Neap:

Trial Examination 2018

VCE Mathematical Methods Units 3&4

Written Examination 2

Suggested Solutions

SECTION A – MULTIPLE-CHOICE QUESTIONS

1	A
2	B
3	C
4	D
5	A
6	A
7	B
8	C
9	E
10	C

11	E
12	A
13	B
14	D
15	C
16	B
17	A
18	D
19	E
20	D

Question 1 **A**

$x = -5 \rightarrow f(-5) = 3$

$x = 4 \rightarrow f(4) = 0$

$\therefore x \in [0, 3)$

Question 2 **B**

$$\text{average rate of change} = \frac{f(2) - f(-1)}{2 - (-1)}$$
$$= \frac{-1 - 2}{3}$$
$$= -1$$

Question 3 **C**

$$\text{distance} = \sqrt{(x_2 - x_1)^2 + (y_2 - y_1)^2}$$
$$= \sqrt{(2 - (-1))^2 + (-3 - 3)^2}$$
$$= \sqrt{9 + 36}$$
$$= \sqrt{45}$$
$$= 3\sqrt{5}$$

Question 4 **D**

Dilation factor of $\frac{1}{2}$ from x-axis:

$$y_1 = \frac{1}{2}(2^{x+1} + 2)$$
$$= 2^{-1} \times 2^{x+1} + 1$$
$$= 2^x + 1$$

Reflection in the y-axis:

$$y_2 = 2^{-x} + 1$$

Question 5 **A**

Let $g(x) = 2x^3 - 5x^2 + ax$.
If $(2x + a)$ is a factor, then $g\left(-\frac{a}{2}\right) = 0$.

$$\Rightarrow 2\left(-\frac{a}{2}\right)^3 - 5\left(-\frac{a}{2}\right)^2 + a\left(-\frac{a}{2}\right) = 0$$
$$-\frac{a^3}{4} - \frac{7a^2}{4} = 0$$
$$a^2(a + 7) = 0$$
$$\therefore a = -7 \text{ as } a \neq 0$$

Question 6 A

$$\begin{aligned}\mathrm{E}(X) &= a \times 0 + b \times 1 + 0.15 \times 2 + 0.04 \times 3 + 0.01 \times 4 \\ &= 0.91\end{aligned}$$

$b = 0.45$

$$\begin{aligned}a + b + 0.15 + 0.04 + 0.01 &= 1 \\ a + 0.45 + 0.2 &= 1 \\ a &= 0.35\end{aligned}$$

Question 7 B

$$\begin{aligned}\int_1^4 2(1 - f(x))dx &= \int_1^4 2dx - 2\int_1^4 f(x)dx \\ &= 6 - 2 \times 10 \\ &= -14\end{aligned}$$

Question 8 C

$$\begin{aligned}y &= x^2 - ax \\ &= \left(x - \frac{a}{4}\right)^2 - \frac{a^2}{4}\end{aligned}$$

$$\begin{aligned}\Rightarrow -\frac{a^2}{4} &= -4 \\ a &= \pm 4 \\ \therefore a &= 4 \text{ (only option)}\end{aligned}$$

Question 9 E

$$\begin{aligned}y_{\text{average}} &= \frac{1}{b - a}\int_a^b f(x)dx \\ &= \frac{1}{2 - 0}\int_0^2 \log_e(2x + 4)dx \\ &= 4\log_e(2) - 1 \\ &= \log_e(2^4) - 1 \\ &= \log_e(16) - 1\end{aligned}$$

