The Mathematical Association of Victoria

Trial Examination 2018

MATHEMATICAL METHODS

Written Examination 2

STUDENT NAME ____

Reading time: 15 minutes
Writing time: 2 hours

QUESTION AND ANSWER BOOK

Structure of examination

Section	*Number of questions*	*Number of questions to be answered*	*Number of marks*
A	20	20	20
B	4	4	60
			Total 80

- Students are permitted to bring into the examination room: pens, pencils, highlighters, erasers, sharpeners, rulers, a protractor, set-squares, aids for curve sketching, one bound reference, one approved technology (calculator or software) and, if desired, one scientific calculator. Calculator memory DOES NOT need to be cleared. For approved computer-based CAS, full functionality may be used.
- Students are NOT permitted to bring into the examination room: blank sheets of paper and/or correction fluid/tape.

Materials supplied
- Question and answer book of 22 pages.
- Formula sheet.
- Answer sheet for multiple-choice questions.

Instructions
- Write your **name** in the space provided above on this page.
- Write your **name** on the multiple-choice answer sheet.
- Unless otherwise indicated, the diagrams in this book are **not** drawn to scale.
- All written responses must be in English.

At the end of the examination
- Place the answer sheet for multiple-choice questions inside the front cover of this book.

Students are NOT permitted to bring mobile phones and/or any other unauthorised electronic devices into the examination room.

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SECTION A- Muliple-choice questions

Instructions for Section A

Answer **all** questions in pencil on the answer sheet provided for multiple – choice questions.
Choose the response that is **correct** for the question.
A correct answer scores 1; an incorrect answer scores 0.
Marks will **not** be deducted for incorrect answers.
No marks will be given if more than one answer is completed for any question.
Unless otherwise indicated, the diagrams in this book are **not** drawn to scale.

Question 1

Let $f: R \to R,\ f(x) = -2\sin(3x) + 1$. The amplitude and period of this function are respectively

A. -2 and 3
B. 3 and 2
C. -2 and $\frac{2\pi}{3}$
D. 2 and $\frac{2\pi}{3}$
E. 2 and $\frac{\pi}{3}$

Question 2

The graph of *f* with equation $f(x) = \frac{1}{(1-x)^2} + 2$ undergoes the following transformations

- a reflection in the *x*-axis followed by
- a dilation of a factor of 2 from the *y*-axis.

The equation of the graph of the image, *g* is

A. $g(x) = \frac{-4}{(2-x)^2} - 2$

B. $g(x) = \frac{2}{(1+x)^2} + 4$

C. $g(x) = \frac{-1}{(1-2x)^2} + 2$

D. $g(x) = \frac{-2}{(2-x)^2} - 2$

E. $g(x) = \frac{-4}{(2-x)^2} + 2$

SECTION A - continued
TURN OVER

Question 3

Let $g(x)=\frac{1}{1-x^2}$. Which one of the following is true for all $x, y \in R$?

A. $g(x\times y)=g(x)\times g(y)$

B. $g(x+y)=g(x)\times g(y)$

C. $g(x)=-g(-x)$

D. $g(x\times y)=g(x)+g(y)$

E. $g(x)=g(-x)$

Question 4

The range of the function $h:(-2,7]\to R, h(x)=2x^2-4x+7$ is

A. $(-2,7]$

B. $(5,77]$

C. $[5,77]$

D. $(23,77]$

E. $[23,77]$

Question 5

Consider $g(x)=\frac{1}{1+3x}$ and $f(x)=(x)^{\frac{2}{3}}$ over their maximal domains. Then

