YEAR 12 *Trial Exam Paper*

2019

MATHEMATICAL METHODS

Written examination 1

Worked solutions

This book presents:

- worked solutions
- mark allocations
- tips.

Question 1a.

Worked solution

$\frac{d}{dx}(2x+1)=2$ and $\frac{d}{dx}\left(\log_e(2x+1)\right)=\frac{2}{2x+1}$

$$\begin{aligned}\frac{dy}{dx}&=(2x+1)\times\frac{2}{(2x+1)}+(2)\times\log_e(2x+1)\\&=2+2\log_e(2x+1)\end{aligned}$$

Mark allocation: 2 marks

- 1 method mark for application of the product rule
- 1 answer mark for $\frac{dy}{dx}=2+2\log_e(2x+1)$, or an equivalent answer such as a factorised expression

Tip

- *Expect the first question in the exam to ask you to evaluate a derivative. This will require you to use the chain, product or quotient rules.*

Question 1b.

Worked solution

$$f'(x)=-8x\sin\left(4x^2\right)$$

$$\begin{aligned}f'\left(\frac{\sqrt{\pi}}{4}\right)&=-8\times\left(\frac{\sqrt{\pi}}{4}\right)\times\sin\left(4\left(\frac{\sqrt{\pi}}{4}\right)^2\right)\\&=-2\sqrt{\pi}\sin\left(\frac{\pi}{4}\right)\\&=-2\sqrt{\pi}\times\frac{\sqrt{2}}{2}\\&=-\sqrt{2\pi}\end{aligned}$$

Mark allocation: 2 marks

- 1 answer mark for calculating the correct derivative $f'(x)=-8x\sin\left(4x^2\right)$
- 1 answer mark for the correct answer $f'\left(\frac{\sqrt{\pi}}{4}\right)=-\sqrt{2\pi}$

Question 2

Worked solution

$y = \int 2 - e^{-x}\,dx = 2x + e^{-x} + c$

Substitute $x = 2$ and $y = 4 - \frac{1}{e^2}$ and solve for c.

$$4 - \frac{1}{e^2} = 2 \times 2 + e^{-2} + c$$

$$= 4 + \frac{1}{e^2} + c$$

$$c = -\frac{2}{e^2}$$

$$y = 2x + e^{-x} - 2e^{-2} = 2x + \frac{1}{e^x} - \frac{2}{e^2}$$

Mark allocation: 3 marks

- 1 answer mark for antidifferentiation of $\frac{dy}{dx}$ to give $y = 2x + e^{-x} + c$, or award this mark for $y = 2x + e^{-x}$ given as the final answer.
- 1 method mark for the substitution of $x = 2$ and $y = 4 - \frac{1}{e^2}$
- 1 answer mark for the equation $y = 2x + e^{-x} - 2e^{-2}$, or equivalent with positive indices

Tip

- *Keep the final answer consistent with the format of the question unless otherwise specified. Sometimes the question will ask you to express the answer with positive indices.*

Question 3a.

Worked solution

The first block has a $\frac{1}{3}$ chance of being red and $\frac{2}{3}$ chance of being blue.

If a blue block is drawn first, then there is a $\frac{1}{2}$ chance of drawing the red block on the second draw.

The probability is $\frac{1}{3}+\frac{2}{3}\times\frac{1}{2}=\frac{2}{3}$.

Note: Alternatively, the probability of drawing the red block can be found by calculating the probability of drawing both blue blocks, with Pr(red drawn) $=1-$ Pr(both blue drawn).

Mark allocation: 1 mark

- 1 mark for the correct probability $\frac{2}{3}$

Tips

- *A tree diagram can be a helpful tool to use for sequential choice probability questions, as it will allow you to visualise the blocks being selected.*
- *Read the question carefully to determine whether the blocks are drawn with or without replacement.*

Question 3b.