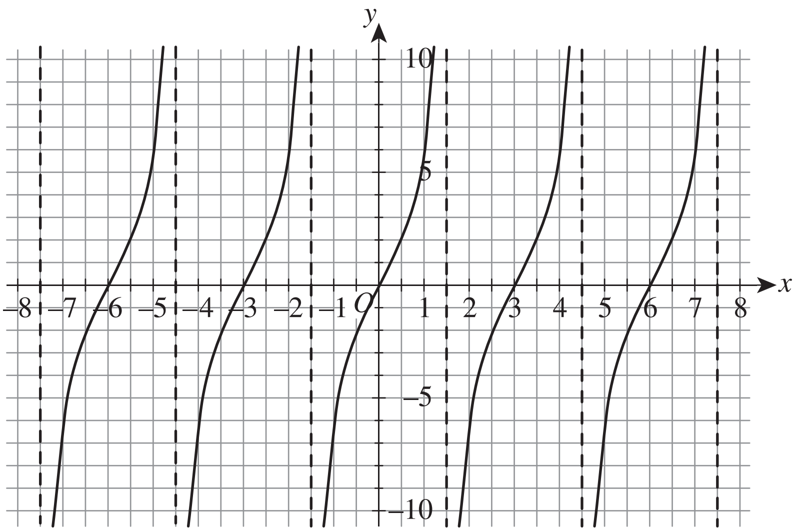
Question 10 C

	Pr(*A*)	**Pr(*A*′)**	
Pr(*B*)	0.4	0.3	0.7
Pr(*B*′)	0.1	0.2	0.3
	0.5	0.5	1

$$\begin{aligned}\Pr(A \cap B) &= \Pr(B) - \Pr(A' \cap B)\\ &= 0.7 - 0.3\\ &= 0.4\end{aligned}$$

$$\begin{aligned}\Pr(A) &= \Pr(A \cap B) + \Pr(A \cap B')\\ &= 0.4 + 0.1\\ &= 0.5\end{aligned}$$

Question 11 E



Vertical asymptotes are separated by new period and starting at $x = \frac{3}{2}$.

$$\text{period} = \frac{\pi}{\frac{\pi}{3}}$$

$$= 3$$

$\therefore$ option **E** expands to $R\backslash\left\{\frac{3}{2} + 3n\right\}$

Question 12 A

- reflection in the x-axis: $f_1(x) = -2\log_e(x) + 1$
- dilation factor of 4 from the y-axis: $f_2(x) = -2\log_e\left(\frac{x}{4}\right) - 1$
- translation of 2 units right: $g(x) = -2\log_e\left(\frac{x-2}{4}\right) - 1$

Question 13 B

$2x - ay = a - 2$

$\Rightarrow y_1 = \frac{2}{a}x + \frac{2-a}{a}$

$ax - 8y = a$

$\Rightarrow y_2 = \frac{a}{8}x - \frac{a}{8}$

For infinite solutions:

$m_1 = m_2$

$\Rightarrow \frac{a}{8} = \frac{2}{a}$

$a = \pm 4$

$c_1 = c_2$

$\Rightarrow \frac{2-a}{a} = -\frac{a}{8}$

$a = 4$

Question 14 D

$$\text{area} = \int_0^b (f(x) - g(x))dx + \int_b^c (g(x) - f(x))dx - \int_c^d g(x)dx$$

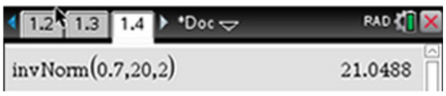
$$= \int_0^b (f(x) - g(x))dx + \int_c^b (f(x) - g(x))dx + \int_d^c g(x)dx$$

Question 15 C

$\Pr(X > a) = 0.3$

$\Rightarrow \Pr(X < a) = 0.7$

$X \sim N(20, 2^2)$



Question 16 B

The derivative graph indicates three turning points at approximately –0.6, 0 and 0.6, so the solution could be either **B** or **C**.

For $x > 0.6$, $f'(x) > 0$, option **B** is thus the correct solution.

Question 17 A

p = probability that traveller does not have ticket

$= 0.04$

$n = 500$

$$\text{sd}(\hat{p}) = \sqrt{\frac{p(1-p)}{n}}$$

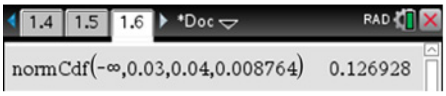
$$= \sqrt{\frac{0.04(1-0.04)}{500}}$$

$$= 0.008764$$

$X \sim N(0.04, 0.008764^2)$

$$\Pr\left(\hat{P} \leq \frac{3}{100}\right) = \Pr(X < 0.03)$$

$$= 0.1269$$



Question 18 D

$g(f(x))$ is defined if $\text{range}_f \subseteq \text{domain}_g$.

$$\text{domain}_g = \left(-\infty, \frac{1}{2}\right)$$

We need to restrict range of f to $\left(-\infty, \frac{1}{2}\right)$.

$$f(x) < \frac{1}{2}$$

$$x^2 < \frac{1}{2}$$

$$x \in \left(-\frac{\sqrt{2}}{2}, \frac{\sqrt{2}}{2}\right)$$

Question 19 E

require new period $> 2\pi$

$\Rightarrow -1 < k < 1$

This is relevant for both options **D** and **E**.

