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YEAR 12 *Trial Exam Paper*

2019

MATHEMATICAL METHODS

Written examination 2

Reading time: 15 minutes
Writing time: 2 hours

STUDENT NAME:

QUESTION AND ANSWER BOOK

Structure of book

Section	*Number of questions*	*Number of questions to be answered*	*Number of marks*
A	20	20	20
B	5	5	60
			Total 80

- Students are permitted to bring into the examination room: pens, pencils, highlighters, erasers, sharpeners, rulers, a protractor, set squares, aids for curve sketching, one bound reference, one approved technology (calculator or software) and, if desired, one scientific calculator. Calculator memory DOES NOT need to be cleared. For approved computer-based CAS, full functionality may be used.
- Students are NOT permitted to bring into the examination room: blank sheets of paper and/or correction fluid/tape.

Materials provided

- Question and answer book of 23 pages
- Formula sheet
- Answer sheet for multiple-choice questions

Instructions

- Write your **name** in the box provided above on this page, and on your answer sheet for multiple-choice questions.
- Unless otherwise indicated, the diagrams in this book are **not** drawn to scale.
- All written responses must be in English.

Students are NOT permitted to bring mobile phones and/or any other unauthorised electronic devices into the examination.

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SECTION A – Multiple-choice questions

Instructions for Section A

Answer **all** questions in pencil on the answer sheet provided for multiple-choice questions.

Choose the response that is **correct** for the question.

A correct answer scores 1; an incorrect answer scores 0.

Marks will **not** be deducted for incorrect answers.

No marks will be given if more than one answer is completed for any question.

Unless otherwise indicated, the diagrams in this book are **not** drawn to scale.

Question 1

Let $f: R \to R$, $f(x) = 3\cos\left(\frac{2\pi x}{5}\right) - 2$.

The period and range of this function is

A. $\frac{1}{5}$ and $[-3, 3]$

B. $\frac{1}{5}$ and $[-5, 1]$

C. 5 and $[-3, 3]$

D. 5 and $[-5, 1]$

E. 5 and $[-3, -1]$

Question 2

The length of the line segment that joins (1, 4) to (3, *a*) is

A. $\sqrt{a^2 + 8a + 20}$

B. $\sqrt{a^2 - 8a + 20}$

C. $\sqrt{a^2 - 2a + 10}$

D. $\sqrt{a^2 + 2a + 10}$

E. $\sqrt{a^2 + 8a - 20}$

Question 3

Let $f:[-2,3) \to R$, $f(x) = 4 - 2x$. The range of f is

A. $(-2, 8]$

B. $[-2, 8]$

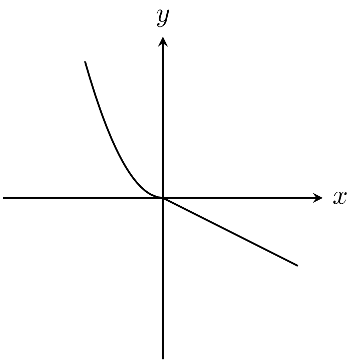
C. $[-2, 8)$

D. $(-2, 8)$

E. $(-2, 6]$

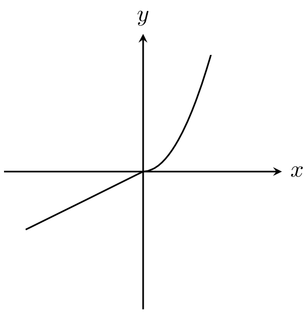
Question 4

Part of the graph of $y = f(x)$ is shown below.



Which one of the following could be the graph of $y = f^{-1}(x)$ where f^{-1} is the inverse of f ?

A.



