2019 Mathematical Methods Trial Exam 2 Solutions

SECTION A – Multiple-choice questions

1	2	3	4	5	6	7	8	9	10
C	C	D	A	B	B	C	E	E	A

11	12	13	14	15	16	17	18	19	20
C	E	E	A	A	A	C	B	D	D

Q1 $(b-x)^2 = a^c$, $b-x = \pm a^{\frac{c}{2}}$, $x = b \pm a^{\frac{c}{2}}$ C

Q2 $\cos\left(x+\frac{\pi}{4}\right)=0$, $x=-\frac{7\pi}{4},-\frac{3\pi}{4},\frac{\pi}{4},\frac{5\pi}{4}$

or $\tan\left(x-\frac{\pi}{4}\right)=0$, $x=-\frac{7\pi}{4},-\frac{3\pi}{4},\frac{\pi}{4},\frac{5\pi}{4},\frac{9\pi}{4}$ C

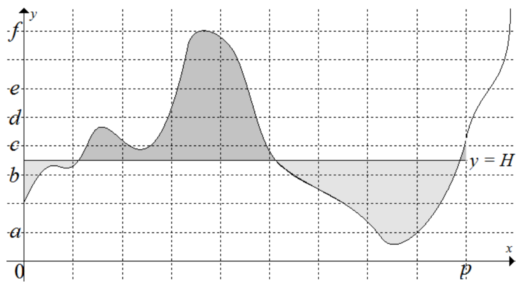
Q3 The domain of the inverse is the range of the function.
$D=\{-3,-2,-1,2\}$ D

Q4 $x=a\left(1\pm\frac{1}{y-a}\right)$, $x=a\pm\frac{a}{y-a}$, $\pm(x-a)=\frac{a}{y-a}$

$y-a=\pm\frac{a}{x-a}$, $y=a\left(1\pm\frac{a}{x-a}\right)$ A

Q5 B

Q6 B

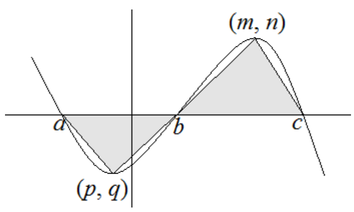


Q7 Average rate $=\frac{c-\frac{a+b}{2}}{p-0}=\frac{2c-a-b}{2p}$ C

Q8 $f(a-x)=-f(x-a)$, $f(-x)=-f(x)$
$\therefore$ $f(x)$ is an odd function. E

Q9 E

Q10 Area $\approx\frac{1}{2}(-q)(b-a)+\frac{1}{2}n(c-b)=\frac{1}{2}(nc-(n+q)b+qa)$ A



Q11 Binomial:
$n=18$, $p=\frac{1}{3}$, $\Pr(X=5)+\Pr(X=6)\approx 0.3774$ C

Q12 E

Q13 $\frac{p(1-p)}{256}=0.025^2$, $p^2-p+0.16=0$, $p=0.20$ E

Q14 $\hat{P}$: mean ≈ 0.25, standard dev $\approx\sqrt{\frac{0.25\times0.75}{300}}=0.025$

Proportion of random samples $\approx\Pr(\hat{P}>0.2)\approx 0.9772$ A

Q15 $y=f(x)\to y+b=f(x)\to -y+b=f(x)\to -\frac{y}{a}+b=f(x)$

$\therefore$ $y=ab-a\,f(x)$ A

Q16 $\sum\Pr=1$, $a^2+0.2a+0.85=1$, $a^2+0.2a-0.15=0$

$\therefore$ $a=0.3$, $\overline{X}=0.50\times1+0.3^2\times2+0.35\times3+0.2\times0.3\times4=1.97$

$\text{Var}(X)=0.50\times1^2+0.3^2\times2^2+0.35\times3^2+0.2\times0.3\times4^2-1.97^2$
$=1.0891$

$\therefore$ $\text{sd}(X)=\sqrt{1.0891}\approx1.0436$ A

Q17 $\int_0^a\left(\frac{1}{1+a-x}\right)dx=1$, $[-\log_e(1+a-x)]_0^a=1$, $a=e-1$ C

Q18 $D=\sqrt{x^2+y^2}=\sqrt{x^2+\frac{(x-2)^6}{9}}$, min $D\approx0.99$ B

Q19 $\frac{dy}{dx}=9x^2+2ax+b^2$

No stationary points, $\Delta=4a^2-36b^2<0$, $-6b<2a<6b$ D

Q20 $y=\frac{a}{x-a}+b$, $\therefore$ $(x-a)(y-b)=a$

If $a=b$, the inverse is the function itself, $\therefore$ infinitely many solutions.

