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MATHS METHODS 3 & 4 TRIAL EXAMINATION 1 SOLUTIONS 2019

Question 1 (3 marks)

a.

$$y = \log_e(4-2x^3)$$

$$\frac{dy}{dx} = \frac{-6x^2}{4-2x^3}$$

$$= \frac{-6x^2}{2(2-x^3)}$$

$$= \frac{-3x^2}{2-x^3}$$

(1 mark)

b.

$$f(x) = \frac{e^x}{\tan(x)}$$

$$f'(x) = \frac{\tan(x)\times e^x - \sec^2(x)\times e^x}{\tan^2(x)}$$

(1 mark)

$$f'\left(\frac{\pi}{4}\right) = \frac{\tan\left(\frac{\pi}{4}\right)e^{\left(\frac{\pi}{4}\right)} - \sec^2\left(\frac{\pi}{4}\right)\times e^{\left(\frac{\pi}{4}\right)}}{\tan^2\left(\frac{\pi}{4}\right)}$$

$$= e^{\left(\frac{\pi}{4}\right)} - 2\times e^{\left(\frac{\pi}{4}\right)}$$

$$= -e^{\left(\frac{\pi}{4}\right)}$$

Now $\tan\left(\frac{\pi}{4}\right)=1$, so $\tan^2\left(\frac{\pi}{4}\right)=1$

and $\sec^2\left(\frac{\pi}{4}\right) = \frac{1}{\cos^2\left(\frac{\pi}{4}\right)}$

$$= \frac{1}{\left(\frac{1}{\sqrt{2}}\right)^2}$$

$$= \frac{1}{1/2}$$

$$= 2$$

(1 mark)

Question 2 (3 marks)

$$g'(x)=\frac{1}{3x-1}-\frac{1}{4}, \qquad x>\frac{1}{3}$$

$$g(x)=\int\left(\frac{1}{3x-1}-\frac{1}{4}\right)dx$$

$$=\frac{1}{3}\log_e(3x-1)-\frac{x}{4}+c$$ **(1 mark)**

Since $g(1)=\frac{3}{4}$,

$$\frac{3}{4}=\frac{1}{3}\log_e(2)-\frac{1}{4}+c$$

$$c=1-\frac{1}{3}\log_e(2)$$ **(1 mark)**

So $g(x)=\frac{1}{3}\log_e(3x-1)-\frac{x}{4}+1-\frac{1}{3}\log_e(2)$

(1 mark)

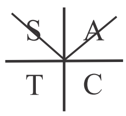
Question 3 (2 marks)

$$\left(\cos^2(\theta)-\frac{1}{2}\right)\left(\sin(\theta)-\frac{1}{2}\right)=0, \qquad 0\le\theta\le\pi$$

$$\left(\cos(\theta)-\frac{1}{\sqrt{2}}\right)\left(\cos(\theta)+\frac{1}{\sqrt{2}}\right)\left(\sin(\theta)-\frac{1}{2}\right)=0$$

$\cos(\theta)=\frac{1}{\sqrt{2}}$, or $\cos(\theta)=-\frac{1}{\sqrt{2}}$, or $\sin(\theta)=\frac{1}{2}$

$\theta=\frac{\pi}{4}$, $\qquad\theta=\frac{3\pi}{4}$, $\qquad\theta=\frac{\pi}{6},\frac{5\pi}{6}$



(1 mark) for solutions from first bracket
(1 mark) for solutions from second bracket

Question 4 (2 marks)

$$\text{sd}(\hat{P}) = \sqrt{\frac{p(1-p)}{n}}$$

We require $\sqrt{\frac{p(1-p)}{n}} \geq 0.04$

$$\sqrt{\frac{0.2 \times 0.8}{n}} \geq 0.04$$ **(1 mark)**

$$\frac{0.16}{n} \geq 0.0016$$

$$0.16 \geq 0.0016n \qquad (n > 0)$$

$$n \leq \frac{0.16}{0.0016}$$

$$n \leq \frac{1\,600}{16}$$

$$n \leq 100$$

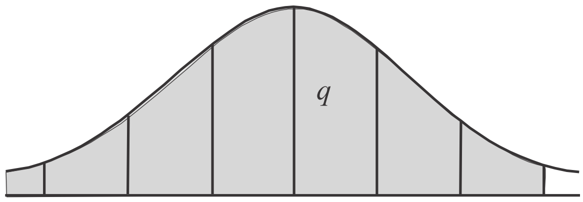
So the largest sample size, i.e. the largest value of n is 100.