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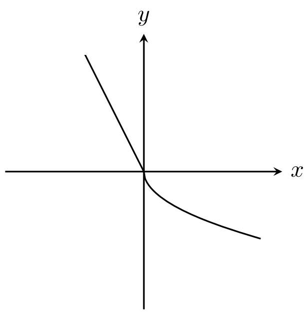

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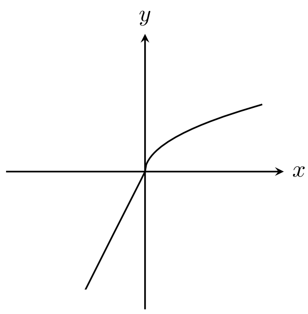
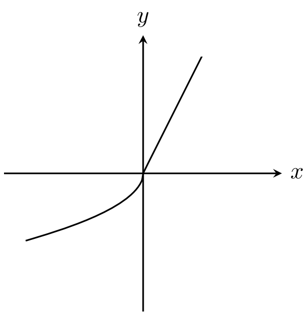
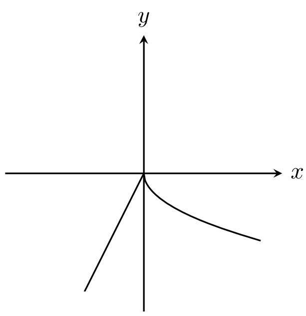


D.



E.



Question 5

Consider the perpendicular line to the graph of $y = x^2$ at the point $(-3, 9)$.

This line passes through the x-axis when

A. $x = 57$

B. $x = 12$

C. $x = -57$

D. $x = -55$

E. $x = \frac{19}{2}$

Question 6

The average value of the function with rule $f(x) = -6(x-1)^2(x-a)$ over the interval $[1, a]$ where a is a real number and $a > 1$ is 4.

The value of a is

A. 7

B. 6

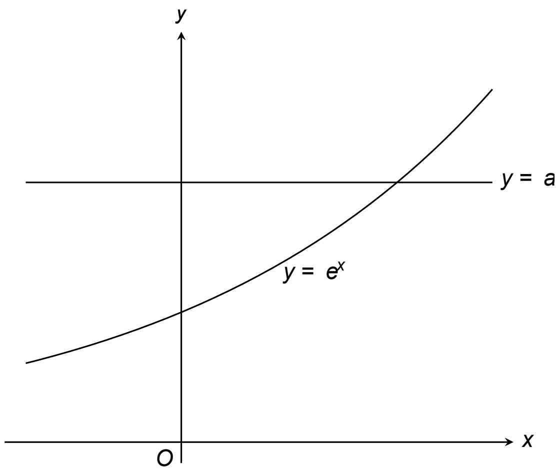
C. 5

D. 4

E. 3

Question 7

Parts of the graphs with equations $y = e^x$ and $y = a$, $a > 1$ are shown below.



The total area of the region bounded by the y-axis, the line $y = a$, $a > 1$, and the curve with equation $y = e^x$ is equal to

A. $1 + a^2 - e^a$

B. $a - 1 - a\log_e(a)$

C. $a\log_e(a) + 1 - a$

D. $\dfrac{e^a - 1}{\log_e(a)}$

E. $\dfrac{a^{\log_e(a)} - 1}{\log_e(a)}$

Question 8

Suppose $P(x) = ax^3 + bx^2 + x - 5$, $P(1) = -3$ and $P(3) = -83$. The sum of a and b equals

A. –1

B. 0

C. 1

D. 2

E. 3

Question 9

The continuous random variable X has a probability density function given by

$$f(x)=\begin{cases} ax^2(2-x) & 0\le x\le 2 \\ 0 & \text{elsewhere} \end{cases}$$

where $a\in R$.

The median of X is closest to

A. 1.2

B. 1

C. 1.23

D. 1.33

E. 1.43

Question 10

If $\int_0^2 f(x)dx=5$, then $\int_{-4}^{0} 3f\left(-\frac{x}{2}\right)dx$ is equal to

A. 30

B. 10

C. 15

D. 20

E. 60

Question 11

The function f has the property $f(x+1)+f(x-1)-2f(x)=1$ for all $x\in R$.

Which one of the following is a possible rule for the function?

A. $f(x)=\frac{2}{x^2}$

B. $f(x)=\frac{1}{x^2}$

C. $f(x)=x^2$

D. $f(x)=2x^2$

E. $f(x)=\frac{1}{2}x^2$

Question 12

Consider $f(x)=a^2x-a\sqrt{x}$, $a>0$.

There is a stationary point on the graph of f when $x=1$.

