THE SCHOOL FOR EXCELLENCE (TSFX)
UNITS 3 & 4 MATHEMATICAL METHODS 2018
WRITTEN EXAMINATION 2 – SOLUTIONS

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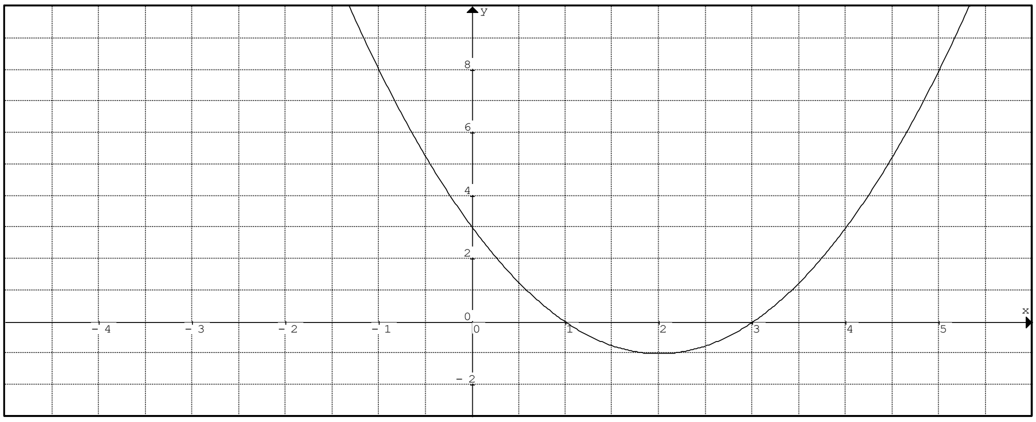
SECTION A

1	2	3	4	5	6	7	8	9	10
D	C	E	C	A	E	C	D	D	E

11	12	13	14	15	16	17	18	19	20
B	D	B	A	A	D	C	D	A	C

QUESTION 1 **Answer is D**

From the graph, $x \in (-\infty, 0\,]$ for $y \ge 0$.



QUESTION 2 Answer is C

$f(x) = (x-1)(x+3)P(x) + m(x-1) + n$

Let $x = 1$: $-10 = (1-1)(1+3)P(1) + m(1-1) + n$
$n = -10$

Let $x = -3$: $2 = (-3-1)(-3+3)P(-3) + m(-3-1) + n$
$2 = -4m + n$
$2 = -4m - 10$
$m = -3$

QUESTION 3 Answer is E

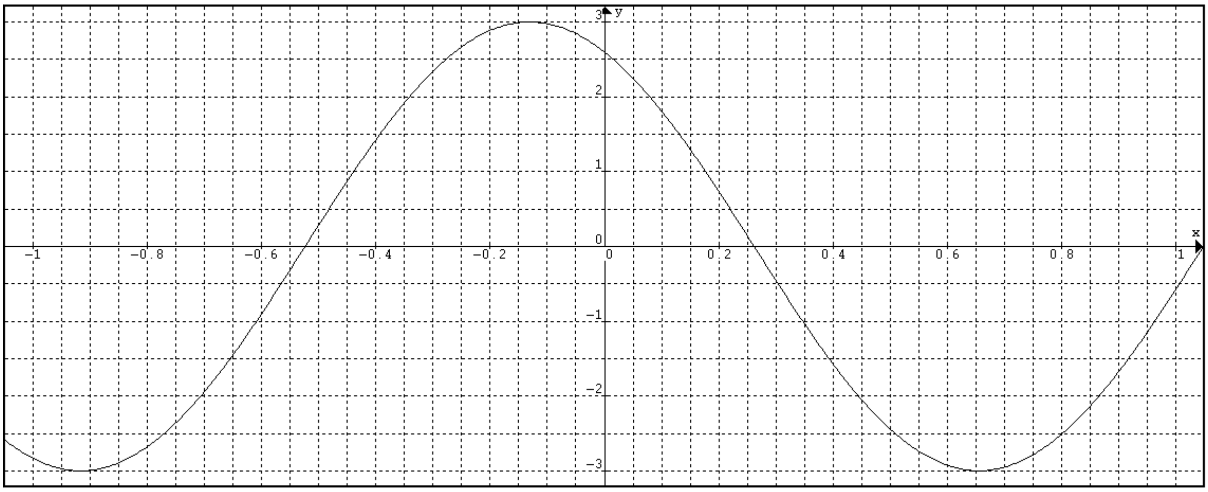
The equations could be written as: $y = -(a+2)x + 3$ and $y = -\frac{a}{2}x + \frac{b}{2}$

Linear equations will have a unique solution if the gradients are **not** equal.

Equating the gradients: $-(a+2) = -\frac{a}{2}$ if $a = -4$.

A unique solution will exist if $a \in R \setminus \{-4\}$ and b can take any value.

QUESTION 4 Answer is C



The maximum value on the graph occurs at $x = -\frac{\pi}{24}$ and so going any further to the left would cause the function to not be 1-to-1. Hence the largest value of a is $\frac{\pi}{24}$.

QUESTION 5 Answer is A

$g(x) = x^3 - 4x$

Dilation by a factor of 2 from the *y*-axis: $g_1(x) = \left(\frac{x}{2}\right)^3 - 4\left(\frac{x}{2}\right) = \frac{x^3}{8} - 2x$

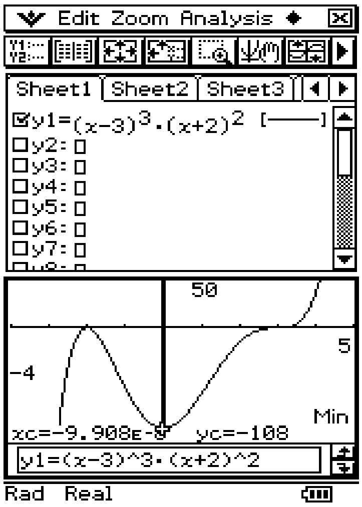
Reflection in the *y*-axis: $g_2(x) = \frac{(-x)^3}{8} - 2(-x) = -\frac{x^3}{8} + 2x$

Translation of 2 units in the negative direction of the x-axis: $g_3(x) = -\frac{(x+2)^3}{8} + 2(x+2)$

Hence $f(x) = 4 + 2x - \frac{(x+2)^3}{8}$.

QUESTION 6 Answer is E

The minimum turning point for the graph of $y = (x+2)^2(x-3)^3$ occurs at $(0,-108)$. Therefore, if anything greater than 108 is added to the equation there would only be one *x*-intercept.