(1 mark)

Question 5 (3 marks)

a. $\mu(X) = 20$, $\text{var}(X) = 9$ so $\text{sd}(X) = 3$

Draw a diagram. Note that the shaded area represents $\Pr(X < 29)$ which equals *q*.



$$\begin{aligned}\Pr(Z < -3) &= \Pr(Z > 3)\\ &= \Pr(X > 29)\\ &= 1 - q\end{aligned}$$

(1 mark)

b. Method 1 – using the diagram

The sample space is reduced to $Z < 3$ i.e. $X < 29$ i.e. q. Also $\Pr(Z < 0) = 0.5$

So, $\Pr(Z > 0 \mid Z < 3)$

$$= \frac{q - 0.5}{q}$$

(1 mark) for numerator **(1 mark)** for denominator

Method 2 – using the formula

$\Pr(Z > 0 \mid Z < 3)$

$$= \frac{\Pr(Z > 0 \cap Z < 3)}{\Pr(Z < 3)}$$

$$= \frac{\Pr(0 < Z < 3)}{\Pr(Z < 3)}$$ **(1 mark)**

$$= \frac{q - 0.5}{q}$$ **(1 mark)**

Question 6 (6 marks)

a. horizontal asymptote: $y=1$

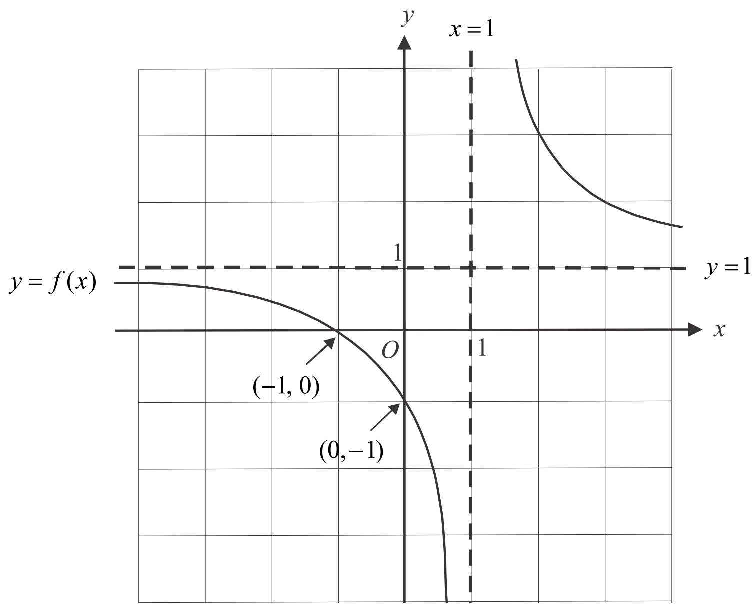
vertical asymptote: $x=1$

x-intercepts occur when $y=0$

$$0=1+\frac{2}{x-1}$$
$$-1=\frac{2}{x-1}$$
$$-1(x-1)=2$$
$$-x+1=2$$
$$x=-1$$

y-intercepts occur when $x=0$

$$y=1+\frac{2}{0-1}$$
$$y=1-2$$
$$y=-1$$



(1 mark) – correct intercepts **(1 mark)** – correct asymptotes **(1 mark)** – correct shapes

b. $$f(x)=1+\frac{2}{x-1}$$

Let $y=1+\dfrac{2}{x-1}$

Swap x and y for inverse.

$$x=1+\frac{2}{y-1}$$
$$x-1=\frac{2}{y-1}$$
$$y-1=\frac{2}{x-1}$$
$$y=1+\frac{2}{x-1}$$
$$f^{-1}(x)=1+\frac{2}{x-1}$$ **(1 mark)**