The value of a is

A. $\frac{1}{\sqrt{2}}$

B. $\frac{1}{2}$

C. $\frac{1}{2\sqrt{2}}$

D. $\frac{1}{4}$

E. 0

Question 13

Let $f: D \to R$, $f(x)=\frac{9-2x}{x-3}$, where D is the maximal domain.

The graph of f has asymptotes

A. $x=-3$, $y=2$

B. $x=-3$, $y=-2$

C. $x=3$, $y=2$

D. $x=3$, $y=-2$

E. $x=-2$, $y=3$

Question 14

The graph of the function $f(x)=ke^x$ intersects the graph of $g(x)=3-e^{-x}$ twice if

A. $k>-\frac{9}{4}$

B. $k<-\frac{9}{4}$

C. $k=\frac{9}{4}$

D. $k>\frac{9}{4}$

E. $0<k<\frac{9}{4}$

Question 15

The discrete random variable X has the following probability distribution.

The mean of X is 3.2.

x	0	2	3	4	6
$\Pr(X = x)$	a	$\frac{1}{10}$	$\frac{1}{5}$	b	$\frac{1}{5}$

The values of a and b are

A. $a = \frac{1}{10}$, $b = \frac{2}{5}$

B. $a = \frac{1}{5}$, $b = \frac{3}{10}$

C. $a = \frac{3}{20}$, $b = \frac{7}{20}$

D. $a = \frac{2}{5}$, $b = \frac{1}{10}$

E. $a = \frac{3}{10}$, $b = \frac{1}{5}$

Question 16

X is a normally distributed random variable with mean 100. If $\Pr(X > 120) = 0.2$ then the standard deviation is closest to

A. 20

B. 21

C. 22

D. 23

E. 24

Question 17

A probability density function, f, is given by

$$f(x)=\begin{cases}\frac{4}{27}(x-1)^2(4-x) & 1\le x\le 4\\ 0 & \text{otherwise}\end{cases}$$

The standard deviation of this function is

A. $\frac{3}{5}$

B. $\frac{9}{25}$

C. $\frac{2}{5}$

D. $\frac{1}{25}$

E. $\frac{3}{25}$

Question 18

If A and B are independent events from a sample space such that $\Pr(A)=0.4$ and $\Pr(B)=0.3$, $\Pr(A\cup B')$ is equal to

A. 0.42

B. 0.82

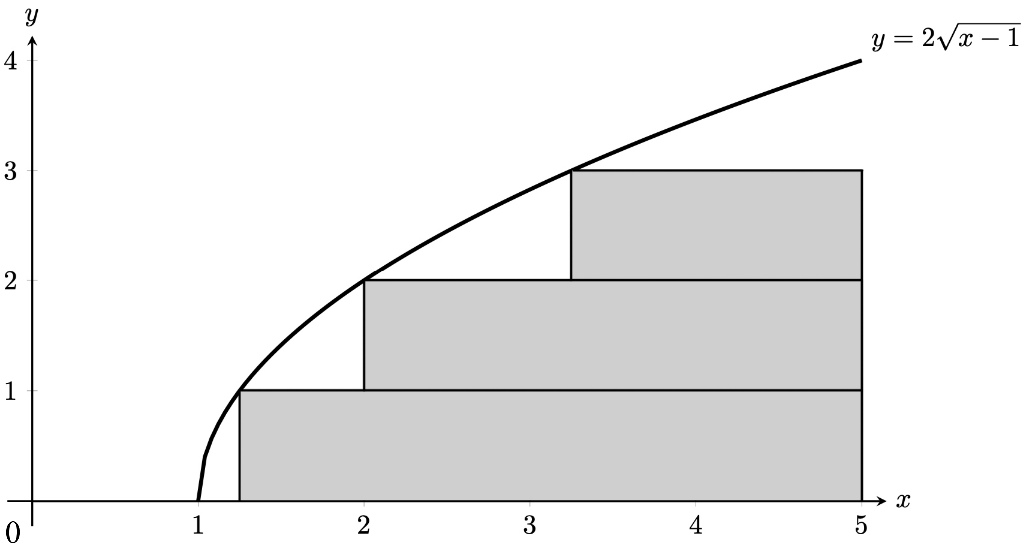
C. 0.36

D. 0.92

E. 0.98

Question 19

An approximation to the area between the graph of $y=2\sqrt{x-1}$ and the x-axis between $x=1$ and $x=5$ is found using three rectangles each of height 1 as shown below.