Hence $-p > 108$ and so $p < -108$.

Also, if $p > 0$, the maximum turning point $(-2,0)$ and stationary point of inflection $(3,0)$ will be translated down under the x-axis.

Hence $p > 0$ and $p < -108$ satisfy.

QUESTION 7 Answer is C

Consider each function in the functional equation $f(x+y) = f(x)f(y)$:

Option A: $f(x) = \log_e(x)$

L.H.S. $= \log_e(x+y)$
R.H.S. $= \log_e(x)\log_e(y)$. L.H.S. $\neq$ R.H.S.

Option B: $f(x) = e^{(x-1)}$

L.H.S. $= e^{(x+y-1)}$
R.H.S. $= e^{(x-1)} \times e^{(y-1)} = e^{(x+y-2)}$. L.H.S. $\neq$ R.H.S.

Option C: $f(x) = e^{-x}$

L.H.S. $= e^{-(x+y)}$
R.H.S. $= e^{-x} \times e^{-y} = e^{-(x+y)}$. L.H.S. = R.H.S

QUESTION 8 Answer is D

$f(x)=x^3g(x)-3x$

$f'(x)=x^3g'(x)+3x^2g(x)-3$

$f'(3)=(3)^3g'(3)+3(3)^2g(3)-3$

$f'(3)=(27)(2)+(27)(-1)-3$

$f'(3)=54-27-3=24$

QUESTION 9 Answer is D

Average rate of change = $\dfrac{f(4)-f(2)}{4-2}=\dfrac{60-6}{4-2}=27$

QUESTION 10 Answer is E

The graph of $y=f(x)$ has a stationary point of inflection at $x=1$ and a local minimum at $x=6$ so B, C and D are incorrect. A is incorrect as it requires the inclusion at $x=6$.
The answer is E.

QUESTION 11 Answer is B

$$\int\left(\frac{3}{2x-1}\right)dx=\int\left(\frac{3\times2\times\frac{1}{2}}{2x-1}\right)dx=\frac{3}{2}\int\left(\frac{2}{2x-1}\right)dx=\frac{3}{2}\log_e(2x-1)+c$$

$$\therefore\int_1^k\left(\frac{3}{2x-1}\right)dx=\frac{3}{2}\left[\log_e(2x-1)\right]_1^k$$

$$\frac{3}{2}\left[\log_e(2x-1)\right]_1^k=\frac{3\log_e 5}{2}$$

$$\therefore\frac{3}{2}\left(\log_e(2k-1)-\log_e 1\right)=\frac{3\log_e 5}{2}$$

$$\frac{3}{2}\log_e(2k-1)=\frac{3}{2}\log_e 5$$

$$2k-1=5$$

$$k=3$$

QUESTION 12 Answer is D

$$\int_{-1}^{3}\left(4f(x)-3x\right)dx=\int_{-1}^{3}\left(4f(x)\right)dx-3\int_{-1}^{3}\left(x\right)dx$$

$$=2\int_{-1}^{3}\left(2f(x)\right)dx-3\left[\frac{x^2}{2}\right]_{-1}^{3}$$

$$=-2\int_{3}^{-1}\left(2f(x)\right)dx-3\left(\frac{9}{2}-\frac{1}{2}\right)$$

$$=-2(-5)-3(4)$$

$$=10-12=-2$$

QUESTION 13 Answer is B

As $\frac{d}{dx}(x\log_e x)=1+\log_e x$ then $\int\left(1+\log_e x\right)dx=x\log_e x+c$

$$\int(1)dx+\int\left(\log_e x\right)dx=x\log_e x+c$$
$$x+\int\left(\log_e x\right)dx=x\log_e x+c$$
$$\int\left(\log_e x\right)dx=x\log_e x-x+c$$

QUESTION 14 Answer is A

$$Area=\int_{0}^{2}\left(\sqrt{4-x^2}\right)dx-\int_{0}^{1}2\sqrt{(1-x)}\,dx$$
$$=\pi-\frac{4}{3}$$
$$=\frac{3\pi-4}{3}\text{ square units.}$$

QUESTION 15 Answer is A

The probability that they are all blue is $\frac{5}{8}\times\frac{4}{7}\times\frac{3}{6}$.

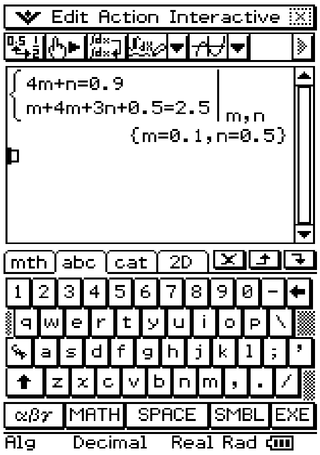
The probability that they are all red is $\frac{3}{8}\times\frac{2}{7}\times\frac{1}{6}$.

The probability that they are all the same colour $=\frac{60+6}{8\times7\times6}=\frac{11}{56}$.

So the probability that they are NOT all the same colour is $1-\frac{11}{56}=\frac{45}{56}$.

QUESTION 16 Answer is D

Using simultaneous equations for the sum of the probabilities and the mean, $E(X) = 2.5$ gives the values $m = 0.1$ and $n = 0.5$.



$$E(X^2) = m + (4\times 2m) + 9n + (25\times 0.1)$$
$$= m + 8m + 9n + 2.5$$
$$= 9m + 9n + 2.5$$
$$= 7.9$$

Variance $= E(X^2) - [E(X)]^2$
$$= 7.9 - 2.5^2$$
$$= 1.65$$

QUESTION 17 Answer is C

Binomial: $n = 10,\ p = 0.7$

Let $X =$ number that gain entry within 6 minutes

$$\Pr(X \ge 8) = 0.3827828$$
$$\Pr(X = 9) = 0.1210608$$

$$\Pr(9\ enter\ |\ge 8\ enter) = \frac{\Pr(9\ enter \cap \ge 8\ enter)}{\Pr(\ge 8\ enter)}$$

$$= \frac{\Pr(9\ enter)}{\Pr(\ge 8\ enter)}$$

$$= \frac{0.12106.....}{0.38278.....} = 0.316265$$

QUESTION 18 Answer is D

$\mu = 50$ and $\sigma = 2$

Required probability is $\Pr(X < 42) + \Pr(X > 54)$

When $X = 42$ then $Z = -4$
When $X = 54$ then $Z = 2$

So the probability is: $\Pr(Z < -4) + \Pr(Z > 2)$

QUESTION 19 Answer is A

$z_1 = \frac{5-\mu}{\sigma}$ and $z_2 = \frac{8-\mu}{\sigma}$

$\Pr(X < 5) = 0.15$

$\Pr\left(z_1 < \frac{5-\mu}{\sigma}\right) = 0.15$ therefore, $z_1 = -1.03643$

$\Pr(X < 8) = 0.70$

$\Pr\left(z_2 < \frac{8-\mu}{\sigma}\right) = 0.70$ therefore, $z_2 = 0.524401$

$-1.03643 = \frac{5-\mu}{\sigma}$

$\therefore -1.03643\sigma = 5-\mu$ Equation 1

$0.524401 = \frac{8-\mu}{\sigma}$

$\therefore 0.524401\sigma = 8-\mu$ Equation 2

Equation 1 – Equation 2:

$1.56083\sigma = 3$

$\therefore \sigma = 1.92205$

Substitute $\sigma = 1.92205$ into Equation 1: $\mu = 6.99208$

QUESTION 20 Answer is C

$p = 0.45$

A 95% confidence interval gives $z = 1.96$

$M = 0.01$

$$M = Z_{\alpha/2}\sqrt{\frac{\hat{p}\left(1-\hat{p}\right)}{n}}$$

$$0.01 = 1.96\sqrt{\frac{0.45(1-0.45)}{n}}$$

$n = 9507.96$

Therefore, 9508 people need to be surveyed.

SECTION B

Marking Legend:

- $\left(\text{A}\frac{1}{2}\times 4\downarrow\right)$ means four answer half-marks rounded **down** to the next integer.
  Rounding occurs at the end of a part of a question.
- **M**1 = 1 **M**ethod mark.
- **A**1 = 1 **A**nswer mark (it **must** be this or its equivalent).
- **H**1 = 1 consequential mark (**H**is/**H**er mark…correct answer from incorrect statement or slip, arithmetic slip preventing an **A** mark).

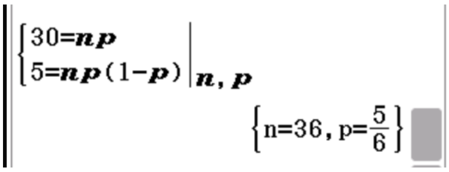
QUESTION 1 (18 marks)

a. $S_1 \sim N(60,2)$

$\Pr(X<58)=0.0786$ **A1**

b. **(i)**

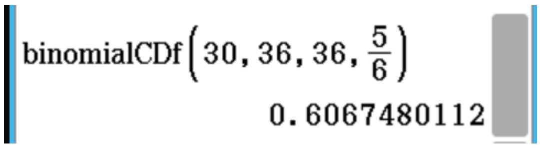
$S_2 \sim Bi(n,p)$



Giving $n=36,\ p=\frac{5}{6}$ **A2**

(ii)

$\Pr(30<X<36)=0.6067$



A1

(iii) $S_2 \sim Bi\left(n,\frac{5}{6}\right)$ **M1**

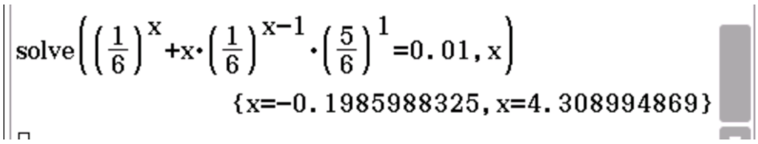
$\Pr(X\geq 2)\leq 0.99$

$1-\left[\Pr(X=0)+\Pr(X=1)\right]\leq 0.99$

Solve $\Pr(X=0)+\Pr(X=1)=0.01$

$^{n}C_0\left(\frac{5}{6}\right)^0\left(\frac{1}{6}\right)^n+{}^{n}C_1\left(\frac{5}{6}\right)^1\left(\frac{1}{6}\right)^{n-1}=0.01$ **M1**

Gives $n = 4.31$
Maximum number of walks is 5.

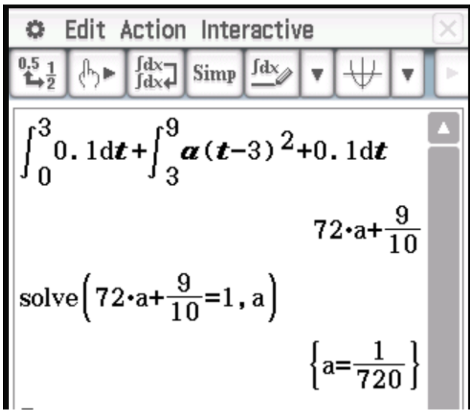


A1

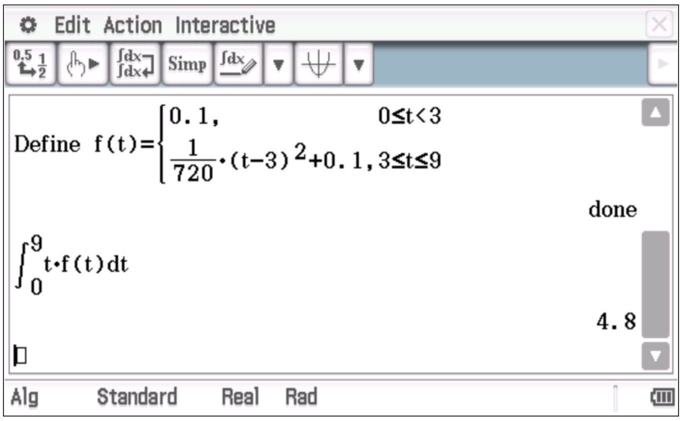
c. **(i)** $\int_0^3 (0.1)\,dt + \int_3^9 \left(a(t-3)^2 + 0.1\right)dt = 1$ **M1**

Giving $72a + 0.9 = 1$

$\therefore a = \dfrac{1}{720}$ **M1**



(ii) $E(X) = \int_0^3 t(0.1)\,dt + \int_3^9 t\left(\dfrac{1}{720}(t-3)^2 + 0.1\right)dt = 4.8$ **A1**



d. **(i)** $E\left(\hat{P}\right) = p = 0.55$

$sd\left(\hat{P}\right) = \sqrt{\dfrac{p(1-p)}{n}} = \sqrt{\dfrac{0.55(1-0.55)}{100}}$ **1M 1A**