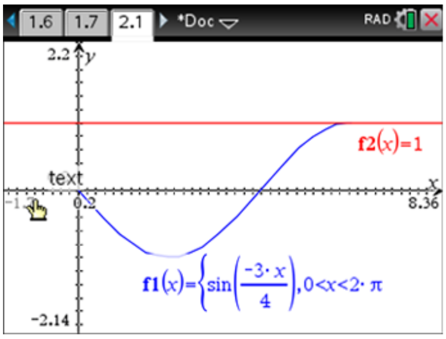
$k = \frac{1}{4}$

$\Rightarrow \sin\left(\frac{1}{4}x\right) = 1$ has a solution for $x = 2\pi$

$\therefore$ option **E** is the correct solution.

Note that $k \neq -\frac{3}{4}$.

The graph for $k = -\frac{3}{4}$ is shown below.



Question 20 D

$f(u) - f(-u) = e^{2u} - e^{-2u}$

$$\frac{(f(u))^2 - 1}{f(u)} = \frac{(e^{2u})^2 - 1}{e^{2u}}$$

$$= \frac{e^{4u} - 1}{e^{2u}}$$

$$= e^{2u} - \frac{1}{e^{2u}}$$

$$= e^{2u} - e^{-2u}$$

SECTION B

Question 1 (14 marks)

a. **i.** Let $f(x) = h$.

$$x(2 - x) = h$$

$$x^2 - 2x + h = 0 \qquad \text{M1}$$

$$x = \frac{2 \pm \sqrt{4 - 4h}}{2}$$

$$= 1 \pm \sqrt{1 - h}$$

$x_A < x_B$

$\therefore x_A = 1 - \sqrt{1 - h}$ M1

point A: $(1 - \sqrt{1 - h}, h)$

ii. $x_B = 1 + \sqrt{1 - h}$

$x_B - x_A = 1 + \sqrt{1 - h} - (1 - \sqrt{1 - h})$ M1

$\therefore \overline{AB} = 2\sqrt{1 - h}$

b. area $= lw$

$= 2\sqrt{1 - h} \times h$

$= 2h\sqrt{1 - h}$ A1

c. $A(h) = 2h\sqrt{1 - h}$

$$A'(h) = 2\sqrt{1 - h} - \frac{h}{\sqrt{1 - h}} \qquad \text{A1}$$

Let $A'(h) = 0$.

$$2\sqrt{1 - h} - \frac{h}{\sqrt{1 - h}} = 0$$

$$h = \frac{2}{3} \qquad \text{A1}$$

$$A\left(\frac{2}{3}\right) = \frac{4\sqrt{3}}{9}$$

$$\therefore \text{maximum area} = \frac{4\sqrt{3}}{9} \qquad \text{A1}$$

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d. Let $g(x) = h$.

$x(k - x) = h$

$$x = \frac{k \pm \sqrt{k^2 - 4h}}{2}$$ M1

$$\overline{QR} = \frac{k + \sqrt{k^2 - 4h}}{2} - \left(\frac{k - \sqrt{k^2 - 4h}}{2}\right)$$

$$= \sqrt{k^2 - 4h}$$ A1

e. **i.** $A(h) = h\sqrt{k^2 - 4h}$ A1

$$A'(h) = \sqrt{k^2 - 4h} - \frac{2h}{\sqrt{k^2 - 4h}}$$ A1

Let $A'(h) = 0$.

$$\sqrt{k^2 - 4h} - \frac{2h}{\sqrt{k^2 - 4h}} = 0$$

$$k^2 - 4h = 2h$$

$$h = \frac{k^2}{6}$$ M1

$$A\left(\frac{k^2}{6}\right) = \frac{k^2}{6}\sqrt{k^2 - \frac{4k^2}{6}}$$

$$= \frac{k^2}{6}\sqrt{\frac{k^2}{3}}$$

$$= \frac{\sqrt{3}\left|k^3\right|}{18}$$

$k > 0$, $\therefore$ maximum area $= \dfrac{\sqrt{3}k^3}{18}$ A1

ii. $A = h^2$ and $h = \dfrac{k^2}{6}$ for maximum area.

$\Rightarrow A = \dfrac{k^4}{36}$

From **part e. i.**, $A = \dfrac{\sqrt{3}k^3}{18}$.