The error in the approximation as a percentage of the actual area is closest to

A. 20.3%

B. 24.7%

C. 25.1%

D. 27.5%

E. 32.3%

Question 20

Consider the transformation T, defined as

$$T: R^2 \to R^2,\ T\left(\begin{bmatrix} x \\ y \end{bmatrix}\right) = \begin{bmatrix} -2 & 0 \\ 0 & 3 \end{bmatrix}\begin{bmatrix} x \\ y \end{bmatrix} + \begin{bmatrix} 1 \\ 0 \end{bmatrix}.$$

The transformation T maps the graph of $y = f(x)$ onto the graph of $y = g(x)$.

If $g(x) = 3\sqrt{4-x}$ then $f(x)$ is equal to

A. $\sqrt{3-2x}$

B. $\sqrt{3+2x}$

C. $\sqrt{5-4x}$

D. $\sqrt{3+x}$

E. $\sqrt{5+4x}$

END OF SECTION A
TURN OVER

SECTION B

Instructions for Section B

Answer **all** questions in the spaces provided.

In all questions where a numerical answer is required, an exact value must be given unless otherwise specified.

In questions where more than one mark is available, appropriate working **must** be shown.

Unless otherwise indicated, the diagrams in this book are **not** drawn to scale.

Question 1 (13 marks)

a. Consider the function $g(x)=\frac{4}{3}x^3-x^2-2x$.

i. Given that $f'(x)=g(x)$ and $f(0)=-1$, show that $f(x)=\frac{1}{3}x^4-\frac{1}{3}x^3-x^2-1$.

1 mark

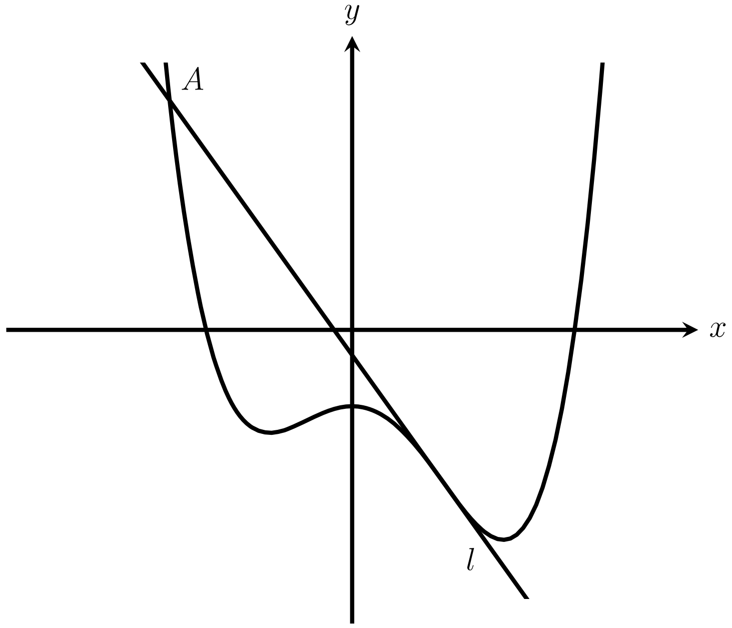
The graph of $y=f(x)$ has stationary points at $x=0$ and at two other points.

ii. Find the other values of x for which the graph of $y=f(x)$ has stationary points.

Express your answer in the form $\frac{a\pm\sqrt{b}}{c}$ where a, b and c are integers.

2 marks

b. The diagram below shows part of the graph of $f(x)$ and the tangent l to the graph at $x = 1$. The tangent crosses the graph at A.



i. Find the equation of the tangent l.