$\therefore sd\left(\hat{P}\right) = 0.0497$

```
√(0.55(0.45)/100)
                        0.04974937186
```

(ii) $X \sim Bi(100, 0.55)$ **1M**

$\Pr(X > 60) = \Pr(X \geq 61) = 0.1343$ **1A**

```
binomialCDf(61,100,100,0.55)
                        0.1342540411
```

(iii) $Y \sim N\left(0.55, 0.0497...^2\right)$ **1M**

$\Pr(X > 60)$ = 0.157 **1A**

```
normCDf(0.6,1,0.049749,0.55)
                        0.1574375121
```

e. From **part b (i)** $S_2 \sim Bi\left(n, \dfrac{5}{6}\right)$

$E\left(\hat{P}\right) = p = \dfrac{5}{6}$

$sd\left(\hat{P}\right) = \sqrt{\dfrac{p(1-p)}{n}} = \sqrt{\dfrac{\frac{5}{6}\left(1-\frac{5}{6}\right)}{200}} = 0.02635$ **1M**

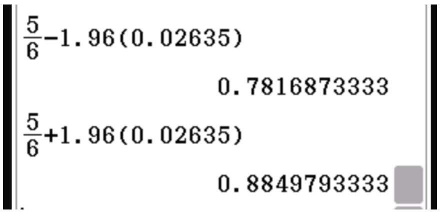
```
√((5/6)(1/6)/200)
                        0.02635231383
```

95% CI = $\left(\mu - 1.96\sigma \le X \le \mu + 1.96\sigma\right)$

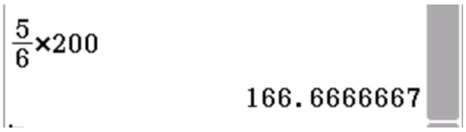
95% CI $= \left(\mu - 1.96\sigma, \mu + 1.96\sigma\right) = \left(\frac{5}{6} - 1.96 \times 0.02635, \frac{5}{6} + 1.96 \times 0.02635\right)$

CI $= \left(0.781687...., 0.884979...\right)$

CI = $\left(0.78, 0.88\right)$ **1A**



OR USING STATISTICS ON CAS: CI $= \left(0.78, 0.89\right)$



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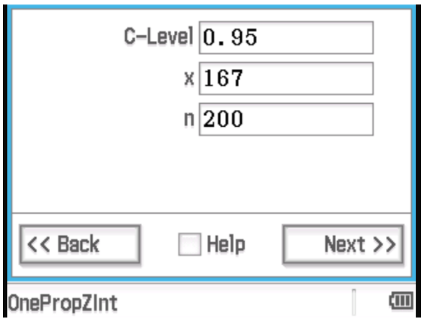
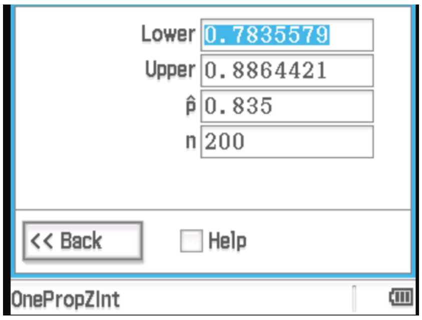
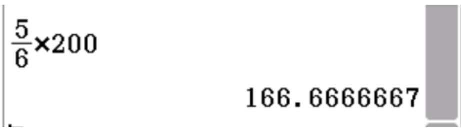




OR IF USED



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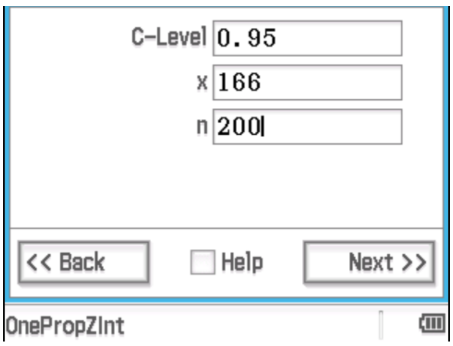
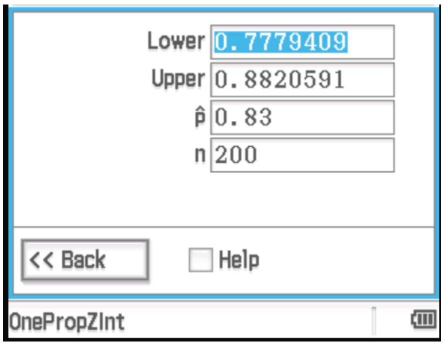
OR IF USED $\frac{5}{6} \times 200$. **CI** $= \left(0.78, 0.88\right)$

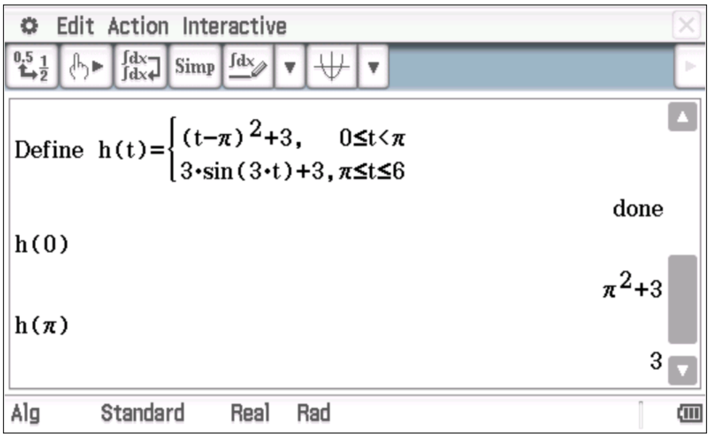




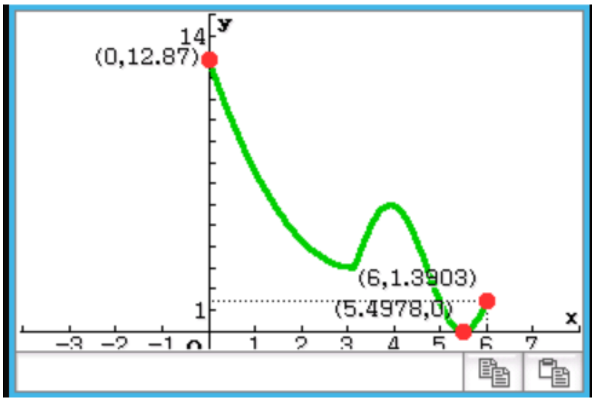
QUESTION 2 (18 marks)

a. **(i)** $h(0) = \pi^2 + 3$ **1A**

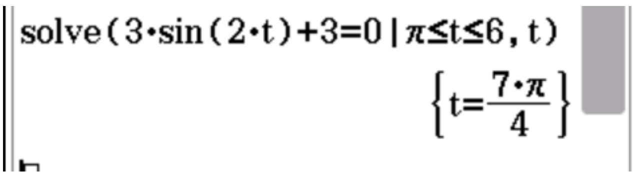
(ii) $h(\pi) = 3$ **1A**



(iii) $\{t : h(t) = 0\}$ is in the second section when $t = \dfrac{7\pi}{4}$. **1A**