For intersections, solve $(x-a)(y-b)=a$ and $y=x$

$\therefore$ $x^2-(a+b)x+(ab-a)=0$, it is a quadratic equation, it cannot have three solutions (intersections)

$\Delta=(a+b)^2-4(ab-a)=(a-b)^2+4a$

No intersections if $\Delta<0$, e.g. $b=0$ and $a=-1$

One intersection if $\Delta=0$, e.g. $b=0$ and $a=-4$

Two intersections if $\Delta>0$, e.g. $b=0$ and $a=1$ D itute.com

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SECTION B

Q1a $y = a(x-2)^2$ and $(3, 1)$

$\therefore\ a = 1$ and $y = (x-2)^2 = x^2 - 4x + 4$

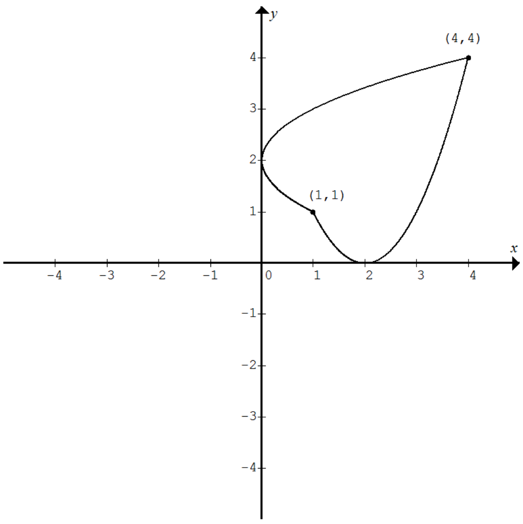
Q1b Inverse $x = (y-2)^2$, $y = 2 \pm \sqrt{x}$

$\therefore\ f(x) = 2 + \sqrt{x}$ and $g(x) = 2 - \sqrt{x}$

Q1c $y = x$ and $y = x^2 - 4x + 4$

$\therefore\ x = x^2 - 4x + 4$, $x^2 - 5x + 4 = 0$, $x = 1, 4$, $\therefore\ (1, 1)$, $(4, 4)$

Q1di



Q1dii Area $= 2\times\int_1^4 \left(x - (x-2)^2\right)dx = 2\times\left[\frac{x^2}{2} - \frac{(x-2)^3}{3}\right]_1^4 = 9$

Q1e Same as $x = d$ dividing the bounded region.

$\int_d^4 \left(2 + \sqrt{x} - (x-2)^2\right)dx = \frac{9}{2}$, $d \approx 2.08$

Q1f Maximum length of line segment $y = -x + c$ occurs when its endpoint is the point where the gradient of the parabola is 1.

Let $\frac{dy}{dx} = 2(x-2) = 1$, $\therefore\ x = 2.5$ and $y = 0.25$, $(2.5, 0.25)$.

The other endpoint is $(0.25, 2.5)$.

Max length $= \sqrt{(2.5-0.25)^2 + (0.25-2.5)^2} = \frac{9\sqrt{2}}{4}$

$y = -x + c$ and $(2.5, 0.25)$, $\therefore\ c = \frac{11}{4}$

Q2a $\left\{x : x = \frac{n}{\cos 20°},\ n = 1, 2, 3, ..., 22\right\}$

Q2b $a^2 = (3 + x\cos 20°)^2 + (x\sin 20° + 1.5 - 3)^2$

$= 9 + 6x\cos 20° + x^2\cos^2 20° + x^2\sin^2 20° - 3x\sin 20° + 2.25$

$= (\sin^2 20° + \cos^2 20°)x^2 + (6\cos 20° - 3\sin 20°)x + 11.25$

$= x^2 + (6\cos 20° - 3\sin 20°)x + 11.25$

Q2c $b^2 = (3 + x\cos 20°)^2 + (8 + 3 - 1.5 - x\sin 20°)^2$

$= 9 + 6x\cos 20° + x^2\cos^2 20° + 90.25 - 19x\sin 20° + x^2\sin^2 20°$

$= (\sin^2 20° + \cos^2 20°)x^2 + (6\cos 20° - 19\sin 20°)x + 99.25$

$= x^2 + (6\cos 20° - 19\sin 20°)x + 99.25$

Q2di $\cos\theta$

$$= \frac{x^2 + (6\cos 20° - 3\sin 20°)x + 11.25 + x^2 + (6\cos 20° - 19\sin 20°)x + 99.25 - 8^2}{2\sqrt{x^2 + (6\cos 20° - 3\sin 20°)x + 11.25}\sqrt{x^2 + (6\cos 20° - 19\sin 20°)x + 99.25}}$$

