THE SCHOOL FOR EXCELLENCE (TSFX)
UNITS 3 & 4 MATHEMATICAL METHODS 2017
WRITTEN EXAMINATION 2 – SOLUTIONS

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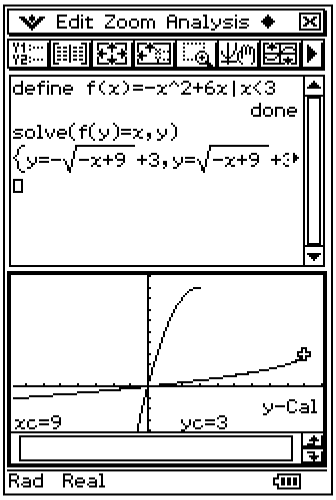
SECTION 1 – MULTIPLE CHOICE QUESTIONS

1	**2**	**3**	**4**	**5**	**6**	**7**	**8**	**9**	**10**	**11**
D	D	A	C	C	B	B	D	C	C	C

12	**13**	**14**	**15**	**16**	**17**	**18**	**19**	**20**	**21**	**22**
D	A	D	D	C	E	A	C	D	A	E

QUESTION 1 **Answer is D**

Sketch f^{-1} and $f:(-\infty,3)\to R,\ f(x)=-x^2+6x$ together, noting the domain and range.



Answer is: $f^{-1}:(-\infty,9)\to R,\ f^{-1}(x)=3-\sqrt{9-x}$

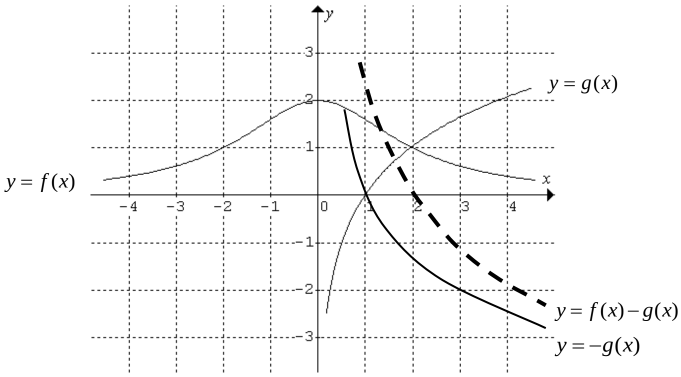
QUESTION 2 Answer is D

The vertical asymptote of a rational function occurs when the denominator (bottom line) of the fraction is equal to zero. As the vertical asymptote is $x = -b$ then the denominator must be $(x+b)$.

The horizontal asymptote of a rational function is given by the part of the equation that is separate to the fraction containing x. As the horizontal asymptote is $y = a$, the correct answer will have a "$+ a$" either before or after the fraction.

QUESTION 3 Answer is A

Subtract ordinates across the common domain i.e. $(0, \infty)$. Alternatively, draw the graph of $y = -g(x)$ then add the ordinates of $y = f(x)$ and $y = -g(x)$ across the common domain.



QUESTION 4 Answer is C

$\log_e(x^2)$ is defined for $R / \{0\}$.

$\log_e(1-x)$ is defined for $(-\infty, 1)$.

The given expression is defined across the intersection of the two individual domains:

i.e. $(-\infty, 0) \cup (0,1)$.

QUESTION 5 Answer is C

$T: R^2 \rightarrow R^2 : \begin{bmatrix} -1 & 0 \\ 0 & 2 \end{bmatrix} \left(\begin{bmatrix} x \\ y \end{bmatrix} + \begin{bmatrix} -2 \\ 1 \end{bmatrix} \right)$ gives the following equations:

$-(x-2) = x_{new}$ giving $x = 2 - x_{new}$

$2(y+1) = y_{new}$ giving $y = \dfrac{y_{new}}{2} - 1$

Substitute x and y into $y = \sqrt{x}$: $\dfrac{y_{new}}{2} - 1 = \sqrt{2 - x_{new}}$

QUESTION 6 **Answer is B**

The point $\left(0,3\right)$ is transformed to the point $\left(3,3+\sqrt{2}\right)$ by a translation of 3 units to the right and $\sqrt{2}$ units up. i.e. $y=f(x-3)+\sqrt{2}$.