A. $f(g(x)):R\to R,$ where $f(g(x))=\left(\frac{1}{1+3x}\right)^{\frac{2}{3}}$

B. $f(g(x)):R\setminus\left\{\frac{1}{3}\right\}\to R,$ where $f(g(x))=\left(\frac{1}{1+3x}\right)^{\frac{2}{3}}$

C. $g(f(x)):R\to R,$ where $g(f(x))=\frac{1}{1+3x^{\frac{2}{3}}}$

D. $g(f(x)):R\setminus\left\{-\frac{1}{3}\right\}\to R,$ where $g(f(x))=\frac{1}{1+3x^{\frac{2}{3}}}$

E. $f(g(x)):[0,\ \infty)\to R,$ where $f(g(x))=\left(\frac{1}{1+3x}\right)^{\frac{2}{3}}$

Question 6

The product of the solutions to the equation $2\sin(2x)+1=0, x\in[0,\pi]$ is

A. $\frac{3\pi}{2}$

B. $\frac{77}{144}\pi$

C. $\frac{5}{144}\pi^2$

D. 17325

E. $77\left(\frac{\pi}{12}\right)^2$

Question 7

The equation $(k+1)x^2-2kx-(k-1)=0$ has at least one distinct real root when

A. $k\in\left(-\frac{\sqrt{2}}{2},\frac{\sqrt{2}}{2}\right)$

B. $k\in\left(-\infty,-\frac{\sqrt{2}}{2}\right)\cup\left(\frac{\sqrt{2}}{2},\infty\right)$

C. $k\in\left(-\infty,-\frac{\sqrt{2}}{2}\right]\cup\left[\frac{\sqrt{2}}{2},\infty\right)$

D. $k\in\left(-\infty,-\sqrt{2}\right]\cup\left[\sqrt{2},\infty\right)$

E. $k=-\frac{\sqrt{2}}{2}, k=\frac{\sqrt{2}}{2}$ only

Question 8

A transformation $T:R^2\to R^2$ with the rule $T\left(\begin{bmatrix}x\\y\end{bmatrix}\right)=\begin{bmatrix}-1&0\\0&4\end{bmatrix}\begin{bmatrix}x\\y\end{bmatrix}+\begin{bmatrix}1\\-2\end{bmatrix}$ maps the graph of $y=2\sqrt{x}$ onto the graph of

A. $y=4\sqrt{\frac{1}{4}-\frac{x}{4}}-2$

B. $y=8\sqrt{\frac{1}{4}-\frac{x}{4}}-8$

C. $y=4\sqrt{1-x}+2$

D. $y=4\sqrt{1-x}-2$

E. $y=8\sqrt{1-x}-2$

SECTION A - continued
TURN OVER

Question 9

For the function with the rule $f(x) = 2\sin(3x) + 2\cos(3x)$, the average rate of change of $f(x)$ with respect to *x* on the interval $\left[-\frac{\pi}{2}, \pi\right]$ is

A. -2
B. 2
C. $-\frac{8}{\pi}$
D. $-\frac{8}{3\pi}$
E. 0

Question 10

If the tangent to the graph of *f* with equation $f(x) = \sqrt{m - x^2}$ has a gradient of 3 when $x = -2$ then

A. $m = \frac{40}{9}$
B. $m = \frac{45}{4}$
C. $m = 40$
D. $m \notin R$
E. $m = 0$

Question 11

Consider $g(x) = \frac{-2}{(2x-3)^2} + 1$ over its maximal domain. The intersection point of the graphs of g^{-1} and *g* is

A. $\left(\frac{3}{2}, \frac{3}{2}\right)$
B. $\left(\frac{1}{2}, \frac{1}{2}\right)$
C. $\left(\frac{1}{2}, \frac{5}{2}\right)$
D. $(1, -1)$
E. $\left(\frac{1}{2}, 1\right)$

SECTION A - continued

Question 12

Given that g is a differentiable function, $h(x)=\log_e\left(kx^2+2\right)$ where k is a real number, and $g\left(h(x)\right)$ exists, then the derivative of $g\left(h(x)\right)$ is