Worked solution

From **part a.**, $\Pr(X=1)=\frac{2}{3}$. Hence, $\Pr(X=0)=\frac{1}{3}$.

$$\mathrm{E}(X)=0\times\frac{1}{3}+1\times\frac{2}{3}=\frac{2}{3}$$

Mark allocation: 1 mark

- 1 mark for $\mathrm{E}(X)=\frac{2}{3}$

Tip

- *You may choose to draw a probability table to summarise the probability distribution. This can assist you with finding the mean and variance.*

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Question 3c.

Worked solution

$\text{Var}(X) = \sum (x-\mu)^2 \times \Pr(X = x)$

x	**0**	**1**
$(x-\mu)^2$	$\left(0-\frac{2}{3}\right)^2 = \frac{4}{9}$	$\left(1-\frac{2}{3}\right)^2 = \frac{1}{9}$
$\Pr(X = x)$	$\frac{1}{3}$	$\frac{2}{3}$

$$\begin{aligned}\text{Var}(X) &= \frac{4}{9}\times\frac{1}{3}+\frac{1}{9}\times\frac{2}{3}\\ &= \frac{4}{27}+\frac{2}{27}\\ &= \frac{6}{27} = \frac{2}{9}\end{aligned}$$

Alternatively:

$$\begin{aligned}\text{Var}(X) &= \text{E}\left(X^2\right)-\left[\text{E}\left(X\right)\right]^2\\ &= \left[0\times\frac{1}{3}+1^2\times\frac{2}{3}\right]-\left[\frac{2}{3}\right]^2\\ &= \frac{2}{3}-\frac{4}{9}\\ &= \frac{2}{9}\end{aligned}$$

Mark allocation: 2 marks

- 1 method mark for a substitution into $\text{Var}(X) = \sum (x-\mu)^2 \times \Pr(X = x)$ or $\text{Var}(X) = \text{E}\left(X^2\right)-\left[\text{E}\left(X\right)\right]^2$
- 1 answer mark for $\text{Var}(X) = \frac{2}{9}$

Tip

- *You should always give answers in the simplest form. A final answer of* $\frac{6}{27}$ *would not be awarded the answer mark.*

Question 4a.

Worked solution

The quadratic function is strictly decreasing over the given domain, so the range can be found by evaluating the end points of the domain.

$$f\left(-\frac{1}{2}\right)=2\left(-\frac{1}{2}-1\right)^2-3=2\left(-\frac{3}{2}\right)^2-3=2\times\frac{9}{4}-3=\frac{9}{2}-3=\frac{3}{2}\ \text{ and }\ f(1)=2(1-1)^2-3=-3$$

The range of f is $\left[-3,\frac{3}{2}\right)$.

Mark allocation: 1 mark

- 1 mark for the correct range $\left[-3,\frac{3}{2}\right)$

Note: An answer of $\left(\frac{3}{2},-3\right]$ is incorrect.

Tips

- *A quick sketch to visualise the end points and turning point of the function might help with your reasoning in this question.*
- *The y-values of the end points do not always give the range of the function. In this case, the end point is the turning point (hence, the smallest value) but this is not always the case.*

Question 4b.

Worked solution

Let $y = -2(x-1)^2 - 3$.

To find the inverse, create a new equation by swapping the *x* and *y*.

$x = -2(y-1)^2 - 3$

Rearrange this equation to solve for *y*.

$$\begin{aligned} 2(y-1)^2 - 3 &= x \\ 2(y-1)^2 &= x+3 \\ (y-1)^2 &= \frac{x+3}{2} \\ y-1 &= \pm\sqrt{\frac{x+3}{2}} \\ y &= \pm\sqrt{\frac{x+3}{2}} + 1 \end{aligned}$$

The range of f^{-1} is $\left(-\frac{1}{2}, 1\right]$; therefore, the negative root is required.

This gives $f^{-1}(x) = -\sqrt{\frac{x+3}{2}} + 1$.

Mark allocation: 2 marks

- 1 method mark for interchanging variables and rearranging
- 1 answer mark for $f^{-1}(x) = -\sqrt{\frac{x+3}{2}} + 1$

Tips

- *Ensure that you define the variable y before using it in your method. This can be done using a statement such as 'Let $y = f(x)$'.*
- *When determining the correct root, consider either the range of f^{-1} or the domain of f.*
- *Ensure that the final answer is in terms of $f^{-1}(x)$ and is not left in terms of y. If the answer is left in terms of y, you will not be awarded the answer mark.*

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Question 5a.