The equation of curve 2 is obtained from curve 1 using the same set of transformations i.e. $y=f(x-3)+\sqrt{2}$.

Equation of curve 1: $y=f(x)$
Equation of curve 2: $y=f(x-3)+\sqrt{2}$

Therefore, the equation of the tangent to curve 2 can be determined by applying the same transformations to the equation of the tangent to curve 1:

Equation of tangent to curve 1 at $\left(0,3\right)$: $y=3-x\sqrt{2}$
Equation of tangent to curve 2 at $\left(3,3+\sqrt{2}\right)$ is:

$y=3-x\sqrt{2}$ subjected to the transformations described by $y=f(x-3)+\sqrt{2}$

$\therefore y=3-(x-3)\sqrt{2}+\sqrt{2}$

QUESTION 7 **Answer is B**

From the graph, the period of the function is 4π. By definition, the period of a sine function is $\dfrac{2\pi}{k}$. Equating periods gives:

$$\frac{2\pi}{k}=4\pi$$
$$\therefore k=\frac{1}{2}$$
$$\therefore y=\sin\left(\frac{x}{2}\right)$$

The graph of $y=\sin\left(\dfrac{x}{2}\right)$ has been reflected in the X axis. Therefore, there will be a negative sign in front of the sine function. i.e. $y=-\sin\left(\dfrac{x}{2}\right)$

From the graph, the amplitude is b. Therefore, $y=-b\sin\left(\dfrac{x}{2}\right)$.

a is the vertical translation in the positive direction. Therefore, $y=a-b\sin\left(\dfrac{x}{2}\right)$.

QUESTION 8 Answer is D

Test each option using $f(x)=5x-7$. The option whose RHS is equal to its LHS is the correct answer. For example:

Option D: $f(x+y)=f(x)+f(y)+7$

LHS: Find $f(x+y)$ given $f(x)=5x-7$

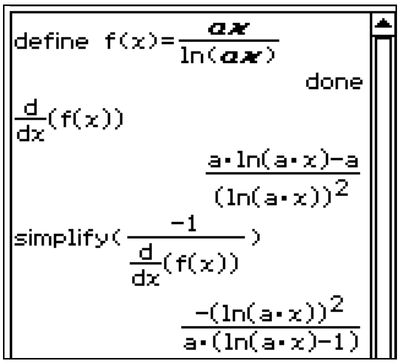
$$f(x+y)=5(x+y)-7=5x+5y-7$$

RHS: Find $f(x)+f(y)+7$ given $f(x)=5x-7$

$$f(x)+f(y)+7=(5x-7)+(5y-7)+7=5x+5y-7$$

As LHS = RHS, option D is true.

QUESTION 9 Answer is C



QUESTION 10 Answer is C

If $x=4+5\sin\left(\frac{\pi}{5}t\right)$, then $velocity=\frac{dx}{dt}=\pi\cos\left(\frac{\pi}{5}t\right)$ and $acceleration=\frac{d^2x}{dt^2}=-\frac{\pi^2}{5}\sin\left(\frac{\pi}{5}t\right)$.

Let $-\frac{\pi^2}{5}\sin\left(\frac{\pi}{5}t\right)=\frac{3\pi^2}{25}$

$$\therefore sin\left(\frac{\pi}{5}t\right)=-\frac{3}{5}$$

$$\therefore x=4+5\times\left(-\frac{3}{5}\right)=1$$

QUESTION 11 Answer is C

A function is not differentiable at cusps, sharp corners, endpoints and points of discontinuity.

The conditions that need to be taken into consideration for the given functions are continuity across all values of x and ensuring that a sharp corner does not arise at the point where the curves join.