$$= \frac{x^2 + (6\cos 20° - 11\sin 20°)x + 23.25}{\sqrt{\left(x^2 + (6\cos 20° - 3\sin 20°)x + 11.25\right)\left(x^2 + (6\cos 20° - 19\sin 20°)x + 99.25\right)}}$$

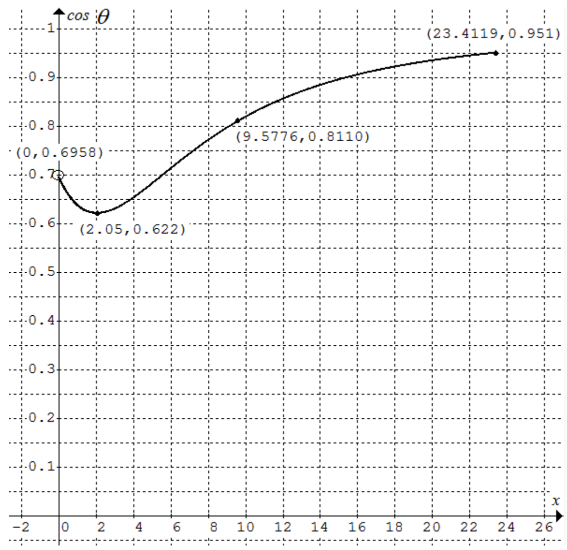
Q2dii $b = 6\cos 20° - 19\sin 20° + 6\cos 20° - 3\sin 20° \approx 3.75$

Q2e $n = 9$ for the tenth row, $x = \frac{9}{\cos 20°} \approx 9.5776$

$\therefore\ \cos\theta \approx \frac{x^2 + 1.88x + 23.25}{\sqrt{x^4 + bx^3 + 106.53x^2 + 448.07x + 1116.56}} \approx 0.8110$

$\theta \approx 36°$

Q2f



Q2g θ is the greatest when $\cos\theta$ is the smallest.

$\cos\theta \approx 0.622$, greatest $\theta \approx 52°$ when $x \approx 2$

Q2h $n \approx x\cos 20° \approx 2\cos 20° \approx 2$, $\therefore$ the third row.

Q3a $f(x)=e^{x}-mx=0$ has exactly one solution if the graphs of $y=e^{x}$ and $y=mx$ intersect at one point only, the point where the two curves have the same gradient, .: $f'(x)=e^{x}-m=0$, $m=e^{x}$

.: $e^{x}-e^{x}x=0$, $e^{x}(1-x)=0$, .: $x=1$ and $m=e$.

Q3b $m>e$

Q3c $\int_{x_1}^{x_2}(-f(x))dx=\int_{x_1}^{x_2}(mx-e^{x})dx$

Q3d $\left[\frac{mx^2}{2}-e^{x}\right]_{x_1}^{x_2}=\frac{mx_2^{\,2}}{2}-\frac{mx_1^{\,2}}{2}-e^{x_2}+e^{x_1}$

Q3e $\frac{mx_2^{\,2}}{2}-\frac{mx_1^{\,2}}{2}-e^{x_2}+e^{x_1}>0$, $e^{x_1}-mx_1=0$ and $e^{x_2}-mx_2=0$

$\frac{m(x_2^{\,2}-x_1^{\,2})}{2}-e^{x_2}+e^{x_1}>0$, $\frac{m(x_2-x_1)(x_2+x_1)}{2}-e^{x_2}+e^{x_1}>0$

$\frac{(e^{x_2}-e^{x_1})(x_2+x_1)}{2}-(e^{x_2}-e^{x_1})>0$, $(e^{x_2}-e^{x_1})\left(\frac{(x_2+x_1)}{2}-1\right)>0$

Since $e^{x_2}-e^{x_1}>0$, .: $\frac{(x_2+x_1)}{2}-1>0$, .: $x_1+x_2>2$

Q3f $g(x)=\log_e x-nx^2=0$ has exactly one solution if the graphs of $y=\log_e x$ and $y=nx^2$ intersect at one point only, where $x=a$, the point where the two curves have the same gradient

.: $f'(a)=\frac{1}{a}-2na=0$ and $g(a)=\log_e a-na^2=0$

.: $na^2=\frac{1}{2}$, $\log_e a=\frac{1}{2}$.: $a=\sqrt{e}$ and $n=\frac{1}{2e}$.