$d_f=R\backslash\{1\}$ and $r_f=R\backslash\{1\}$

So $d_{f^{-1}}=R\backslash\{1\}$ **(1 mark)**

c. Note that f and f^{-1} are self-inverses so $f(x)=f^{-1}(x)$ for $x\in R\backslash\{1\}$. **(1 mark)**

Question 7 (5 marks)

a. Set up a table showing the discrete distribution.

x	0	1	2	3
$\Pr(X = x)$	k	k	$4k$	$9k$

(1 mark)

$$k+k+4k+9k=1$$
$$15k=1$$
$$k=\frac{1}{15}$$

(1 mark)

b. $\text{E}(X)=\frac{12}{5}$ (given)

$$\text{E}(X^2)=0^2\times\frac{1}{15}+1^2\times\frac{1}{15}+2^2\times\frac{4}{15}+3^2\times\frac{9}{15}$$
$$=\frac{1}{15}+\frac{16}{15}+\frac{81}{15}$$
$$=\frac{98}{15}$$

(1 mark)

$$\text{var}(X)=\text{E}(X^2)-\{\text{E}(X)\}^2$$
$$=\frac{98}{15}-\left(\frac{12}{5}\right)^2$$
$$=\frac{98}{15}-\frac{144}{25}$$
$$=\frac{980-864}{150}$$
$$=\frac{116}{150}$$
$$=\frac{58}{75}$$

(1 mark)

c. Method 1 – using the table

x	0	1	2	3
$\Pr(X = x)$	$\frac{1}{15}$	$\frac{1}{15}$	$\frac{4}{15}$	$\frac{9}{15}$

$$\Pr(X\le 2\,|\,X>0)$$
$$=\left(\frac{1}{15}+\frac{4}{15}\right)\div\left(\frac{1}{15}+\frac{4}{15}+\frac{9}{15}\right)$$
$$=\frac{5}{15}\div\frac{14}{15}$$
$$=\frac{5}{15}\times\frac{15}{14}$$
$$=\frac{5}{14}$$

(1 mark)

Method 2 – using the formula

$$\Pr(X\le 2\,|\,X>0)$$
$$=\frac{\Pr(X\le 2\cap X>0)}{\Pr(X>0)}\quad\text{(formula sheet}$$
$$=\frac{\Pr(X=1)+\Pr(X=2)}{\Pr(X>0)}$$
$$=\left(\frac{1}{15}+\frac{4}{15}\right)\div\left(\frac{1}{15}+\frac{4}{15}+\frac{9}{15}\right)$$
$$=\frac{5}{15}\times\frac{15}{14}$$
$$=\frac{5}{14}$$

(1 mark)

Question 8 (5 marks)

a. We have $P(x, y)$ i.e. $P\left(x, \sqrt{x}\right)$ and $Q(1,0)$.

Let D = the distance from P to Q.

$$D = \sqrt{(x-1)^2 + \left(\sqrt{x} - 0\right)^2} \quad \text{(distance formula)}$$
$$= \sqrt{x^2 - 2x + 1 + x}$$
$$= \sqrt{x^2 - x + 1}$$
$$= (x^2 - x + 1)^{\frac{1}{2}}$$

(1 mark)

$$\frac{dD}{dx} = \frac{1}{2}(x^2 - x + 1)^{-\frac{1}{2}} \times (2x - 1) \quad \text{(chain rule)}$$
$$= \frac{2x-1}{2\sqrt{x^2 - x + 1}}$$

$$\frac{dD}{dx} = 0 \text{ for min/max.}$$
$$\frac{2x-1}{2\sqrt{x^2-x+1}} = 0$$

So $2x - 1 = 0$ (Note, if the denominator equals zero then $\frac{dD}{dx}$ is undefined.)

$$x = \frac{1}{2}$$

From the graph we see that we must have a minimum rather than a maximum at $x = \frac{1}{2}$.