$f(x)$ requires a 'join' at $x=0$:

$$(0-a)^3+2=b(0)+\cos(0)$$
$$\therefore -a^3+2=1$$
$$\therefore a^3=1$$
$$\therefore a=1$$

This join must be smooth i.e. the gradients at the point of contact must be the same.

$$f'(x)=\begin{cases}3(x-a)^2, & x\le 0\\ b-\sin x, & x>0\end{cases}$$

Let $3(x-a)^2=b-\sin x$. Then substitute in the value of x at the join i.e. $x=0$:

$$3(0-a)^2=b-\sin(0)$$
$$\therefore 3a^2=b$$

Substitute $a=1$ into $3a^2=b$:

$\therefore\ b=3$

QUESTION 12 Answer is D

If $f(x)=a\sin(x)-b\sqrt{3}\cos(x)$ then $f'(x)=a\cos(x)+b\sqrt{3}\sin(x)$

Let $a\cos(x)+b\sqrt{3}\sin(x)=0$ to find the turning points.

$$b\sqrt{3}\sin(x)=-a\cos(x)$$

$$\frac{\sin(x)}{\cos(x)}=-\frac{a}{b\sqrt{3}}$$

$$\therefore \tan(x)=-\frac{a}{b\sqrt{3}}$$

As the turning point occurs at $x=\frac{\pi}{3}$:

$$\tan\left(\frac{\pi}{3}\right)=-\frac{a}{b\sqrt{3}}$$
$$\therefore \sqrt{3}=-\frac{a}{b\sqrt{3}}$$
$$\therefore a=-3b$$

The answer is option C or D. Using the CAS, sketch the graph of $f(x)$ using the conditions specified in C and D. Only alternative D gives a minimum value at the required value of x.

QUESTION 13 **Answer is A**

$$\int_b^a 3g(x) - x - h(x)\,dx = 3\int_b^a g(x)\,dx - \int_b^a x\,dx - \int_b^a h(x)\,dx$$

$$= 3(2) - \int_b^a x\,dx - (-3)$$

$$= 9 - \int_b^a x\,dx$$

$$= 9 - \left[\frac{1}{2}x^2\right]_b^a$$

$$= 9 - \left(\frac{a^2}{2} - \frac{b^2}{2}\right)$$

$$= 9 - \frac{a^2}{2} + \frac{b^2}{2}$$

$$= 9 + \frac{b^2 - a^2}{2}$$

QUESTION 14 **Answer is D**

$$f(x) = \int\left(x^2 + \frac{1}{x}\right)dx = \frac{x^3}{3} + \log_e|x| + c$$

As $f(1) = \frac{4}{3}$: $\quad \frac{4}{3} = \frac{1}{3} + \log_e(1) + c$

$$\therefore c = 1$$

$$\therefore f(x) = \frac{x^3}{3} + \log_e|x| + 1$$

The average rate of change $= \dfrac{f(4) - f(2)}{4-2}$

$$\frac{f(4) - f(2)}{4-2} = \frac{1}{2}\left(\frac{64}{3} + \log_e(4) + 1 - \frac{8}{3} - \log_e(2) - 1\right)$$

$$= \frac{1}{2}\left(\frac{56}{3} + \log_e(2)\right)$$

$$= \frac{28}{3} + \log_e\left(\sqrt{2}\right)$$

QUESTION 15 Answer is D

$$\frac{d}{dx}\left(x\log_e(3x)\right)=1+\log_e(3x)$$

Integrate both sides:

$$x\log_e(3x)=x+c_1+\int\log_e(3x)\,dx$$

$$\therefore\int\log_e(3x)\,dx=x\log_e(3x)-x-c_1$$

$$\therefore\int2\log_e(3x)\,dx=2(x\log_e(3x)-x-c_2)$$

$$=2x(\log_e(3x)-1)-c$$

QUESTION 16 Answer is C

Average value of the function $=\dfrac{1}{\left(\frac{\pi}{2}-\frac{\pi}{3}\right)}\displaystyle\int_{\frac{\pi}{3}}^{\frac{\pi}{2}}\sin\left(\frac{x}{2}\right)\mathrm{dx}$

$$=\frac{6}{\pi}\left[-2\cos\frac{x}{2}\right]_{\frac{\pi}{3}}^{\frac{\pi}{2}}$$

$$=\frac{6}{\pi}\left[-2\cos\frac{\pi}{4}+2\cos\frac{\pi}{6}\right]$$

$$=\frac{6(\sqrt{3}-\sqrt{2})}{\pi}$$

QUESTION 17 Answer is E

	B	B'	
A	$3x$	x	$3x+x$
A'	$\frac{1}{3}$	$\frac{2}{5}$	$\frac{1}{3}+\frac{2}{5}=\frac{11}{15}$