A. $\dfrac{k}{kx^2+2}$

B. $\dfrac{2kx}{kx^2+2}$

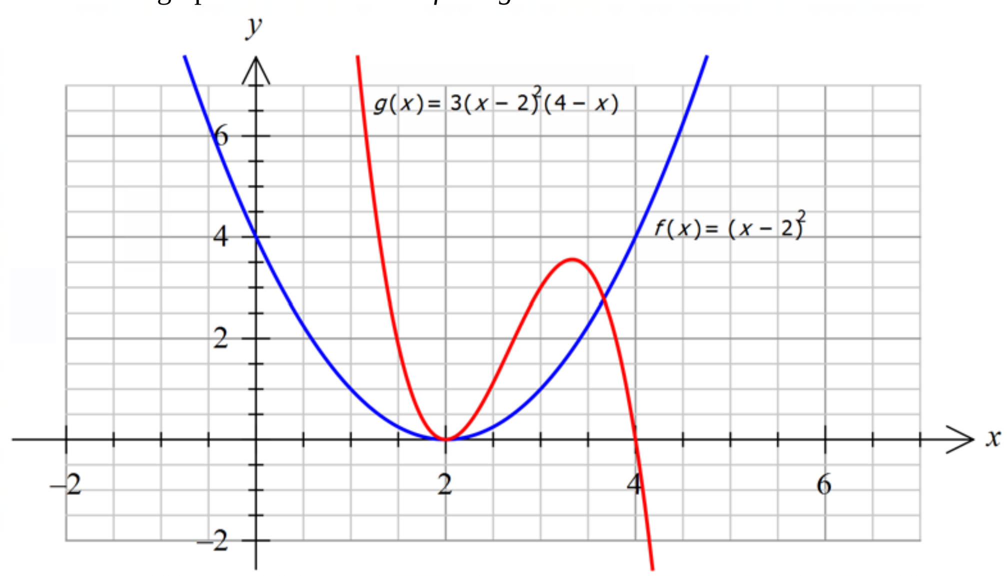
C. $\left(2kx+2\right)g'(x)$

D. $\dfrac{1}{kx^2+2}g'\left(h(x)\right)$

E. $\dfrac{2kx}{kx^2+2}g'\left(h(x)\right)$

Question 13

Consider the graphs of the functions f and g shown below.



The area enclosed by the graphs of f and g could be represented by

A. $\int_2^{\frac{11}{3}}\left(f(x)-g(x)\right)dx$

B. $\int_{\frac{11}{3}}^{2}\left(f(x)-g(x)\right)dx$

C. $\int_0^2\left(f(x)-g(x)\right)dx+\int_2^{\frac{11}{3}}\left(g(x)-f(x)\right)dx$

D. $\int_0^2\left(g(x)-f(x)\right)dx+\int_2^{\frac{11}{3}}\left(f(x)-g(x)\right)dx$

E. $\int_2^{\frac{25}{9}}\left(g(x)-f(x)\right)dx$

Question 14

Using the right endpoint method with strips of width 0.5, the approximate area contained between the curve with equation $y = \frac{2}{1+x^2}$ and the x-axis between $x = 0$ and $x = 2$ is

A. $\frac{339}{130}$

B. $\frac{339}{65}$

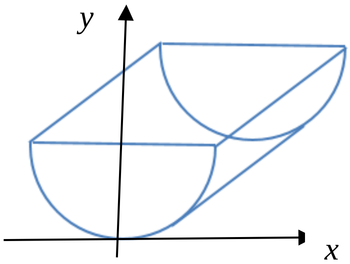
C. 2.214

D. $\frac{47}{26}$

E. $\frac{47}{13}$

Question 15

A trough, which is half of a cylinder, has a cross-section in the shape of a semicircle with equation $x^2 + (y-2)^2 = 4$ for $0 \le y \le 2$ as shown below. The dimensions of the trough are in centimetres.



The depth of water, d in cm, when the trough is one third full can be found by solving which one of the following equations?

A. $\frac{1}{3} \times \frac{1}{4} \times \pi \times 4 = \int_0^d \left(\sqrt{4-x^2} + 2\right) dx$

B. $\frac{1}{3} \times \frac{1}{4} \times \pi \times 4 = \int_0^d \sqrt{4-(y-2)^2}\, dy$

C. $\frac{1}{3} \times \frac{1}{4} \times \pi \times 4 = \int_0^d \left(-\sqrt{4-x^2} + 2\right) dx$

D. $\frac{1}{3} \times \frac{1}{4} \times \pi \times 16 = \int_0^d \sqrt{4-(y-2)^2}\, dy$

E. $\frac{1}{3} \times \frac{1}{2} \times \pi \times 4 = \int_0^d \sqrt{4-(y-2)^2}\, dy$

Question 16

A tyre manufacturer, Burnrubber, claims that 90% of their tyres last longer than 45 000 km. Customers start to complain, claiming that this is not true. Quality control inspectors decide to take a random sample of 300 tyres and find that 265 of them last longer than 45 000 km.