(1 mark)

Substitute $x = \frac{1}{2}$ into

$$D = \sqrt{x^2 - x + 1}$$
$$= \sqrt{\frac{1}{4} - \frac{1}{2} + 1}$$
$$= \sqrt{\frac{1}{4} - \frac{2}{4} + \frac{4}{4}}$$
$$= \sqrt{\frac{3}{4}}$$
$$= \frac{\sqrt{3}}{2}$$

Minimum distance is $\frac{\sqrt{3}}{2}$ units.

(1 mark)

b. Let an image point be (x', y').

$$\begin{bmatrix} x' \\ y' \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & -2 \end{bmatrix}\begin{bmatrix} x \\ y \end{bmatrix} + \begin{bmatrix} 0 \\ 2 \end{bmatrix}$$

$$= \begin{bmatrix} x \\ -2y \end{bmatrix} + \begin{bmatrix} 0 \\ 2 \end{bmatrix}$$

$$= \begin{bmatrix} x \\ -2y+2 \end{bmatrix}$$

So $x' = x$ and $y' = -2y + 2$

$x = x'$ $\quad 2y = 2 - y'$

$$y = \frac{2 - y'}{2}$$

Since $f(x) = \sqrt{x}$

let $y = \sqrt{x}$

The image equation is

$$\frac{2 - y'}{2} = \sqrt{x'}$$

$$2 - y' = 2\sqrt{x'}$$

$$-y' = -2 + 2\sqrt{x'}$$

$$y' = 2 - 2\sqrt{x'}$$

(1 mark)

So $h(x) = 2 - 2\sqrt{x}$

Q is the point $(1,0)$.

$$h(1) = 2 - 2\sqrt{1}$$

$$= 0$$

So Q lies on h.

(1 mark)

Question 9 (5 marks)

a. $f(x)=2x\cos(x), \quad x\ge 0$

$$\frac{d}{dx}(2x\sin(x))=f(x)+2\sin(x)$$

$$\begin{aligned} LS &= \frac{d}{dx}(2x\sin(x)) \\ &= 2\sin(x)+2x\cos(x) \quad \text{(product rule)} \\ &= 2\sin(x)+f(x) \\ &= RS \end{aligned}$$

as required.

(1 mark)

b. $\int_0^{n\pi} f(x)\,dx$

$=\int_0^{n\pi}\frac{d}{dx}(2x\sin(x))\,dx-\int_0^{n\pi}2\sin(x)\,dx$ (using part **a.**) **(1 mark)**

$=\left[2x\sin(x)\right]_0^{n\pi}+\left[2\cos(x)\right]_0^{n\pi}$

$=(2n\pi\sin(n\pi)-0)+(2\cos(n\pi)-2\cos(0))$

$=2n\pi\times 0+(2\times 1-2\times 1)$

$=2-2$

$=0$

(1 mark)

Note that since n is a positive, even integer,
$\sin(n\pi)=0$ and $\cos(n\pi)=1$. Also, $\sin(0)=0$ and $\cos(0)=1$.

c. Method 1 – using part **b.**

From part **b.**, $\int_0^{n\pi} f(x)\,dx=0$ where $n=2, 4, 6....$

So, $\int_0^{2\pi} f(x)\,dx=0$

From the graph we have $\int_0^{a} f(x)\,dx+\int_a^{2\pi} f(x)\,dx=0$ **(1 mark)**

$$-(3\pi+2)+\int_a^{2\pi} f(x)\,dx=0$$

$$\int_a^{2\pi} f(x)\,dx=3\pi+2$$

Required area is $3\pi+2$ square units.