Worked solution

$$\begin{aligned}\text{RHS} &= \left(4\sin^2(x)-3\right)(\sin(x)-1)\\ &= 4\sin^2(x)\times\sin(x)+4\sin^2(x)\times(-1)+(-3)\times\sin(x)+(-3)\times(-1)\\ &= 4\sin^3(x)-4\sin^2(x)-3\sin(x)+3\\ &= f(x)=\text{LHS}\end{aligned}$$

Alternatively:

Let $p=\sin(x)$.

$$\begin{aligned}f(p) &= 4p^3-4p^2-3p+3\\ &= 4p^2(p-1)-3(p-1)\\ &= (4p^2-3)(p-1)\end{aligned}$$

$f(x)=(4\sin^2(x)-3)(\sin(x)-1)$, as required.

Alternatively:

Let $p=\sin(x)$.

$$f(p)=4p^3-4p^2-3p+3$$
$$f(1)=0$$

Hence, $(p-1)$ is a factor.

$$\begin{aligned}f(p) &= 4p^3-4p^2-3p+3\\ &= (a\cdot p^2+b\cdot p+c)(p-1)\end{aligned}$$

Equating coefficients gives $a=4$, $c=-3$, and $-3p-b\cdot p=-3p$, $b=0$.

Hence, $f(p)=(4p^2-3)(p-1)$ and $f(x)=(4\sin^2(x)-3)(\sin(x)-1)$.

Note: This type of question, where an expression is turned into a polynomial by a substitution, is also commonly seen with exponential equations.

Mark allocation: 1 mark

- 1 method mark for clearly shown correct reasoning

Tip

- *You may choose to simplify the expression by equating the common function,* $\sin(x)$*, to a variable. This will make factorising the expression easier.*

Question 5b.

Worked solution

From **part a.**, $\left(4\sin^2(x)-3\right)(\sin(x)-1)=0$.

Either $4\sin^2(x)-3=0$ or $\sin(x)-1=0$.

$\sin(x)=\pm\frac{\sqrt{3}}{2},1$ (solutions in all four quadrants and on the *y*-axis.)

For $x\in[0,\,2\pi]$, this gives

$$x=\frac{\pi}{3},\frac{\pi}{2},\frac{2\pi}{3},\frac{4\pi}{3},\frac{5\pi}{3}$$

Mark allocation: 3 marks

- 1 method mark for $\sin(x)=\pm\frac{\sqrt{3}}{2},1$
- 1 answer mark for at least one of the correct solutions, such as only $x=\frac{\pi}{2}$ or $x=\frac{\pi}{3},\frac{2\pi}{3}$ or the correct solutions with extraneous solutions
- 1 answer mark for the exact set of correct solutions

Tip

- *You are expected to have knowledge of the exact values of circular functions and symmetry properties of the unit circle for Exam 1.*

Question 6a.

Worked solution

$f(x)$ is defined when both $\log_2(x+1)$ and $\log_2(4-x)$ are defined.

$\log_2(x+1)$ implies $x > -1$ and $\log_2(4-x)$ implies $x < 4$; hence, $-1 < x < 4$.
The domain of *f* is $(-1, 4)$.

Alternatively:

Rewrite the equation as $f(x) = \log_2\left(\dfrac{x+1}{4-x}\right)$.

This function is defined when the argument of the logarithm is positive.

The denominator is positive when $x < 4$. The numerator is positive when $x > -1$.
Hence, the domain of *f* is $(-1, 4)$.

Note: The logarithm of zero or a negative number is undefined.

Mark allocation: 1 mark

- 1 answer mark for $(-1, 4)$ or $-1 < x < 4$.

Question 6b.