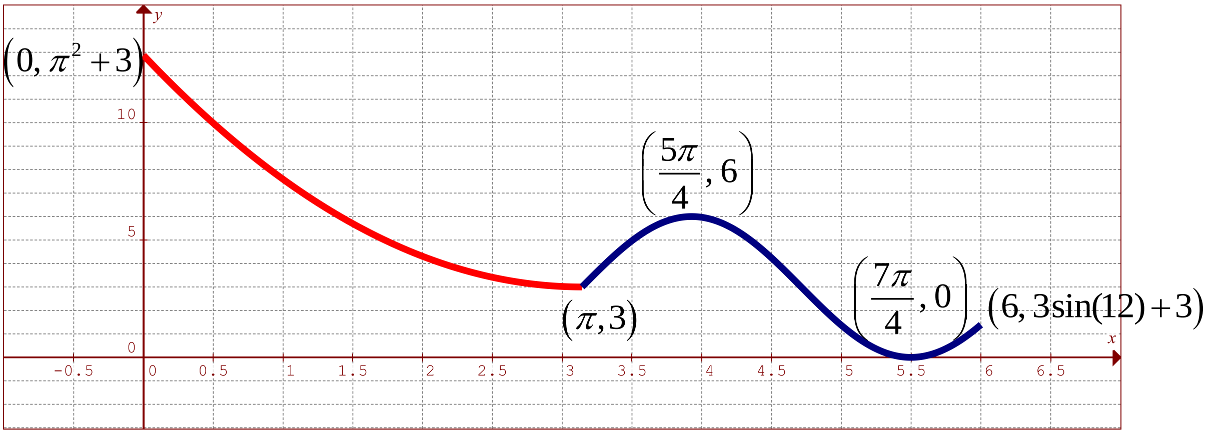
only 1 root found



b. **(i)** Yes because all points exist for $0 \le t < 6$. In particular, the two sections of the graph meet at the point $(\pi, 3)$. **2A**

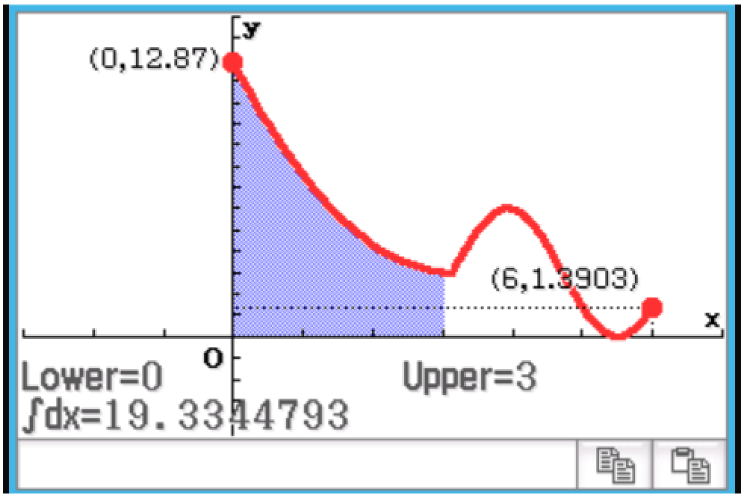
(ii) Cusp at $(\pi, 3)$, Differentiable for $t \in (0, \pi) \cup (\pi, 6)$ **1A**

c.



0.5 marks for each of the following: X intercept, Y intercept, coordinates of the endpoint, point of contact, maximum stationary point, shape. Round down to the nearest integer.

d. (i)



1A

(ii) Area = $\int_0^3 h(t)dt = \int_0^3 \left(\left(t-\pi\right)^2 + 3\right)dt$

$= \left[\frac{\left(t-\pi\right)^3}{3} + 3t\right]_0^3$ **1M**

$= \left(\frac{\left(3-\pi\right)^3}{3} + 9\right) - \left(\frac{\left(0-\pi\right)^3}{3} + 0\right)$

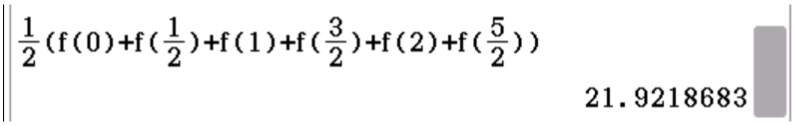
$= \frac{\left(3-\pi\right)^3}{3} + \frac{\pi^3}{3} + 9$ **1M**

(iii) Six equal width strips, width of each strip 0.5 units.

Let $f(t) = (t-\pi)^2 + 3$

Left endpoint approximation = $\frac{1}{2}\left(f(0) + f\left(\frac{1}{2}\right) + f(1) + f\left(\frac{3}{2}\right) + f(2) + f\left(\frac{5}{2}\right)\right)$

1M



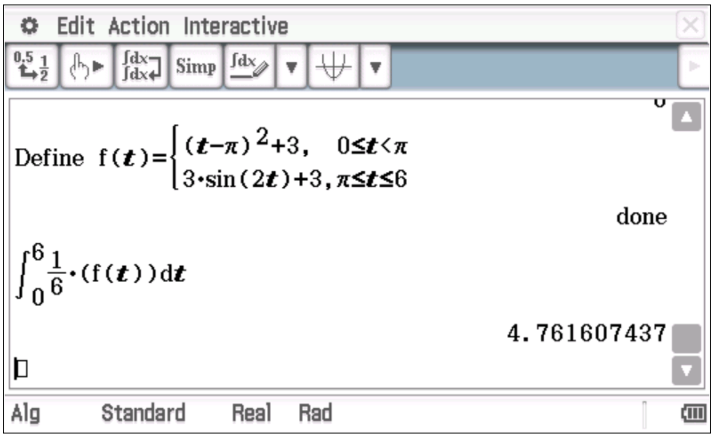
Left endpoint approximation = 21.922 **1A**

This approximation overestimates the area found in **part (ii)** because this section of the graph is concave up so each rectangle overestimates the area.