1 mark

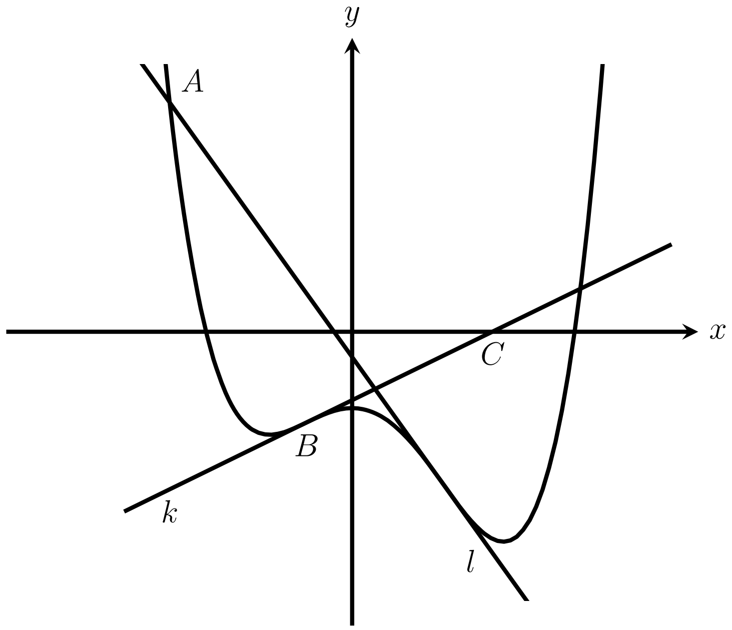
ii. State the coordinates of A.

1 mark

iii. Find the area bounded by the tangent l and $y = f(x)$.

2 marks

c. The tangent k to $f(x)$ at B, where B is in the third quadrant, has gradient $\frac{7}{12}$ and crosses the x-axis at C.



i. Find the coordinates of B and C.

3 marks

ii. There is another tangent to f that is parallel to the tangent k.
Determine the equation of this tangent.

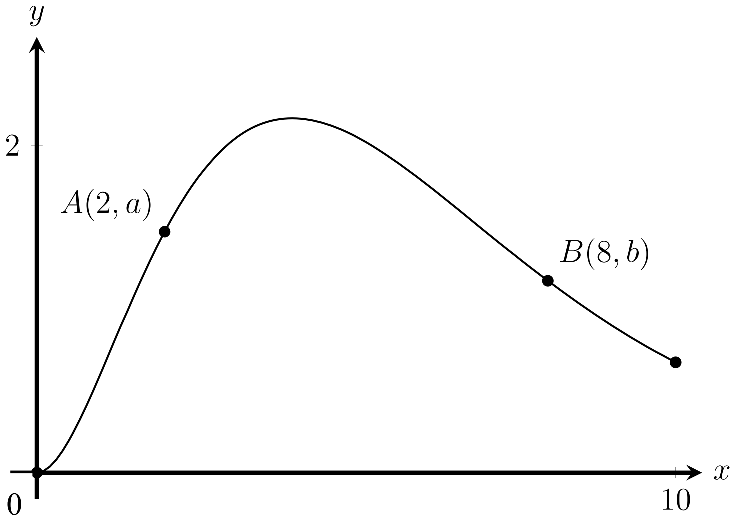
1 mark

d. Determine the acute angle between the tangents k and l. Give your answer in degrees correct to two decimal places.

2 marks

Question 2 (11 marks)

Let $f:[0,10]\to R$, $f(x)=x^2e^{-\frac{x}{2}}$. The graph of f is shown below.



a. State the interval for which the graph of f is strictly increasing.

1 mark

b. The points $A(2, a)$ and $B(8, b)$ are labelled on the diagram.

Find m, the gradient of the line which passes through A and B. Give your answer in the form $\frac{p}{qe^4}+\frac{r}{se}$ where p, q, r and s are integers.

2 marks

c. Determine the value of x for which $f'(x)$ is equal to m. Give your answer correct to three decimal places.

2 marks

d. Let $g:R\to R$, $g(x)=x^2$ and $h:D\to R$, $h(x)=f(g(x))$ where D is the largest domain for which h is defined.

i. Determine the domain of $h'(x)$, where $h'(x)$ is the derivative of $h(x)$.