Worked solution

$$\log_2(x+1)-\log_2(4-x)=0$$

$$\log_2\left(\frac{x+1}{4-x}\right)=0$$

$$\frac{x+1}{4-x}=2^0$$

$$\frac{x+1}{4-x}=1$$

$$x+1=4-x$$

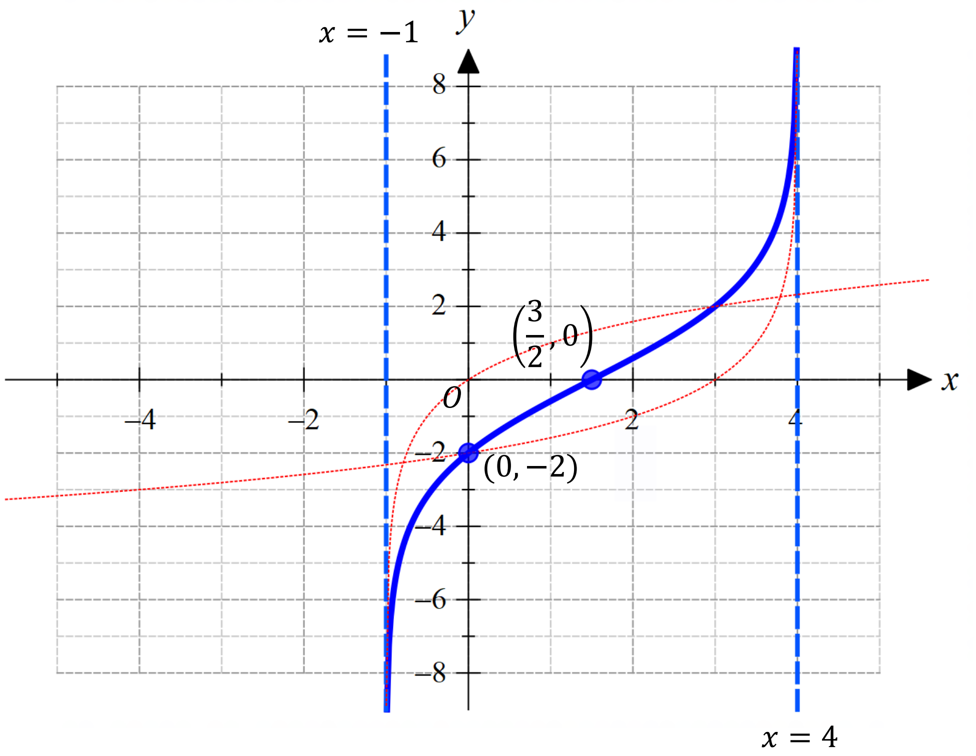
$$2x=3$$

$$x=\frac{3}{2}$$

Mark allocation: 2 marks

- 1 method mark for applying log laws to create the expression $\log_2\left(\frac{x+1}{4-x}\right)$ or equivalent; or, alternatively, for rearranging the equation to $\log_2(x+1)=\log_2(4-x)$ and equating the arguments of the log functions
- 1 answer mark for the correct answer $x=\frac{3}{2}$ or equivalent

Question 6c.



Note: The graphs of $y = \log_2(x+1)$ and $y = -\log_2(4-x)$ are shown as thinly dotted lines. These might be used as part of an addition of ordinates process when answering this question.

Mark allocation: 3 marks

- 3 answer marks: drawing a correctly shaped line with both axis intercepts labelled and with both asymptotes marked and labelled
- 2 answer marks: drawing a correctly shaped line with two or three of the axis intercepts or two asymptotes correctly labelled and drawn
- 1 answer mark: drawing an almost correctly shaped line with one of the two axis intercepts or two asymptotes correctly identified

Tips

- *When sketching your response to a question, it is often the case that the answers to previous parts to the question will assist with the sketch. This is the case here, with **part a.** giving the domain of the function and **part b.** giving the x-axis intercept.*
- *Note that marks are not awarded for roughly and multi-lined drawn curves when it should be one smooth curve. If you do draw other curves as part of your method, ensure that the final answer is clearly labelled.*