$\Pr\left(A'\right)=\dfrac{11}{15}$ therefore, $\Pr\left(A\right)=1-\Pr\left(A'\right)=1-\dfrac{11}{15}=\dfrac{4}{15}$

As $\Pr\left(A\right)$ is also equal to $3x+x$: $3x+x=\dfrac{4}{15}$

$$\therefore x=\frac{1}{15}$$

$$\Pr(B)=3x+\frac{1}{3}=\frac{3}{15}+\frac{1}{3}=\frac{8}{15}$$

QUESTION 18 Answer is A

$Var(X) = E(X^2) - [E(X)]^2$

$(3)^2 = E(X^2) - (4)^2$

$9 = E(X^2) - 16$

$E(X^2) = 25$

QUESTION 19 Answer is C

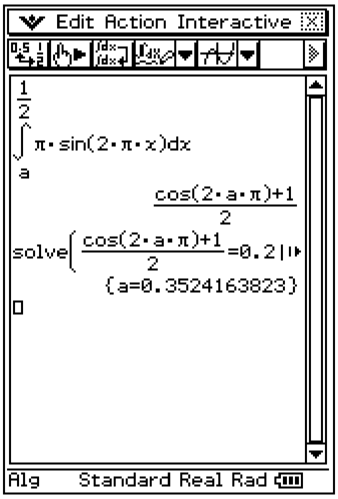
Variance $= \int_0^{\frac{1}{2}} x^2 f(x)\,dx - \mu^2$ where $\mu = \int_0^{\frac{1}{2}} x f(x)\,dx$

Answer: $\pi\int_0^{\frac{1}{2}} x^2 \sin(2\pi x)\,dx - \left(\pi\int_0^{\frac{1}{2}} x\sin(2\pi x)\,dx\right)^2$

QUESTION 20 Answer is D

Find the value of a such that $\int_a^{\frac{1}{2}} \pi\sin(2\pi x)\,dx = 0.2$.

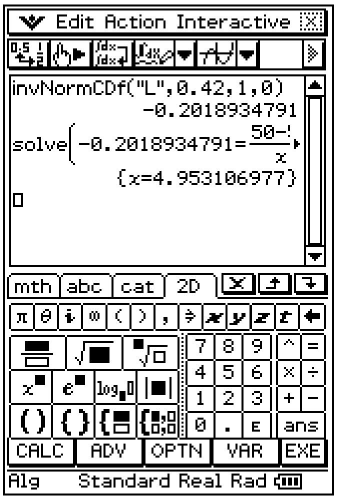
Answer: $a = 0.35$



QUESTION 21 Answer is A

$N(\mu, \sigma^2) = N(51, \sigma^2)$ and $\Pr(X < 50) = 0.42$

Giving $\Pr(Z < -0.20189) = 0.42$ Using $z = \dfrac{x-\mu}{\sigma}$. Answer $\sigma = 4.95$



QUESTION 22 **Answer is E**

The confidence level is not affected by the margin of error.

When the margin of error is small, the confidence level can be low or high or anything in between.

A confidence interval is a type of interval estimate, not a type of point estimate.

A population mean is not an example of a point estimate; a sample mean is an example of a point estimate.