Q3g $0<n<\frac{1}{2e}$

Q3h $g(b)=\log_e b-nb^2=0$, $\log_e b=nb^2$

Since $n<0$ and $b^2>0$, .: $nb^2<0$

.: $\log_e b<0$ and $0<b<1$

Q4a $\tan\theta_1=\frac{30}{25}$, $\tan\theta_2=\frac{30}{20}$, $\tan(\theta_1+\theta_2)=\frac{\frac{30}{25}+\frac{30}{20}}{1-\frac{30}{25}\times\frac{30}{20}}=\frac{-27}{8}$

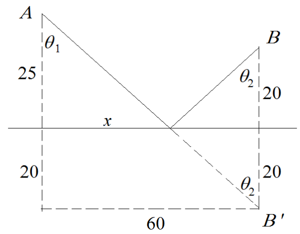
.: $\theta_1+\theta_2=\tan^{-1}\left(\frac{-27}{8}\right)$

Q4b $\tan\theta_1=\frac{x}{25}$, $\tan\theta_2=\frac{60-x}{20}$, $\tan(\theta_1+\theta_2)=\frac{\frac{x}{25}+\frac{60-x}{20}}{1-\frac{x}{25}\times\frac{60-x}{20}}$

When $\theta_1+\theta_2=90°$, $1-\frac{x}{25}\times\frac{60-x}{20}=0$, $500-x(60-x)=0$

.: $x=10, 50$

Q4c When $\theta_1=\theta_2$, AB' is a straight line and it is the shortest.



Q4d $\theta_1=\tan^{-1}\left(\frac{60}{25+20}\right)=\tan^{-1}\left(\frac{4}{3}\right)$

Q4e Minimum total length $=\sqrt{(25+20)^2+60^2}=75$ m

Q5a $k\left(\int_0^1 e^{-2}x\,dx+\int_1^9 e^{-\frac{(x-5)^2}{8}}dx-\int_9^{10}e^{-2}(x-10)dx\right)=1$

$k(0.06766764+4.7851521+0.06766764)=1$, $k\approx0.203232$

Q5b $\Pr(X<6\mid X>1)=\frac{\Pr(1<X<6)}{\Pr(X>1)}$

$\approx\frac{3.3524265k}{(4.7851521+0.06766764)k}\approx0.6908$

Q5c $p=\Pr(X>6)=\Pr(6<X\le9)+\Pr(9<X\le10)$

$\approx k(1.4327256+0.06766764)\approx0.3049$

Q5d Mean $(\hat{P})\approx p\approx0.3049$, sd$(\hat{P})\approx\sqrt{\frac{0.3049(1-0.3049)}{100}}\approx0.0460$

Q5e $\Pr(\hat{P}<0.4)\approx0.9807$ (Normal: $\mu=0.3049, \sigma=0.0460$)

Q5f $\Pr(X>c)=\Pr(c<X\le9)+\Pr(9<X\le10)=p=0.3$

$\Pr(c<X\le9)+0.06766764k\approx0.3$, $\Pr(c<X\le9)\approx0.28625$

$\int_c^9 0.203232e^{-\frac{(x-5)^2}{8}}dx\approx0.28625$, $c\approx6.03$

Q5g $\sqrt{\frac{0.3(1-0.3)}{n}}\le0.01$, $n\ge2100$

Q5h For $Y<8$, $p\approx\hat{p}=\frac{400-100}{400}=0.75$

Q5i $\sqrt{\frac{0.75(1-0.75)}{400}}\approx0.02$

Q5j For 70%, $z=\left|\text{invNorm}\left(\frac{1-0.70}{2}\right)\right|=|\text{invNorm}(0.15)|\approx1.04$

70% interval $\approx(0.75-1.04\times0.02, 0.75+1.04\times0.02)\approx(0.73, 0.77)$

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