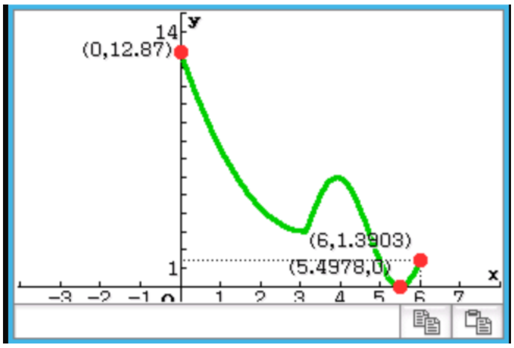
e. **(i)** For $h(t) = \begin{cases} (t-\pi)^2 + 3, 0 \le t < \pi \\ 3\sin(2t) + 3, \pi \le t \le 6 \end{cases}$

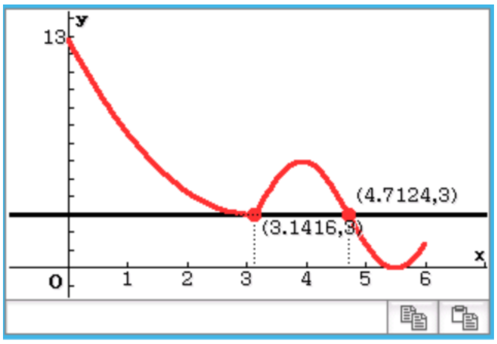
For a continuous function: Average value $= \frac{1}{6-0}\int_0^6 (f(t))\,dt$ **1M**

= $476,000 to nearest $1000 **1A**



(ii) Equation of horizontal line in graphs: $y = 3$





Equation of horizontal line in graphs: $y = 3$

Solve $h(t) < 3$:

Gives the section: $6 - 4.7124 = 1.2876$
1.2876 proportion of 6 months **1M**

Answer = 21.46%
To nearest % = 21% **1A**

QUESTION 3 (11 marks)

$v(t)=a\sin\left(\frac{5\pi}{3}(t+\beta)\right)+c$ with $v(t)$ in km/hr, t in hours and a,β, c are real constants.

a. Range = $[0,120]$, giving amplitude = 60.

Period = $\frac{2\pi}{\frac{5\pi}{3}}=\frac{6}{5}$ so for the graph to rise from $v=0$ to $v=120$ with less than one cycle, the graph must be reflected over the *t*-axis giving $a=-60$. **1M**

Vertical translation will be 60 giving $c=60$, so $c+a=0$ **1M**

b. $v(t)=-60\sin\left(\frac{5\pi}{3}(t+\beta)\right)+60$

Find β such that $v(0)=0$

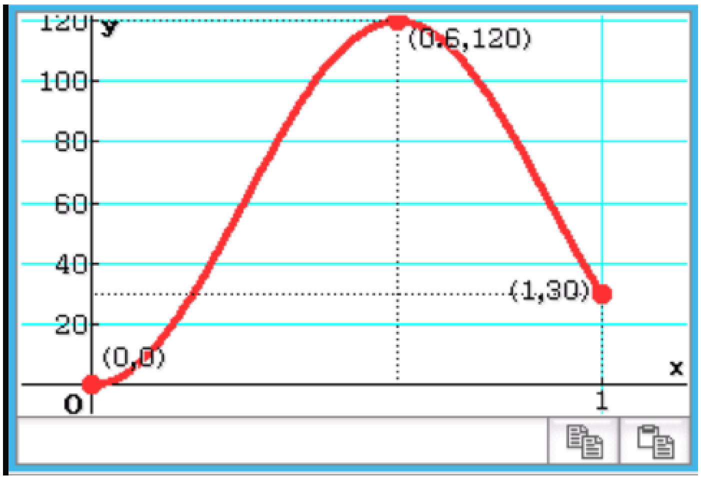
$$0=-60\sin\left(\frac{5\pi\beta}{3}\right)+60$$

$$\therefore \sin\left(\frac{5\pi\beta}{3}\right)=1$$

$$\Rightarrow \frac{5\pi\beta}{3}=\frac{\pi}{2}$$

Giving $\beta=\frac{3}{10}$ as required **1M**

We now have $v(t)=-60\sin\left(\frac{5\pi}{3}\left(t+\frac{3}{10}\right)\right)+60$



c. $x(t)=\int\left(-60\sin\left(\frac{5\pi}{3}\left(t+\frac{3}{10}\right)\right)+60\right)dt$ **1M**

$$=60\times\frac{3}{5\pi}\cos\left(\frac{5\pi}{3}\left(t+\frac{3}{10}\right)\right)+60t+c$$

$$=\frac{36}{\pi}\cos\left(\frac{5\pi}{3}\left(t+\frac{3}{10}\right)\right)+60t+c$$ **1M**

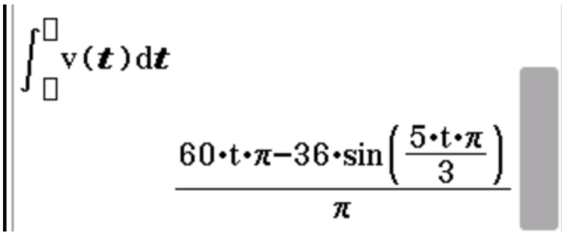
$t=0, x=0$ gives:

$$0=\frac{36}{\pi}\cos\left(\frac{5\pi}{3}\left(\frac{3}{10}\right)\right)+c$$

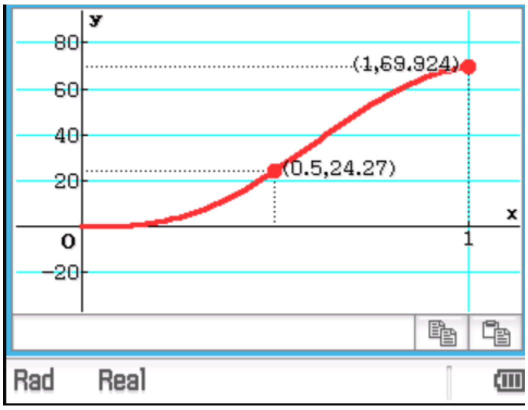
$$\therefore 0=\frac{36}{\pi}\cos\left(\frac{\pi}{2}\right)+c \therefore c=0$$

So $x(t)=\frac{36}{\pi}\cos\left(\frac{5\pi}{3}\left(t+\frac{3}{10}\right)\right)+60t$ **1A**