SECTION 2 – EXTENDED ANSWER QUESTIONS

QUESTION 1

a. (i) $f'(x) = \frac{3}{2}x^{\frac{1}{2}} - 1 = \frac{3}{2}\sqrt{x} - 1$

$\frac{3}{2}\sqrt{x} - 1 = 0$ gives $\sqrt{x} = \frac{2}{3}$

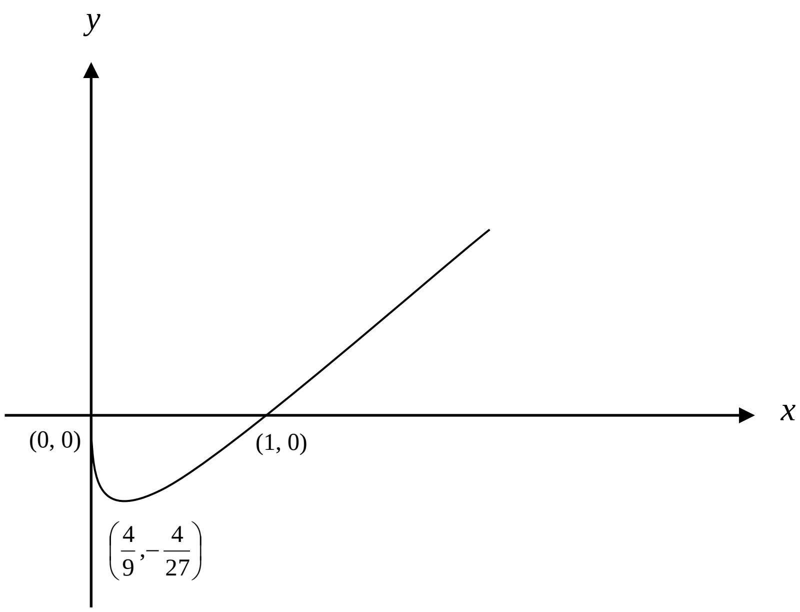
$\therefore x = \frac{4}{9}$

From the graph, a minimum turning point occurs at this value of x.

Answer: $x = \frac{4}{9}$ **1M, 1A**

(ii) Strictly increasing for $x \in \left[\frac{4}{9}, \infty\right)$ **1A**

b.