A 98% confidence interval, correct to three decimal places, for the proportion of tyres lasting longer than 45 000 km is

A. $(0.836,\ 0.931)$
B. $(0.883,\ 0.914)$
C. $(0.860,\ 0.940)$
D. $(0.840,\ 0.926)$
E. $(0.883,\ 0.920)$

Question 17

Events *A* and *B* are independent. If $\Pr(A) = 0.6$ and $\Pr(A \cap B) = 0.1$ then $\Pr(A' \cap B')$ equals

A. $\frac{1}{15}$
B. $\frac{1}{2}$
C. $\frac{1}{3}$
D. $\frac{1}{10}$
E. $\frac{1}{6}$

Question 18

Sam is practising his basketball skills. The probability that Sam shoots a goal is 0.2. The least number of shots Sam should make to ensure a probability of more than 0.8 of shooting a goal **more than once** is

A. 14
B. 13
C. 10
D. 8
E. 7

SECTION A - continued
TURN OVER

Question 19

The gestation period for giraffes is normally distributed.
0.6% have a gestation period greater than 155 days and 2.3% less than 110 days.
The mean and standard deviation, in days, are respectively closest to

A. 130 and 10
B. 135 and 13
C. 127 and 9
D. 120 and 14
E. 124 and 20

Question 20

A random variable *X* has a probability density function *f* with rule

$$f(x)=\begin{cases}\frac{3}{34}x & 0\le x\le 2\\ \frac{3}{68}x^2 & 2<x\le 4\\ 0 & \text{elsewhere}\end{cases}$$

The standard deviation of *X* is

A. $\frac{1153}{1445}$

B. $\frac{\sqrt{5765}}{85}$

C. $\frac{774}{85}$

D. $\frac{3\sqrt{7310}}{85}$

E. 0.893

END OF SECTION A

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TURN OVER

SECTION B

Instructions for Section B

Answer **all** questions in the spaces provided.
In all questions where a numerical answer is required, an exact value must be given unless otherwise specified.
In questions where more than one mark is available, appropriate working **must** be shown.
Unless otherwise indicated, the diagrams in this book are **not** drawn to scale.

Question 1 (12 marks)
Let $f: R \to R, f(x) = 18x^2 - 36x^4$.

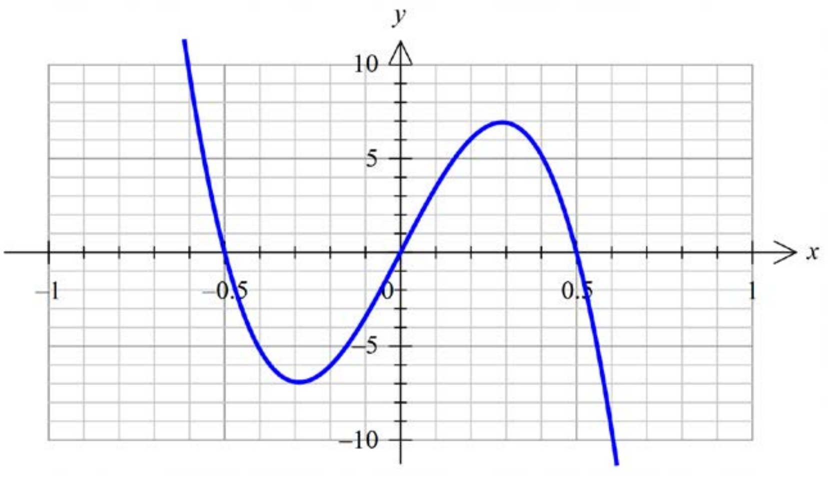
a. State the coordinates of the y-intercept of the graph of f. 1 mark

b. State the coordinates of the x-intercepts of the graph of f. 2 marks

c. Find $f'(x)$ and hence find the coordinates of the turning points of f. 2 marks

SECTION B - Question 1 - continued

Part of the graph of $y = f'(x)$ is shown below.



d. For $x \in \left[-\frac{1}{2}, \frac{1}{2}\right]$ find the *x*-value of the point where the gradient graph is at its maximum. 1 mark

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Now let $g:[-k,k] \to R, g(x) = 18x^2 - 36x^4$, $k \in R^+$.