$\Rightarrow \dfrac{k^4}{36} = \dfrac{\sqrt{3}k^3}{18}$

$\therefore k = 2\sqrt{3}$ A1

Graphically:

$\text{solve}\left(\sqrt{k^2-4\cdot h}=h,k\right)|h=\dfrac{k^2}{6}$

$k=-2\cdot\sqrt{3}$ or $k=0$ or $k=2\cdot\sqrt{3}$

As $k > 0$, $k = 2\sqrt{3}$. A1

Question 2 (13 marks)

a. $X \sim \text{Bi}(n, p)$

$X \sim \text{Bi}(12, 0.6)$ A1

$\Pr(X \geq 6) = 0.8418$ A1

binomCdf(12,0.6,6,12) 0.841788

b.

$$\Pr(X = 8 | X \geq 6) = \frac{\Pr(X = 8 \cap X \geq 6)}{\Pr(X \geq 6)}$$

$$= \frac{\Pr(X = 8)}{\Pr(X \geq 6)}$$ M1

$$= \frac{0.212841\ldots}{0.841788\ldots}$$

$$= 0.2528$$ A1

$\dfrac{\text{binomPdf}(12,0.6,8)}{\text{binomCdf}(12,0.6,6,12)}$ 0.252844

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c. $Y \sim \text{Bi}(n, p)$

$Y \sim \text{Bi}(4, p)$

$\Pr(Y = 2) = {}^4C_2\, p^2(1-p)^2$ M1

$= 0.0486$

$6p^2(1-p)^2 = 0.0486$

$p = 0.1$ as $0 < p < 0.4$ A1

$R \sim \text{Bi}(12, 0.1)$

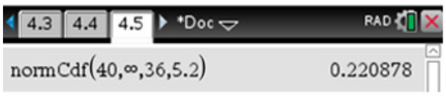
$\text{E}(R) = np$

$= 12 \times 0.1$

$= 1.2$ A1

d. $F \sim \text{N}(36, (5.2)^2)$ A1

$\Pr(F > 40) = 0.2209$ A1

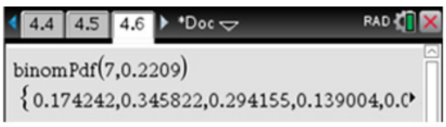


e. $L \sim \text{Bi}(7, 0.2209)$

Number of days (n)	**Pr($L = n$)**
0	0.1742...
1	0.3458...
2	0.2941...
3	
4	
5	
6	
7	

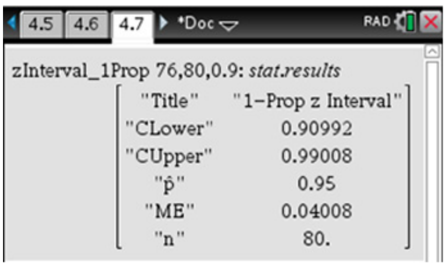
$\Pr(L = 0) + \Pr(L = 1) = 0.5160...$ (> fiftieth percentile) M1

$\therefore$ median = 1 day A1



f. $n = 80, p = \dfrac{76}{80}$

90% confidence interval: [0.9099, 0.9901] A1

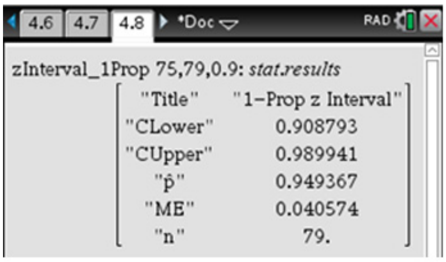


g. The new sample proportion must be either $\frac{75}{79}$ or $\frac{76}{79}$ → test each value.

$n = 79, \hat{p} = \frac{75}{79}$

90% confidence interval: [0.9088, 0.9899]

$\therefore \hat{p} = \frac{75}{79}$ A1



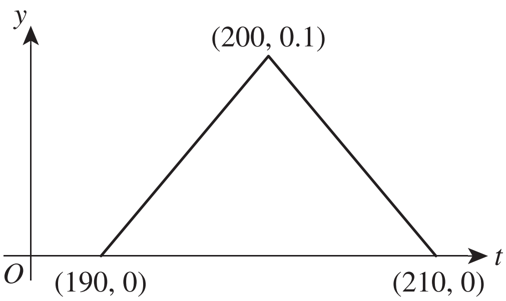
Question 3 (8 marks)

a. $\int_{190}^{200} m(t-190)dt + \int_{200}^{210} -m(t-210)dt = 1$ M1

$$100m = 1$$

$$m = 0.01$$ A1

b.



correct shape A1
correct coordinates A1

c. $\Pr(192 < F < 208) = 1 - 2\Pr(190 < F < 192)$

$$= 1 - 2\int_{190}^{192} 0.01(t-190)dt$$ M1

$$= 1 - 2 \times 0.02$$

$$= 0.9600$$ A1

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d. $\Pr(192 < F < 208 | F > 205) = \dfrac{\Pr(192 < F < 208 \cap F > 205)}{\Pr(F > 205)}$