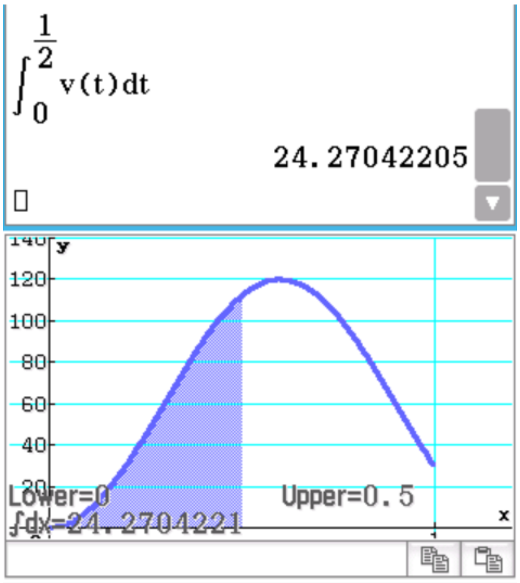
Or from CAS giving a correct but different form:



d.



using displacement graph



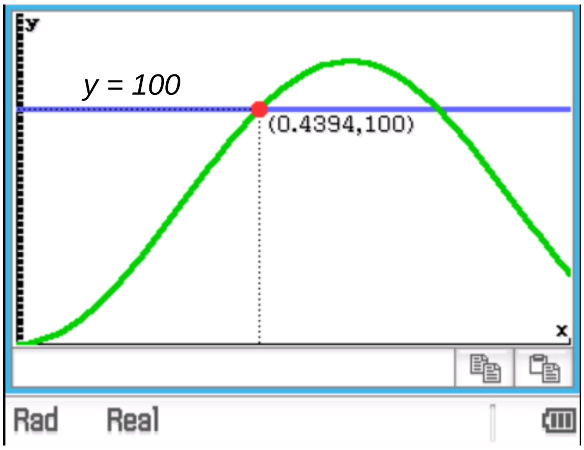
or using velocity graph **2A**

Geoff has travelled 24.3 km halfway through his one hour trip. **1A**

e. Solve $v(t) = 100$

Maximum is at $t = 0.6$
Gives by symmetry: $2 \times (0.6 - 0.4394)$ **1M**
$= 0.3212$ hour.

To the nearest minute: 19 minutes **1A**

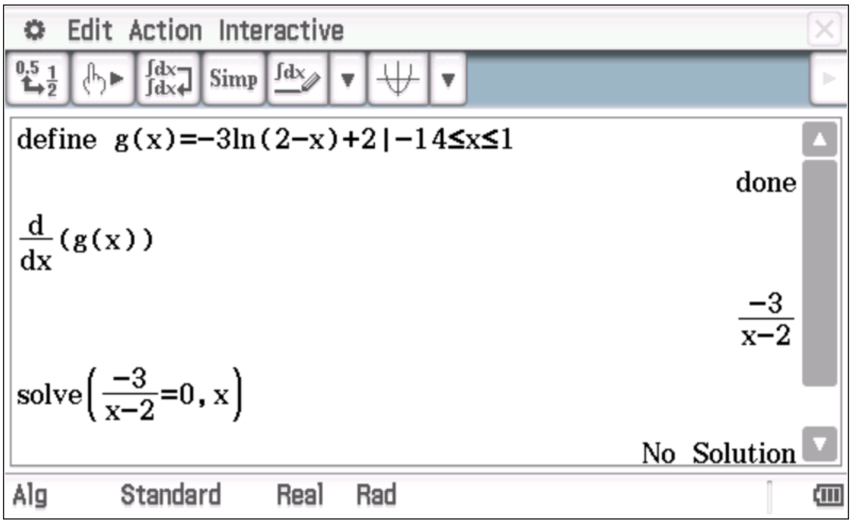


QUESTION 4 (13 marks)

Let $g:[-14,1]\to R, g(x)=-3\log_e(2-x)+2$

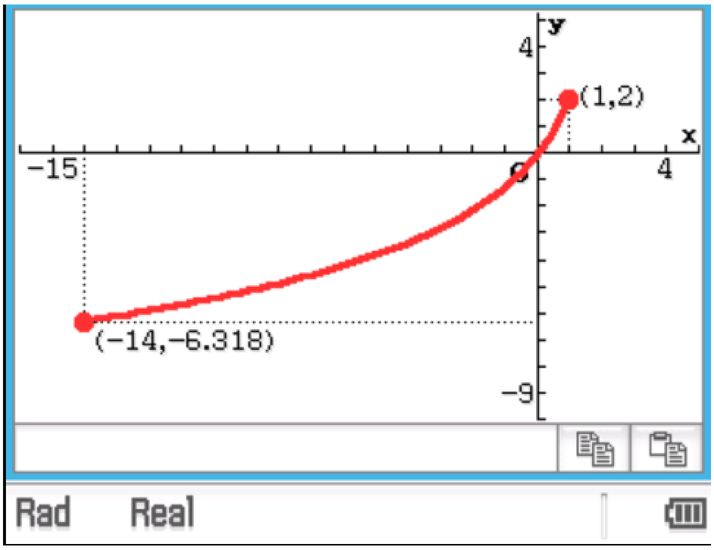
a. **(i)** $g'(x)=-\frac{3}{x-2}$ **1A**

Solve $g'(x)=0$ for stationary point(s) gives no solution. **1A**



(ii) From the graph the maximum point is at the endpoint at $x=1$. **1A**

b. **(i)**



Endpoints = $(-14,-12\log_e(2)+2)$ and $(1,2)$

x-intercept $\left(-e^{\frac{2}{3}}+2,0\right)$

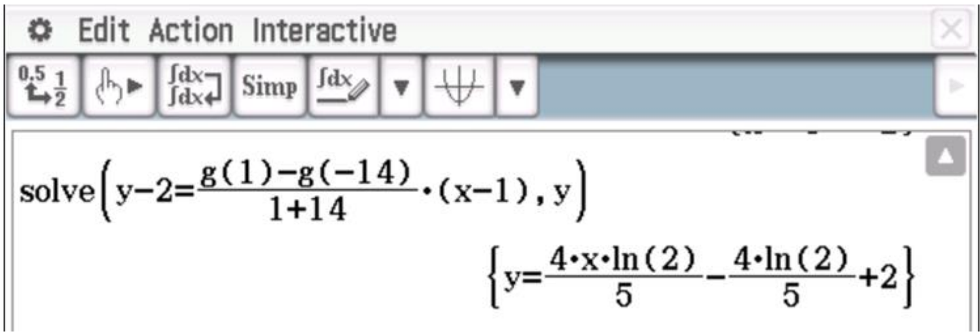
y-intercept $(0,-3\log_e(2)+2)$ **2A**

(ii) Endpoints = $\left(-14, -12\log_e(2)+2\right)$ and $\left(1, 2\right)$

Gradient = $\dfrac{-12\log_e(2)+2-2}{-14-1} = \dfrac{-12\log_e(2)}{-15} = \dfrac{4\log_e(2)}{5}$