Question 7

Worked solution

$$\text{Area} = \int_0^{\frac{2\pi}{3}} \sin(x)+\sin(2x)dx - \int_{\frac{2\pi}{3}}^{\pi} \sin(x)+\sin(2x)dx$$

$$= \left[-\cos(x)-\frac{1}{2}\cos(2x)\right]_0^{\frac{2\pi}{3}} - \left[-\cos(x)-\frac{1}{2}\cos(2x)\right]_{\frac{2\pi}{3}}^{\pi}$$

$$= \left[\left(-\cos\left(\frac{2\pi}{3}\right)-\frac{1}{2}\cos\left(\frac{4\pi}{3}\right)\right)-\left(-\cos(0)-\frac{1}{2}\cos(0)\right)\right]$$

$$-\left[\left(-\cos(\pi)-\frac{1}{2}\cos(2\pi)\right)-\left(-\cos\left(\frac{2\pi}{3}\right)-\frac{1}{2}\cos\left(\frac{4\pi}{3}\right)\right)\right]$$

$$= \left[\left(\frac{1}{2}+\frac{1}{4}\right)-\left(-1-\frac{1}{2}\right)\right]-\left[\left(1-\frac{1}{2}\right)-\left(\frac{1}{2}+\frac{1}{4}\right)\right]$$

$$= \left[\frac{9}{4}\right]-\left[-\frac{1}{4}\right]$$

$$= \frac{10}{4} = \frac{5}{2} \text{ square units}$$

Mark allocation: 3 marks

- 1 method mark for an appropriate expression for calculating the unsigned area, such as by $\int_0^{\frac{2\pi}{3}} f(x)dx - \int_{\frac{2\pi}{3}}^{\pi} f(x)\,dx$, or equivalent
- 1 method mark for calculating the antiderivative of $\sin(x)+\sin(2x)$ as $-\cos(x)-\frac{1}{2}\cos(2x)$
- 1 answer mark for the answer $\frac{5}{2}$ square units

Note: Units may be omitted.

Tip

- *Even if you cannot solve a definite integral, you should still write down the expression that you will solve, as you may get a method mark.*

Question 8a.i.

Worked solution

$$\begin{aligned}
\text{LHS} &= f'(x) \\
&= \frac{d}{dx}(x-k)\times e^{-x} + (x-k)\times\frac{d}{dx}\left(e^{-x}\right) \\
&= e^{-x} + (x-k)\times -e^{-x} \\
&= -(x-1-k)e^{-x} = \text{RHS}, \text{ as required.}
\end{aligned}$$

Mark allocation: 1 mark

- 1 method mark for application of the product rule, leading to the answer

Tips

- *If you have trouble with applying the product rule for differentiation, then break the problem into smaller steps. Begin by identifying parts of the function and finding their derivatives.*
- *In 'Show that …' questions, you should write your final answer in the same format as that shown in the question.*

Question 8a.ii.

Worked solution

At the stationary point $f'(x) = 0$.

$$\begin{aligned} -(x-1-k)e^{-x} &= 0 \\ x-1-k &= 0 \\ x &= k+1 \end{aligned}$$

$$\begin{aligned} f(k+1) &= (k+1-k)e^{-(k+1)} \\ &= e^{-(k+1)} \end{aligned}$$

P is located at $\left(k+1, e^{-(k+1)}\right)$.

Mark allocation: 2 marks

- 1 answer mark for $x = k+1$
- 1 answer mark for the coordinate $\left(k+1, e^{-(k+1)}\right)$

Tips

- *Pay attention to the format in which questions ask for the answer to be provided. A question like this might have asked for the x-value where the stationary point occurs; it might have asked for the maximum value itself, or for the coordinate.*
- *A stationary point implies $f'(x) = 0$; this should be stated in your working.*

Question 8b.i.

Worked solution

The rule for $h(x)$ can be written as $h(x)=3\times(3x)\times e^{-(3x)}=3g(3x)$.

To map g to h requires a dilation by a factor of $\frac{1}{3}$ in the x-direction (from the y-axis) and a dilation by a factor of 3 in the y-direction (from the x-axis).