$= \dfrac{\Pr(205 < F < 208)}{\Pr(F > 205)}$

$= \dfrac{\int_{205}^{208} -0.01(t - 210)dt}{\int_{205}^{210} -0.01(t - 210)dt}$ M1

$= \dfrac{0.105}{0.125}$

$= 0.84$ A1

Question 4 (14 marks)

a. $g(x) = \sqrt{x} \times (\sqrt{x} - 4)$ M1

$= x - 4\sqrt{x}$

b. $g'(x) = 1 - \dfrac{2}{\sqrt{x}}$ A1

Let $g'(x) = 0$.

$1 - \dfrac{2}{\sqrt{x}} = 0$

$\Rightarrow x = 4$ A1

$g(4) = -4$

$\therefore$ SP = (4, –4) A1

c. domain: $[0, \infty)$, range: $[-4, \infty)$ A1

d. $s = 4$ A1

e. Let $y = x - 4\sqrt{x}$.

$y = (\sqrt{x} - 2)^2 - 4$ M1

For inverse, swap x and y.

$x = (\sqrt{y} - 2)^2 - 4$

$(\sqrt{y} - 2)^2 = x + 4$

$\sqrt{y} = \sqrt{x + 4} + 2$ (as $y \geq 4$)

$y = (\sqrt{x + 4} + 2)^2$ A1

$\therefore h^{-1}(x) = (\sqrt{x + 4} + 2)^2$

f. domain: $[-4, \infty)$, range: $[4, \infty)$ A1

g. **i.** domain: $[4, \infty)$ A1

ii. $d(x) = (\sqrt{x+4}+2)^2 - (x - 4\sqrt{x})$

$= (\sqrt{x+4}+2)^2 - x + 4\sqrt{x}$ A1

iii. $d(x) = (\sqrt{x+4}+2)^2 - x + 4\sqrt{x}$

$= 4\sqrt{x+4} + 4\sqrt{x} + 8$ M1

$4\sqrt{x+4} > 0$ and $4\sqrt{x} > 0$ for $x \in [4, \infty)$.

$\Rightarrow d(x) > 0$ for $x \in [4, \infty)$.

$\therefore$ vertical distance > 0 and graphs do not intersect A1

h. The minimum vertical distance between $h(x)$ and $h^{-1}(x)$ occurs at the end point of $h(x)$, where $x = 4$.

$d(4) = 16 + 8\sqrt{2}$

$q(x)$ is a transformation of the graph of $y = h(x)$ by $-c$ units upwards.

$\therefore c = -16 - 8\sqrt{2}$ A1

Question 5 (11 marks)

a. **i.** $f'(x) = (x - k + 1)e^x$ A1

ii. Let $f'(x) = 0$.

$\Rightarrow x = k - 1$ A1

$f(k-1) = -e^{k-1}$

$\therefore$ stationary point: $(k-1, -e^{k-1})$ A1

b. Two solutions occur between the stationary point and x-axis, which is an asymptote for $f(x)$.

$n \in (-e^{k-1}, 0)$ A1

c. **i.** $\frac{d}{dx}[xe^x] = x \times e^x + 1 \times e^x$ *use product rule* M1

$= (x+1)e^x$ as required

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ii. $\int (x+1)e^x dx = xe^x + c$

$\int xe^x dx + \int e^x dx = xe^x + c$

$\int xe^x dx = xe^x - e^x + c$ M1

$\int f(x)dx = \int (x-k)e^x dx$

$= \int (xe^x - ke^x)dx$

$= xe^x - e^x - ke^x + c$

$= (x-k-1)e^x + c$ A1

d. area $= -\int_0^k (x-k)e^x dx$ M1

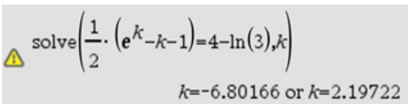
$= -\left[(x-k-1)e^x\right]_0^k$

$= e^k - k - 1$ A1

e. Using transformations: $f(x) \to$ dilation factor of 2 from x-axis and $\frac{1}{4}$ from y-axis $\to g(x)$

Therefore the area is $2 \times \frac{1}{4} = \frac{1}{2}$ of area found in **part d.**

$\Rightarrow \frac{1}{2}(e^k - k - 1) = 4 - \log_e(3)$ M1



As $k \geq 1$, $k = 2.1972$ (correct to four decimal places). A1