begin{bmatrix}x\\y\end{bmatrix}+\begin{bmatrix}0\\2\end{bmatrix}$.

The equation of the image under this transformation from the graph of $y=\dfrac{3}{x}$ is

A. $y=-\dfrac{3}{2x+4}$

B. $y=-\dfrac{3}{2x}+2$

C. $y=-\dfrac{1}{2x+4}+2$

D. $y=\dfrac{3}{4-2x}-2$

E. $y=-\dfrac{3}{2x}-2$

SECTION A - continued
TURN OVER

Question 10

The function f has the property $f(x+2y)=f(x)f(2y)$ for all real numbers of x. Which one of the following is a possible rule for the function?

A. $f(x)=x^2$

B. $f(x)=\frac{1}{x}$

C. $f(x)=e^{2x}$

D. $f(x)=\log_e(2x)$

E. $f(x)=\log_e(x)$

Question 11

The average value of the function $f(x)=e^x\sin(x)$ over the interval $[0,\ 2\pi]$ is

A. $\frac{e^{2\pi}-1}{4\pi}$

B. 0

C. $\frac{1-e^{2\pi}}{2}$

D. $-\frac{e^{2\pi}-1}{4\pi}$

E. $\frac{\pi-\pi e^{2\pi}}{4}$

Question 12

The average rate of change of $y=\frac{1}{2}x+\frac{1}{2x}$ from $x=\frac{1}{2}$ to $x=\frac{3}{2}$ is

A. $-\frac{1}{6}$

B. $\frac{1}{6}$

C. $\frac{\log_e(3)}{2}+\frac{1}{2}$

D. $-\frac{1}{9}$

E. $-\frac{1}{3}$

SECTION A - continued

Question 13

For which one of the following functions is the graph not strictly increasing?

A. $f: R \to R,\ f(x) = x^3$

B. $f: \left(-\frac{\pi}{2}, \frac{\pi}{2}\right) \to R,\ f(x) = \tan(x)$

C. $f: (-\infty,\ 2) \to R,\ f(x) = -\frac{1}{x-2}$

D. $f: (-\infty,\ 3) \to R,\ f(x) = \frac{1}{(x-3)^2}$

E. $f: \left(\frac{3}{2},\ \infty\right) \to R,\ f(x) = -\sqrt{2x-3}$

Question 14

A particle moves in a straight line with an acceleration $a = 5t + 3$. The particle starts from rest 2 metres from a fixed point. Its position, x m at time t seconds could be given by the rule

A. $x = 2$

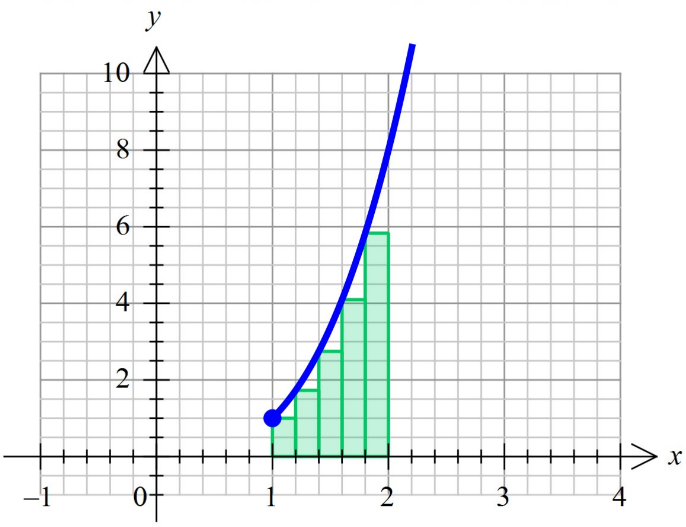
B. $x = \frac{5t^2}{2} + 3t + 2$

C. $x = \frac{5t^3}{6} + \frac{3t^2}{2} + 2$

D. $x = \frac{5t^3}{6} + \frac{3t^2}{2}$

E. $x = 5$

Question 15
Part of the graph of $y = x^3$ is shown below.