Equation of line: $y-2=\dfrac{4\log_e(2)}{5}(x-1)$

$$y=\frac{4\log_e(2)}{5}x-\frac{4\log_e(2)}{5}+2$$ **1A**

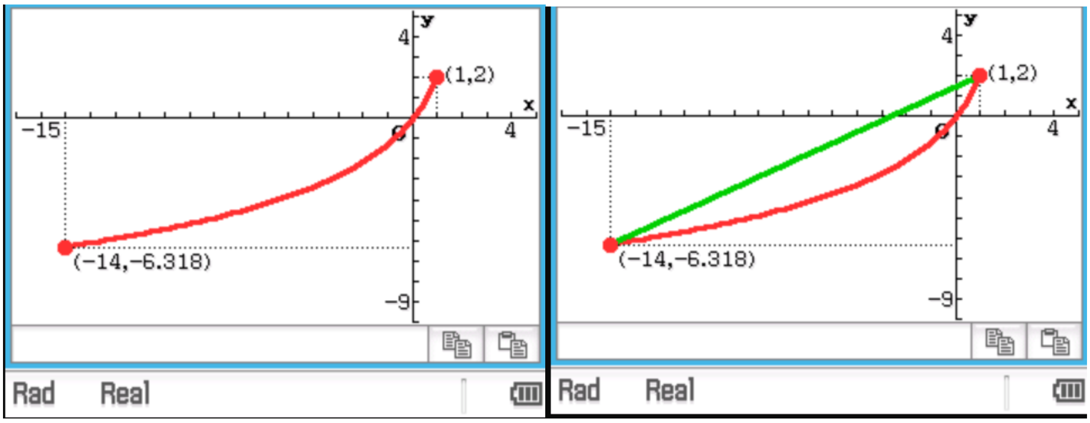


c. $Area=\displaystyle\int_{-14}^{1}\left(\frac{4\log_e(2)}{5}x-\frac{4\log_e(2)}{5}+2\right)-\left(-3\log_e(2-x)+2\right)dx$

$Area=\displaystyle\int_{-14}^{1}\left(\frac{4\log_e(2)}{5}x-\frac{4\log_e(2)}{5}+3\log_e(2-x)\right)dx$ **1M**

$Area=25.7009=25.7\ units^2$ **1A**

d. **(i)** The coordinates on the $g(x)$ graph that illustrate the mean value theorem.



These connect at the points M$\left(-14,-12\log_e(2)+2\right)$ and N$(1,2)$

From **part b (ii)** gradient MN straight = $\dfrac{4\log_e(2)}{5}$ **1M**

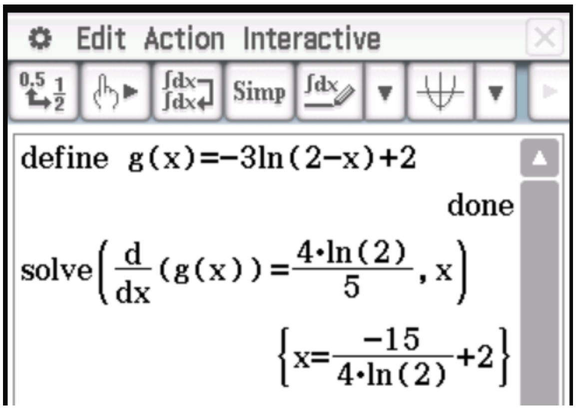
Equation MN curved: $g(x)=-3\log_e(2-x)+2$

$\therefore g'(x)=-\dfrac{3}{x-2}$

Equate $\dfrac{4\log_e(2)}{5}=-\dfrac{3}{x-2}$ **1M**

to get $x=\dfrac{-15}{4\ln(2)}+2$

Coordinates on MN curved = $\left(\dfrac{-15}{4\ln(2)}+2,-3\ln\left(\dfrac{15}{4\ln(2)}+2\right)\right)$ **1A**

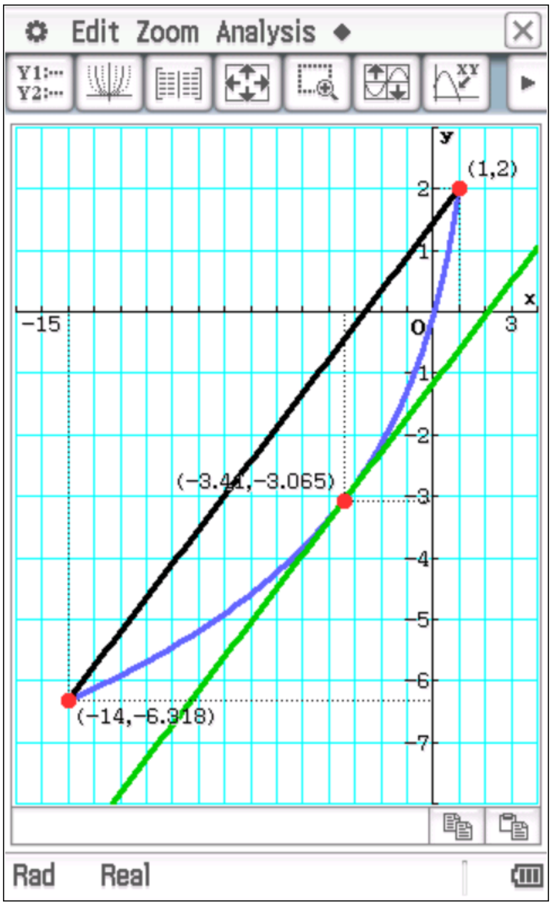


(ii) At $x = \frac{-15}{4\ln(2)} + 2$

Equation of tangent: $y = 0.554x - 1.174$ **1A**

$\text{tanLine}\left(g(x), x, -\frac{15}{4\ln(2)} + 2\right)$

$0.5545177444 \cdot x - 1.173841771$



1A