Therefore, $a=\frac{1}{3}$ and $b=3$.

Mark allocation: 2 marks

- 1 answer mark for $a=\frac{1}{3}$
- 1 answer mark for $b=3$

Tip

- *Remember that when dilating probability density functions, the horizontal and vertical dilation factors will always be reciprocals of each other so that the area under the function remains constant.*

Question 8b.ii.

Worked solution

The probability density function (pdf) of *X* has been dilated by a factor of $\frac{1}{3}$ in the *x*-direction (from the y-axis) to create the pdf of *Y*.

Therefore, $\text{E}(Y)=\frac{1}{3}\times\text{E}(X)=\frac{1}{3}\times 2=\frac{2}{3}$.

Mark allocation: 1 mark

- 1 mark for $\text{E}(Y)=\frac{2}{3}$

Question 8b.iii.

Worked solution

The pdf of *X* has been dilated by a factor of $\frac{1}{3}$ in the *x*-direction to create the pdf of *Y*.

Therefore, $\text{Var}(Y) = \left(\frac{1}{3}\right)^2 \times \text{Var}(X) = \frac{2}{9}$.

Mark allocation: 1 mark

- 1 mark for $\text{Var(Y)} = \frac{2}{9}$

Question 9a.

Worked solution

Let $f(x) = 0$

$$\begin{aligned}\sqrt{-x^3+kx^2} &= 0\\ -x^3+kx^2 &= 0\\ -x^2(x-k) &= 0\\ x &= 0, k\\ a &= k\end{aligned}$$

Mark allocation: 2 marks

- 1 method mark for equating $f(x)$ to 0
- 1 answer mark for $a = k$

Tip

- *Remember to read the question carefully, as a common mistake might be to think that this question is asking for the maximum value of the function.*

Question 9b.i.

Worked solution

P is at $\left(p,\ \sqrt{-p^3+kp^2}\right)$.

Length of chord is given by

$$\begin{aligned}&\sqrt{(p-0)^2+\left(\sqrt{-p^3+kp^2}-0\right)^2}\\ &=\sqrt{p^2+\left(-p^3+kp^2\right)}\\ &=\sqrt{-p^3+(k+1)p^2},\ \text{as required.}\end{aligned}$$

Mark allocation: 1 mark

- 1 mark for clear reasoning, including a substitution into $\sqrt{\Delta x^2+\Delta y^2}$ or equivalent

Tip

- *You must know the formula for the distance between two points in the plane – it is not given on the formula sheet.*

Question 9b.ii.

Worked solution

$$L = \sqrt{-p^3 + (k+1)p^2}$$
$$= \left(-p^3 + (k+1)p^2\right)^{\frac{1}{2}}$$
$$\frac{dL}{dp} = \left(-3p^2 + 2(k+1)p\right) \times \frac{1}{2} \times \left(-p^3 + (k+1)p^2\right)^{-\frac{1}{2}}$$
$$= \frac{-3p^2 + 2(k+1)p}{2\sqrt{-p^3 + (k+1)p^2}}$$

Maximum occurs when $\frac{dL}{dp} = 0.$

$$0 = \frac{-3p^2 + 2(k+1)p}{2\sqrt{-p^3 + (k+1)p^2}}$$
$$0 = -3p^2 + 2(k+1)p$$
$$0 = -p(3p - 2(k+1))$$
$$p = 0, \frac{2(k+1)}{3}$$
$$p > 0$$
$$p = \frac{2(k+1)}{3}$$

Alternatively, minimise the squared distance $-x^3 + (k+1)x^2$ instead of minimising the distance.

Mark allocation: 3 marks

- 1 answer mark for calculating the derivative of $\sqrt{-x^3 + (k+1)x^2}$ or $-x^3 + (k+1)x^2$
- 1 method mark for equating the derivative to zero
- 1 answer mark for $p = \frac{2(k+1)}{3}$

Tips

- *When maximising (or minimising) distance functions, it is often easier to maximise the square distance.*
- *When omitting a solution, you should provide a reason for the omission.*

END OF WORKED SOLUTIONS