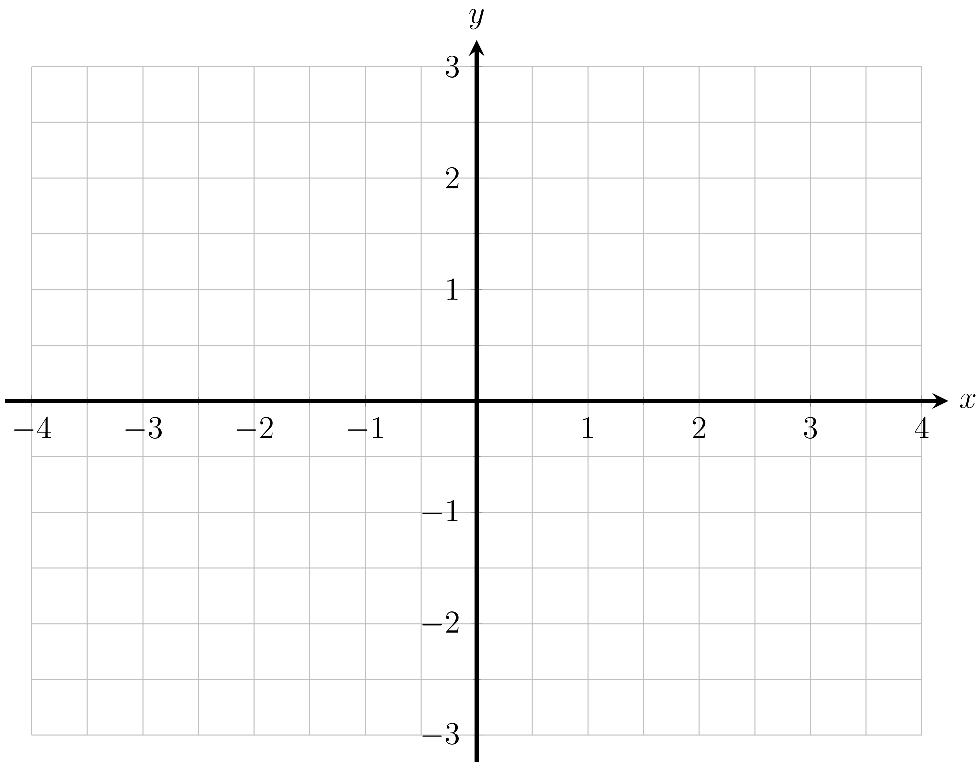
2 marks

ii. Find $h'(x)$.

1 mark

iii. Plot the graph of $h'(x)$ on the axes below. Give the exact coordinates of all axis intercepts and the coordinates of the endpoints correct to two decimal places.

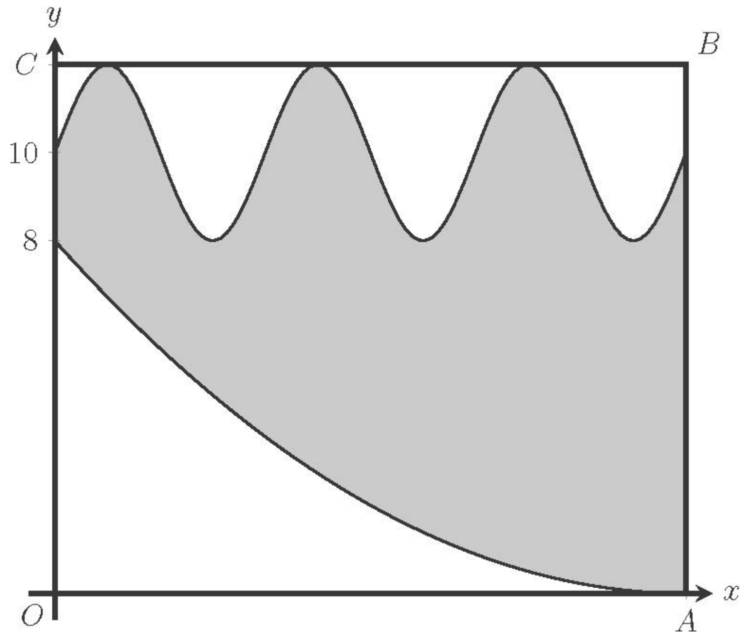
3 marks



Question 3 (10 marks)

A component for a machine is to be cut from a rectangular sheet of plastic *OABC*. The length *OA* is 15 cm. The component is indicated by the shaded region below.