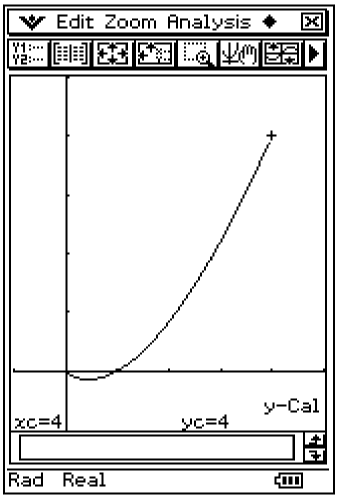


3A

c. (i) Gradient of line: $\dfrac{4-0}{4-1}=\dfrac{4}{3}$

Equation of line: $y-0=\dfrac{4}{3}(x-1)$

Equation of line: $y=\dfrac{4}{3}(x-1)$



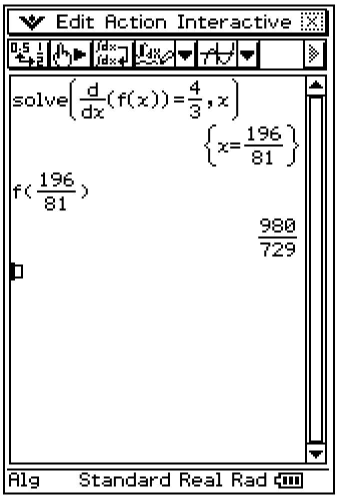
1M, 1A

(ii) Let $g'(x)=\dfrac{3}{2}\sqrt{x}-1=\dfrac{4}{3}$

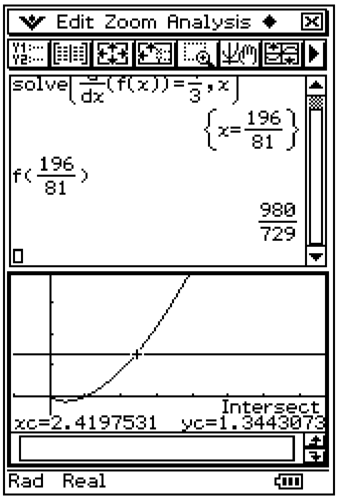
Solve for x: $x=\dfrac{196}{81}$

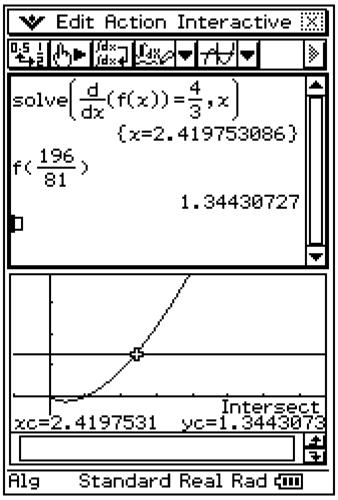
$\therefore (a, \mathrm{g}(a))=\left(\dfrac{196}{81}, \dfrac{980}{729}\right)$

1M, 1A



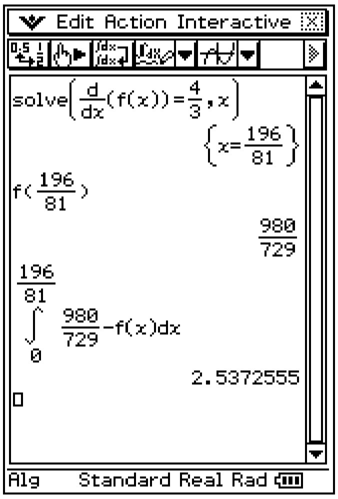
d. (i) $y = g(a)$ gives $y = \dfrac{980}{729}$.





$$\text{Area} = \int_0^{\frac{196}{81}} \frac{980}{729} - \left(x\sqrt{x} - x\right) dx$$ **1M, 1A**

(ii) Area $= 2.54\ units^2$ **1A**



QUESTION 2

a. Equation of the line is $y = mx = \tan\left(\frac{\pi}{12}^{c}\right)x$

$\therefore y = (2-\sqrt{3})x$ **1M**

b. $y = -(x-b)(x+1)^2$

Substitute in $(0, 2)$: $b = 2$

Therefore, the equation of the cubic curve is: $y = -(x-2)(x+1)^2$ **1A**

c. Gradient of the line in **a.** is $2-\sqrt{3}$, therefore, the perpendicular gradient is

$$-\frac{1}{2-\sqrt{3}} = -\frac{1}{(2-\sqrt{3})} \times \frac{(2+\sqrt{3})}{(2+\sqrt{3})} = -2-\sqrt{3}.$$

Let $\frac{d}{dx}\left(-(x-2)(x+1)^2\right) = -2-\sqrt{3}$ and substitute in $x = p$.

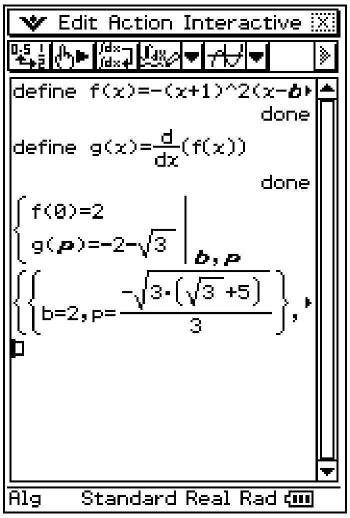
$$\therefore p^2 = \frac{\sqrt{3}+5}{3}$$

$$\therefore p = \pm\sqrt{\frac{\sqrt{3}+5}{3}} = \pm\frac{\sqrt{\sqrt{3}+5}}{\sqrt{3}} \times \frac{\sqrt{3}}{\sqrt{3}} = \pm\frac{\sqrt{3}\times\sqrt{\sqrt{3}+5}}{3} = \pm\frac{\sqrt{3\left(\sqrt{3}+5\right)}}{3} = \pm\frac{\sqrt{3\sqrt{3}+15}}{3}$$