Using left endpoint rectangles with strips of width 0.2, the approximate area contained between the curve with equation $f(x) = x^3$ and the x-axis between $x = 1$ and $x = 2$ as a percentage of the actual area is closest to

A. 3.08
B. 80
C. 82.13
D. 4.48
E. 83.71

Question 16
A probability distribution table is shown below, where a, b and c are real numbers.

x	0	1	2	3
$\Pr(X = x)$	0.2	a	b	c

If the mean of the distribution is 1.9 and the variance 1.29 then the values of a, b and c are respectively

A. 0.4, 0.3 and 0.1
B. 0.2, 0.4 and 0.2
C. 0.1, 0.4 and 0.3
D. 0.1, 0.3 and 0.4
E. 0.29, –0.07 and 0.59

Question 17
A binomial random variable has a mean of 20 and standard deviation of $\sqrt{10}$. The $\Pr(X = 15 \mid X < 18)$ correct to three decimal places is

A. 0.170
B. 0.115
C. 0.037
D. 0.215
E. 0.054

SECTION A - continued

Question 18

Events A and B are independent. If $\Pr(A) = 0.7$ and $\Pr(A' \cap B') = 0.2$ then $\Pr(A \cap B)$ equals

A. 0

B. $\frac{1}{2}$

C. $\frac{7}{30}$

D. $\frac{3}{5}$

E. $\frac{4}{5}$

Question 19

The probability density function of a normal random variable X is given by

$$f(x) = \frac{1}{4\sqrt{2\pi}} e^{-\frac{1}{2}\left(\frac{x-3}{4}\right)^2}, \quad x \in R.$$

The mean and variance are respectively

A. 4 and 3
B. 4 and 9
C. 2 and 9
D. 3 and 4
E. 3 and 16

Question 20

A random variable X has a probability density function f with rule

$$f(x) = \begin{cases} e^{x+2} & 0 \le x \le \log_e\left(e^2+1\right) - 2 \\ 0 & \text{elsewhere} \end{cases}$$

The median of X is

A. $\frac{1}{2}$

B. $\frac{e^{x+2}}{2}$

C. 1

D. $\left(e^2+1\right)\log_e\left(e^2+1\right) - 2e^2 - 3$

E. $\log_e\left(\frac{2e^2+1}{2}\right) - 2$

END OF SECTION A
TURN OVER

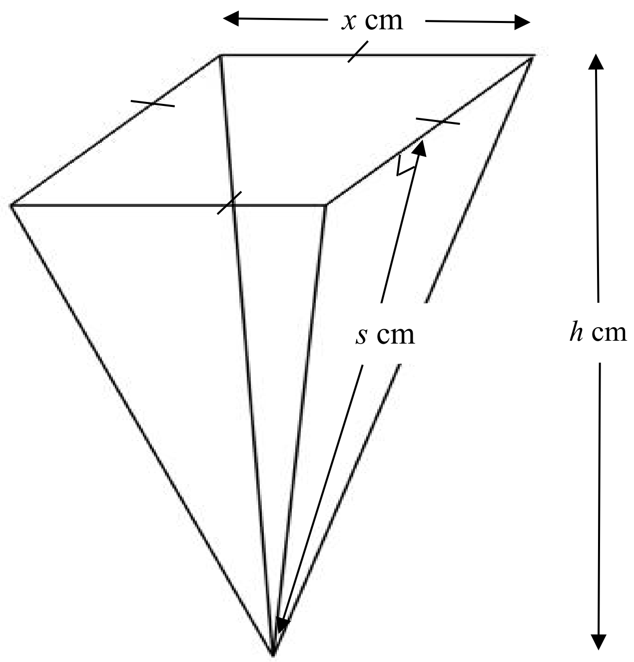
SECTION B

Instructions for Section B

Answer **all** questions in the spaces provided.
In all questions where a numerical answer is required an exact value must be given unless otherwise specified.
In questions where more than one mark is available, appropriate working **must** be shown.
Unless otherwise indicated, the diagrams in this book are **not** drawn to scale.

Question 1 (14 marks)
A rain gauge collects water to measure rainfall. The manufacturers of the rain gauge wanted the shape to be a right inverted square based pyramid as shown below.

The length of the edge of the square base is x cm, the slant height has length s cm and the vertical height h cm as shown in the diagram below.



The rain gauge is to be made out of 1100 cm^2 of a transparent plastic, which is to include a lid of side length x cm.

a. Show that $s = \dfrac{1100 - x^2}{2x}$. 1 mark

b. Find the vertical height, h cm in terms of x. 1 mark

c. Hence show that the volume, V cm^3, of the rain gauge is given by the rule
$V = \dfrac{5x\sqrt{-22(x^2 - 550)}}{3}$. 1 mark