The upper part of the component is modelled by the function $f(x) = 10 + 2\sin\left(\frac{2\pi x}{5}\right)$.



a. Determine the area of the rectangular sheet of plastic *OABC* in square centimetres.

1 mark

b. The lower part of the component is modelled by the function $g(x) = a(x-15)^2$.

Show that the value of a is $\frac{8}{225}$.

1 mark

c. Determine the area of the component in square centimetres.

2 marks

d. Write down a function that gives the vertical distance between the upper and lower parts of the component.

1 mark

e. Find the value of x for which this distance is a maximum. Give your answer, in centimetres, correct to three decimal places.

1 mark

f. Write down the average distance between the upper and lower parts of the component.

1 mark

g. Find all of the values of x for which the distance between the upper and lower parts of the component equals the average distance between the upper and lower parts of the component. Give your answer correct to three decimal places.

3 marks

Question 4 (16 marks)

The apples grown in Victoria have weights that are normally distributed with a mean of 165 grams and a standard deviation of 23 grams.

a. Determine the probability that a randomly selected Victorian apple has a mass between 140 grams and 160 grams. Give your answer correct to four decimal places.

1 mark

b. An apple is considered small if it has a mass between 130 grams and 150 grams. The probability that a randomly selected Victorian apple is small is 0.1931.

i. Find the probability that at most 2 apples from a random sample of 20 Victorian apples are small apples. Give your answer correct to three decimal places.

2 marks

ii. For random samples of 20 Victorian apples, $\hat{P}$ is the random variable that represents the proportion of small apples.

Find $\Pr(\hat{P} \geq 0.1 \mid \hat{P} \leq 0.3)$. Give your answer correct to three decimal places.

3 marks

iii. How many Victorian apples would need to be sampled to ensure that the probability that the sample contained at least five small apples is more than 0.99?

2 marks

c. A random sample of 500 Victorian apples found that 165 were small.

Determine a 95% confidence interval for the population proportion from the sample. Give your result correct to three decimal places.

1 mark

d. Victorian apples are grown at many different altitudes. The continuous random variable A that models the altitude x in metres at which a randomly selected Victorian apple is grown has a probability density function f where

$$f(x)=\begin{cases}\dfrac{1}{1500} & 0\le x\le 1200\\ -\dfrac{1}{1500}\left(\dfrac{x}{600}-3\right) & 1200< x\le 1800\\ 0 & \text{otherwise}\end{cases}$$

i. Sketch the graph of f on the axes provided below showing the coordinates of the axis intercepts.

2 marks

ii. Find $\Pr(X>1000)$.

1 mark

iii. Find the expected value of A in metres.

2 marks

e. Hybrid varieties of apples make up 40% of the apples that are grown in Victoria. It is known that 15% of the apples grown at an altitude greater than 1000 metres are hybrid.

Find the probability that a randomly selected Victorian apple grown at an altitude lower than 1000 metres is a hybrid. Give your answer correct to three decimal places.

2 marks

Question 5 (10 marks)

Let $f: R \to R$, $f(x) = ax(x-k)^2$ and $g: R \to R$, $g(x) = \frac{4ak^2}{9}x$ where a and k are positive real numbers.

a. Find the coordinates of the local maximum of f. Give your answer in terms of a and k.

2 marks

b. Show that the graphs of f and g meet at the local maximum of f.

1 mark

Let $h: \left[0, \frac{k}{3}\right] \to R$, $h(x) = ax(x-k)^2$ where a and k are positive real numbers.

c. Determine the area bounded by the graphs of h and g.

2 marks

d. Show that the maximum value of a in terms of k so that h and h^{-1} meet exactly twice is $\frac{9}{4k^2}$.

2 marks

e. If $a = \dfrac{9}{4k^2}$ and h and h^{-1} meet exactly twice, determine the value of k if the area bounded by h and h^{-1} equals $\dfrac{1}{12}$.

3 marks

END OF QUESTION AND ANSWER BOOK