(1 mark)

Method 2 – "otherwise"

Find a by solving $f(x)=0,$ for $x \geq 0$

$$2x\cos(x)=0$$

$$x=0 \text{ or } \cos(x)=0$$

$$x=\frac{\pi}{2},\frac{3\pi}{2},\frac{5\pi}{2}...$$

From the graph, we see that $a=\frac{3\pi}{2}$.

$$\text{shaded area} = \int_{\frac{3\pi}{2}}^{2\pi} f(x)\,dx$$ **(1 mark)**

$$=\left[2x\sin(x)\right]_{\frac{3\pi}{2}}^{2\pi}+\left[2\cos(x)\right]_{\frac{3\pi}{2}}^{2\pi} \quad \text{(using part **b.**'s solution)}$$

$$=\left(4\pi\sin(2\pi)-3\pi\sin\left(\frac{3\pi}{2}\right)\right)+\left(2\cos(2\pi)-2\cos\left(\frac{3\pi}{2}\right)\right)$$

$$=0-3\pi\times{}^{-}1+2\times1-2\times0$$

$$=3\pi+2$$

Required area is $3\pi+2$ square units.

(1 mark)

Question 10 (6 marks)

a.

$$f(x)-g(x)=1$$

$$2\log_e(x)-\log_e\left(x-\frac{e}{4}\right)=1$$

$$\log_e(x^2)-\log_e\left(x-\frac{e}{4}\right)=1$$

$$\log_e\left(\frac{x^2}{x-\frac{e}{4}}\right)=1$$

$$e^1=\frac{x^2}{x-\frac{e}{4}}$$

(1 mark)

$$e\left(x-\frac{e}{4}\right)=x^2$$

$$ex-\frac{e^2}{4}=x^2$$

$$x^2-ex+\frac{e^2}{4}=0$$

$$\left(x-\frac{e}{2}\right)\left(x-\frac{e}{2}\right)=0$$

$$x=\frac{e}{2}$$

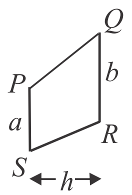
as required

(1 mark)

b. $$A=\int_{\frac{e}{2}}^{\frac{e(e+1)}{4}}\left(f(x)-g(x)\right)dx$$

(1 mark)

c.



area of a trapezium

$= \frac{1}{2}(a+b) \times h$ (where a and b are the parallel sides ie PS and QR)

From part **a.**, when $x = \frac{e}{2}$, $f(x) - g(x) = 1$, so $d(PS) = 1$.

The y-coordinate of Q is $f\left(\frac{e(e+1)}{4}\right) = 2\log_e\left(\frac{e(e+1)}{4}\right)$.

The y-coordinate of R is
$$g\left(\frac{e(e+1)}{4}\right) = \log_e\left(\frac{e(e+1)}{4} - \frac{e}{4}\right)$$
$$= \log_e\left(\frac{e^2+e-e}{4}\right)$$
$$= \log_e\left(\frac{e^2}{4}\right)$$

(1 mark)

$$d\,(QR) = 2\log_e\left(\frac{e\,(e+1)}{4}\right) - \log_e\left(\frac{e^2}{4}\right)$$
$$= \log_e\left(\frac{e^2\,(e+1)^2}{16}\right) - \log_e\left(\frac{e^2}{4}\right)$$
$$= \log_e\left(\frac{e^2\,(e+1)^2}{16} \div \frac{e^2}{4}\right)$$
$$= \log_e\left(\frac{e^2\,(e+1)^2}{16} \times \frac{4}{e^2}\right)$$
$$= \log_e\left(\frac{(e+1)^2}{4}\right)$$

$$\text{area of trapezium} = \frac{1}{2}(a+b) \times h$$
$$= \frac{1}{2}(d(PS) + d\,(QR)) \times \left(\frac{e\,(e+1)}{4} - \frac{e}{2}\right)$$

(1 mark)

$$= \frac{1}{2}\left(1 + \log_e\left(\frac{(e+1)^2}{4}\right)\right) \times \left(\frac{e\,(e+1)}{4} - \frac{e}{2}\right)$$
$$= \frac{1}{2}\left(\frac{e^2+e-2e}{4}\right) \times \left(1 + \log_e\left(\frac{(e+1)^2}{4}\right)\right)$$
$$= \left(\frac{e^2-e}{8}\right) \times \left(1 + \log_e\left(\frac{(e+1)^2}{4}\right)\right)$$

So $c = \frac{e^2-e}{8}$ and $d = \frac{(e+1)^2}{4}$.

(1 mark)