$A(-k, g(-k))$ and $B(k, g(k))$ are two points on the graph of $y = g(x)$.

e. Find the equation of the straight line joining *A* and *B*, in terms of *k*. 1 mark

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f. **i.** The maximum area enclosed by the graphs of g and the straight line in **part e.** for $0 < k \le \frac{1}{2}$ occurs when $k = \frac{1}{2}$. Find this area. 1 mark

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ii. The maximum area enclosed by the graphs of g and the straight line in **part e.** for $0 < k \le \frac{1}{\sqrt{2}}$ occurs when $k = \frac{1}{\sqrt{2}}$. Find this area. 1 mark

iii. Find the value of k which will give the minimum area enclosed by the graphs of g and the straight line in **part e.** for $\frac{1}{2} \le k \le \frac{1}{\sqrt{2}}$. 3 marks

Question 2 (19 marks)
The continuous random variable T, which models the time, t hours, that a gymnast, Georgia, trains each day for the Commonwealth Games has the probability density function g where

$$g(t)=\begin{cases} a\left(-\cos\left(\dfrac{\pi t}{2}\right)+1\right) & 0\le t\le 10 \\ 0 & \text{elsewhere} \end{cases} \quad \text{and } a\in R.$$

a. Write down an equation, involving a definite integral, which when solved will show that $a=\dfrac{1}{10}$. 1 mark

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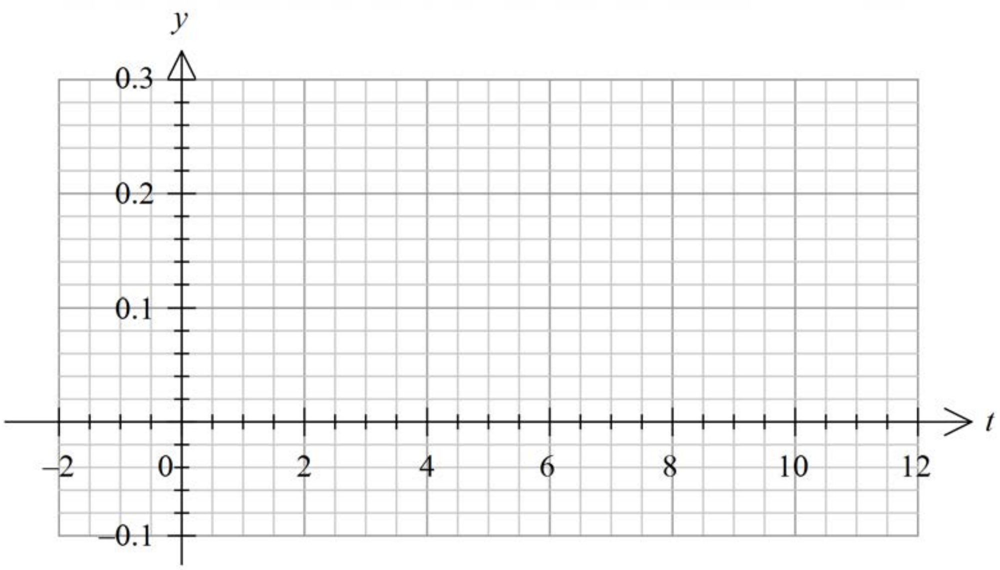
b. Sketch the graph of $y=g(t)$ on the set of axes below. Label the turning points and any points of discontinuity with their coordinates. 3 marks



Georgia's trainer, Don, uses some measures to track how long Georgia is training on any given day.

c. **i.** Determine the mean number of hours. 2 marks

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SECTION B - Question 2 - continued
TURN OVER

ii. Find the median number of hours. Give your answer correct to one decimal place. 2 marks

d. Find $\Pr(T \leq 6)$. 1 mark

e. Given that Georgia trained less than 6 hours on a particular day, find the probability she trained more than 3 hours. Give your answer correct to four decimal places. 2 marks

It frustrates Don that Georgia doesn't seem to have any pattern to her training. The number of hours that Georgia trains on one particular day is independent of the number of hours that she trains on any other day. From the information in **part d**. Don deduces that the probability Georgia trains more than 6 hours on any given day is 0.4.