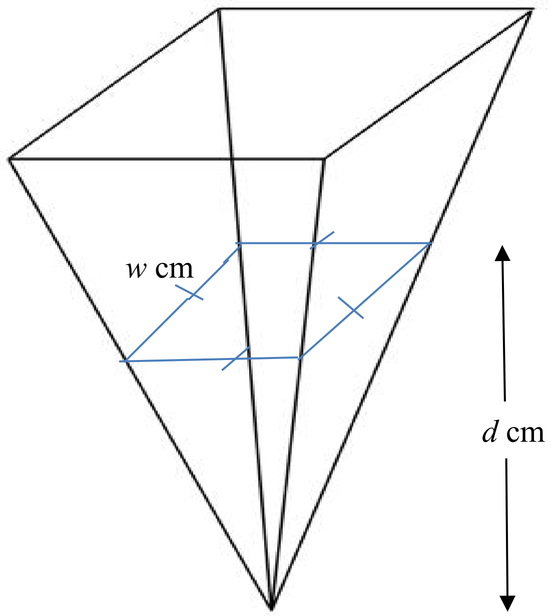
The designers want the rain gauge to hold the maximum amount water possible.

d. What length should the edge of the base be? 1 mark

e. What is the maximum volume of water the rain gauge can hold? 1 mark

The manufacturers use the recommendation of the designers: they produce the gauge which holds the maximum volume.

With the lid open water falls into the rain gauge. Let the vertical height of water be d cm and the length of the edge of the top surface of water be w cm as shown in the diagram below.



On a particularly rainy day, water fills the gauge at a rate of $0.2t \text{ cm}^3/\text{min}$, where t is the time in minutes. The gauge is initially empty.

f. How long will it take the rain gauge to fill? Give your answer in minutes correct to two decimal places. 2 marks

g. Find the average volume of water in the gauge for the time it takes to fill. Give your answer in cm^3, correct to one decimal place. 2 marks

h. Using similar triangles, show that V_w, the volume of water in the gauge in cm^3, in terms of d is $V_w = \frac{d^3}{6}$. Give the domain for the function. 4 marks

i. Find the depth of the water in the gauge when it is half full. Give your answer in cm, correct to one decimal place. 1 mark

Question 2 (12 marks)
A mathematician initially described the height of water at the northern edge of an ocean pool (that part of the ocean which is sectioned off for swimmers by wire mesh) at a certain time of the year as

$$h:[0,\ 60] \to R, h(t) = \frac{1}{2}\cos\left(-\frac{4\pi}{3}\left(t - \frac{1}{2}\right)\right) + 2$$

where h is the height above the sand in metres and t is the time in minutes from 9:00 am each day.

a. What is the initial height of water at the northern edge of the pool? 1 mark

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b. What is the period and range of the graph of h? 2 marks

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c. Write down the transformations that map, in an appropriate order, the function $y = \cos(t)$ to the function h. 2 marks

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The mathematician realised that the initial model was incorrect and applied a transformation given by $T: R^2 \to R^2, T\left(\begin{bmatrix} t \\ h \end{bmatrix}\right) = \begin{bmatrix} b & 0 \\ 0 & c \end{bmatrix}\begin{bmatrix} t \\ h \end{bmatrix}$. The graph of h should be dilated by a factor of $\frac{2}{3}$ from the t axis.

d. State the values of b and c. 1 mark

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e. What is the new rule $h_1(t)$, after the dilation is applied. 1 mark

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SECTION B - Question 2 - continued

f. Display the transformations from the graph of $y = \cos(t)$ to the graph of h_1 as a matrix equation. 1 mark

The mathematician finally decided that the model which best describes the height of the water at the northern edge of the ocean pool is

$$h_2 : [0,\ 60] \to R, h_2(t) = \frac{5}{12}\cos\left(-\frac{4\pi}{3}(t-2)\right) + 2$$

where h_2 is the height of the water above the sand in metres, and t is the time in minutes from 9:00 am each day.

g. Find the value of t at which the rate of change of the height of the water in the pool is first at a maximum? 2 marks

The local council are thinking of putting a temporary platform around the pool so that swimmers can walk around it. The platform is to be 2.1 m high.

h. For what proportion of one wave cycle would the northern path be under water? Give your answer correct to two decimal places. 2 marks

SECTION B - continued
TURN OVER

Question 3 (15 marks)
A researcher has 30 mice in each of 3 cages. She asks her young assistant to select 12 so that she can weigh them. The assistant randomly selects 12 mice from the first cage.

a. Is this an appropriate method to choose a random sample? Give a reason. 1 mark

b. Suggest how the sample could be chosen. 1 mark

c. There are 10 white mice and 20 brown mice in one of the cages. If four mice are selected at random from this cage to be weighed, what is the probability there will be two white mice in the sample? 2 marks

SECTION B - Question 3 - continued

The weights of adult field mice are normally distributed. It is know that 80% of field mice weigh more than 18 g and 70% less than 24 g.

d. Find the mean and standard deviation of the distribution. Give your answer in grams correct to one decimal place. 3 marks

100 such mice are randomly selected.

e. What is the mean and standard deviation of the sample proportion, $\hat{P}$, of mice which weigh more than 18 g. 2 marks

f. What is the probability that the sample proportion is greater than the population proportion? Give your answer correct to three decimal places. 2 marks