f. Find the probability that Georgia trains more than 6 hours on the next three days. 1 mark

g. Given that Georgia trained more than 6 hours on at least three of the next five days, what is the probability she trained more than 6 hours on all five days? 3 marks

From the records kept by many gymnastics coaches it has been found that 55% of gymnasts competing for a place in the Commonwealth Games train for more than 6 hours per day. Don's company, in the lead up to the Commonwealth games, samples 400 such gymnasts.

h. Find the standard deviation of $\hat{P}$. Give your answer correct to four decimal places. 1 mark

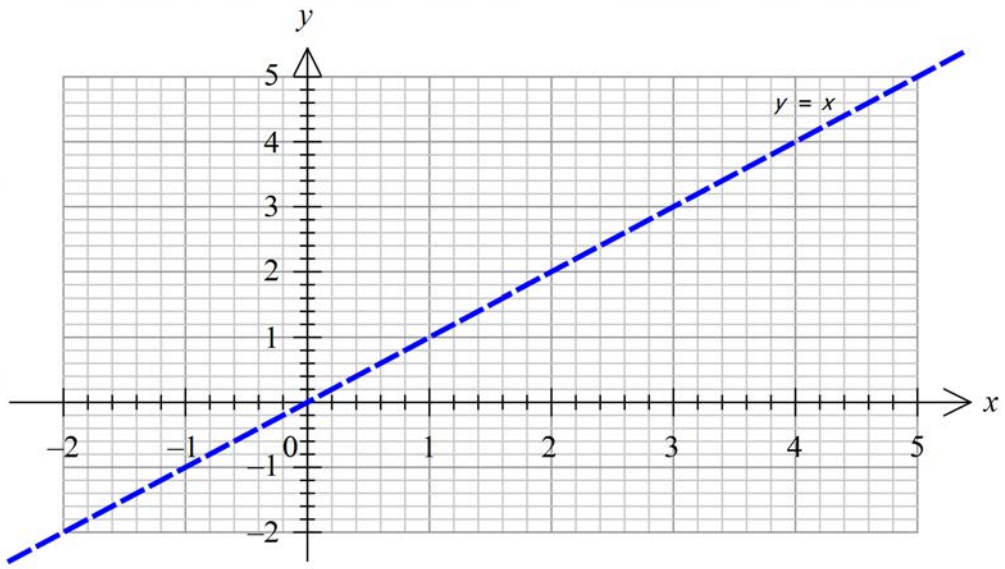
i. Find the probability that more than 60% of them train more than 6 hours per day. Give your answer correct to four decimal places. Do not use the normal approximation. 2 marks

j. Find the probability that more than 60% of them train more than 6 hours per day using the normal approximation. Give your answer correct to three decimal places. 1 mark

Question 3 (13 marks)

Let $f(x) = \log_e(x+3) + \log_e(x)$.

a. Sketch the graph *f* on the set of axes below. Label the asymptote with its equation and the axial intercept with its coordinates correct to two decimal places. 2 marks



b. Find the rule for f^{-1} and sketch the graph f^{-1} on the set of axes above. Label the asymptote with its equation and the axial intercept and the points of intersection with their coordinates correct to two decimal places. 3 marks

Consider the function f_1 where $f_1(x) = \log_e(x+3+k) + \log_e(x+k)$ and *k* is a negative real constant.

c. State the transformation required to get the graph of *f* to the graph of f_1. Give your answer in terms of *k*. 1 mark

d. Find a rule for the horizontal distance, d, between f^{-1} and the line $y = x$. 1 mark

e. Hence or otherwise, find the value of k so that the graphs of f_1 and f_1^{-1} have only one point of intersection. Give your answer correct to three decimal places. 2 marks

Now consider the function f_2 where $f_2(x) = \log_e\left(\frac{x}{a}+3\right)+\log_e\left(\frac{x}{a}\right)$ and a is a positive real constant.

f. State the transformation required to get the graph of f to the graph of f_2. Give your answer in terms of a. 1 mark