Some genetically modified mice were released onto an island several years ago. Scientists want to investigate the weights of these mice. They take a random sample of 200 mice and find 160 of them weigh less than 21 g. Let p be the proportion of genetically modified mice that weigh less than 21 g.

g. Find the 99% confidence interval for p. Give your answers correct to three decimal places. 1 mark

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h. If the scientists took 300 such samples and calculated the 99% confidence intervals, how many of them would they expect to contain p? 1 mark

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i. Find the smallest sample size the scientists would have to take if they wanted the distance between $\hat{p}$ and the endpoints of a 99% confidence interval to be less than 2%. 2 marks

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SECTION B - continued

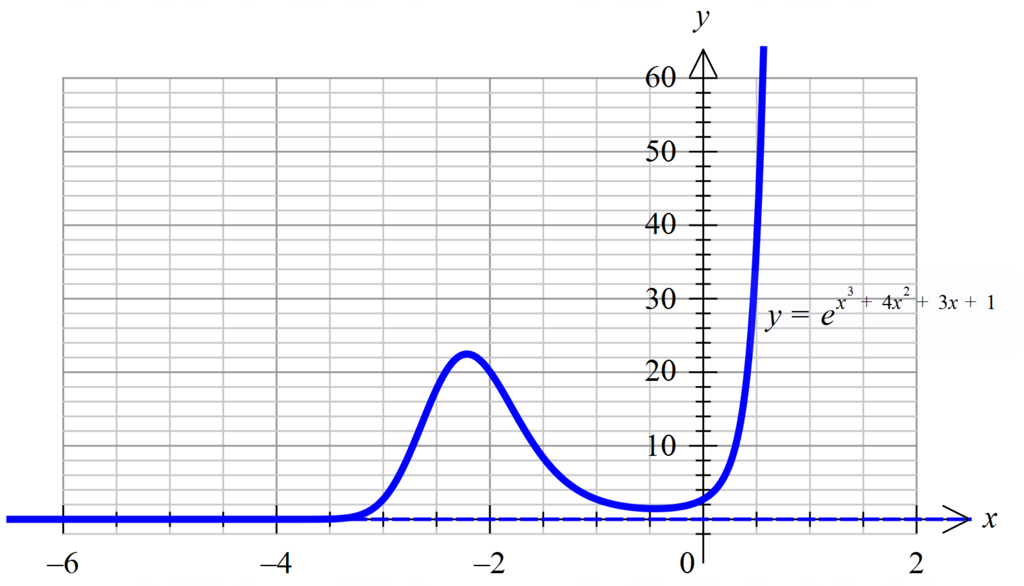
Question 4 (19 marks)

Consider the family of functions $l_b(x) = e^{x^3+bx^2+3x+1}$, where b is a real constant.

a. Find the values of b for which the graph of l_b has two stationary points. 2 marks

b. For what values of b will l_b^{-1} exist. 1 mark

Part of the graph of $l_4(x) = e^{x^3+4x^2+3x+1}$ is shown below.



c. Find the equation of the tangent to the graph of $l_4(x) = e^{x^3+4x^2+3x+1}$ at $x = -3$ and sketch it on the above graph. 2 marks

SECTION B - Question 4 - continued
TURN OVER

d. Find the area bounded by the tangent line in **part c.** and the graph of $l_4(x) = e^{x^3+4x^2+3x+1}$.
Give your answer correct to one decimal place. Write down the expression, involving definite integrals which gives this area. Give any non-integer terminals correct to two decimal places. 2 marks

Now consider the functions $g: R \to R,\ g(x) = e^x$ and $h: [-1, \infty) \to R,\ h(x) = x^2 + 2x + 1$.

e. Explain why $f(x) = g(h(x))$ exists. 1 mark

f. Show that $f^{-1}(x) = \sqrt{\log_e(x)} - 1$. 2 marks

Consider the family of functions $f_a(x) = f(x) + a$, where a is a negative real constant.

g. Find $f_a^{-1}(x)$ in terms of a. 1 mark

h. Find the value of a and the corresponding x value for which the graphs of the functions f_a and f_a^{-1} are tangential, that is touch each other. Give your answers correct to two decimal places. 3 marks

For some values of a the area between the curves of f_a and f_a^{-1} is bounded.

i. Find the value of a for which the bounded area is a maximum. Give the x values of the points of intersection, writing the largest x value correct to three decimal places. 3 marks

j. Find the maximum bounded area, correct to two decimal places. Write down a definite integral which gives the area, giving any non-integer terminals correct to three decimal places. 2 marks

END OF QUESTION AND ANSWER BOOK