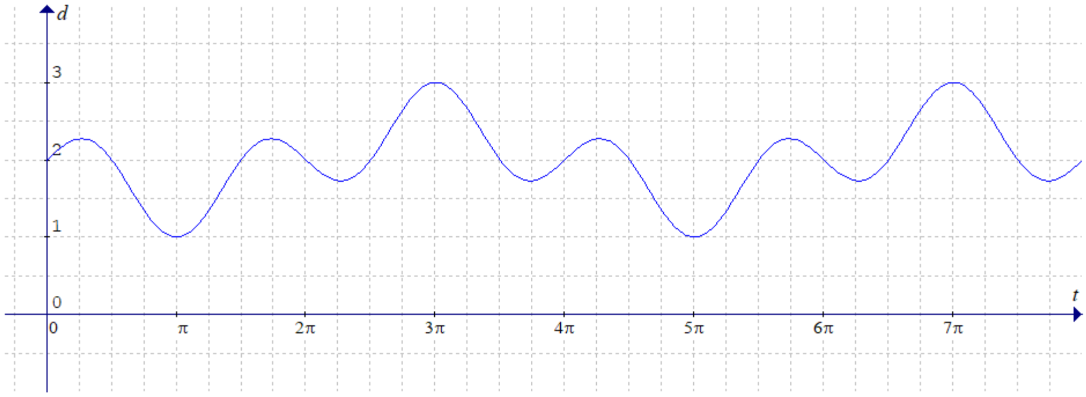
g. Find the value of x in terms of a so that $f_2'(x) = 1$. 1 mark

h. Hence, find the value of a so that the graphs of f_2 and f_2^{-1} have only one point of intersection. Give your answer correct to two decimal places. 2 marks

Question 4 (16 marks)
The depth of water at a particular spot in a wave pool can be modelled by the function

$d(t) = \sin\left(\frac{t}{2}\right)\cos(t) + 2$ where d is the depth in metres and t the time in seconds, $t > 0$.

Part of the graph of d is shown below.



a. State the period and the range of the function. 2 marks

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There is a marker on the inside of the pool 170 cm above the floor of the pool.

b. For what proportion of time, during one wave cycle, is the depth of the water greater than 170 cm? Give your answer correct to three decimal places. 2 marks

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c. Write down the equation of the derivative of d and hence find the times when the depth of water is changing the fastest for the first cycle. Give your answers correct to two decimal places. 2 marks

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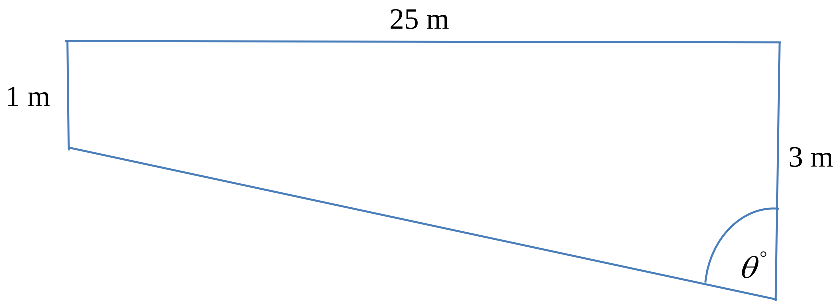
The pool needs to be emptied so that it can be renovated. A pump pumps water out at a rate given by $\frac{dV}{dt} = -\frac{1}{\left(t+\frac{\sqrt{2}}{2}\right)^2} + 2$, where $\frac{dV}{dt}$ is in m^3/min and t is the time in minutes. V is the volume of water being pumped out in m^3.

The cross-section of the pool is in the shape of a trapezium as shown below (not to scale). The height of the wall of the shallow end is one metre and the height of the wall of the deep end three metres. The length of the pool is 25 m and the width 10 m. The pool is initially full. θ° is the angle the wall of the deep end of the pool makes with the floor of the pool.



d. Find θ° correct to one decimal place. 1 mark

e. What volume of water needs to be pumped out? 2 marks

SECTION B - Question 4 continued
TURN OVER

f. How long will it take to empty the pool? Give your answer in minutes correct to two decimal places. 2 marks

g. What will be the maximum depth of water in the pool when it is half full? 1 mark

h. What is the average value of the volume of water **remaining** in the pool for the time it takes to empty the pool. Give your answer in cubic metres correct to two decimal places. 4 marks

END OF QUESTION AND ANSWER BOOK