Using the given domain: $p = \frac{\sqrt{3\sqrt{3}+15}}{3}$

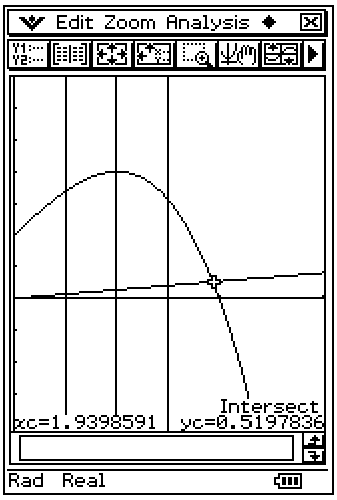
As $p = \frac{\sqrt{\left(m\sqrt{3}+n\right)}}{3}$ then $\frac{\sqrt{3\sqrt{3}+15}}{3} = \frac{\sqrt{\left(m\sqrt{3}+n\right)}}{3}$.

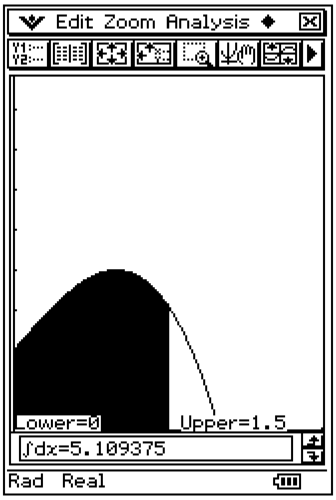
Equating gives $m = 3$ and $n = 15$. **2M, 1A**



d. Vertical strips of width 0.5 units between $x=0$ and $x=1.5$.

(i) Area of the left endpoint rectangles between the cubic curve and the x axis:





$$\text{Area } = \frac{1}{2}\big(y(0)+y(0.5)+y(1)\big) = \frac{75}{16} \; units^2$$

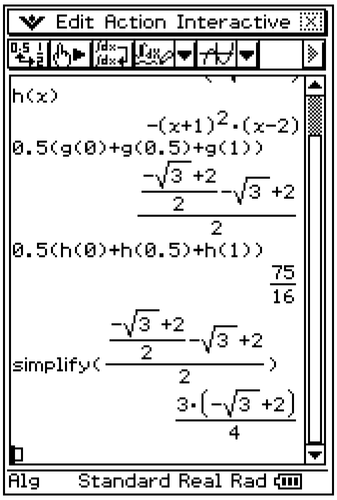
1M, 1A

(ii) Area of the left endpoint rectangles under the straight line:

$$\text{Area } = \frac{1}{2}\big(y(0)+y(0.5)+y(1)\big) = \frac{6-3\sqrt{3}}{4} \; units^2$$ **1M, 1A**

(iii) Approximate area between the curves:

$$\frac{75}{16} - \frac{6-3\sqrt{3}}{4} = 4.49 \; units^2$$ **1A**



e. Area required for painting in yellow:

Area $= \int_0^{1.93986} \left(-(x-2)(x+1)^2 - \left(2-\sqrt{3}\right)x\right) dx = 5.48\ units^2$ **1M, 1A**

f. Graph is half a sine graph with a horizontal but no vertical translation.

Period of graph $= \dfrac{2\pi}{\frac{\pi}{2}} = 4$ units.

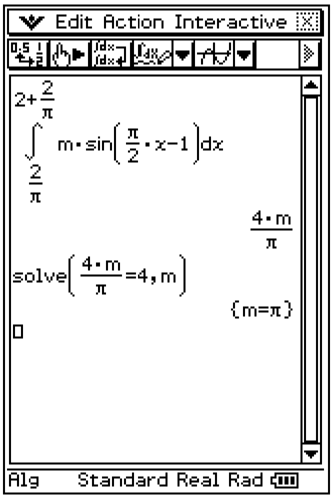
Horizontal translation of $\dfrac{2}{\pi}$ units.

Therefore, the x intercepts could be $0+\dfrac{2}{\pi}$ and $2+\dfrac{2}{\pi}$.

Let the total area be 4 square units:

Area $= m\int_{\frac{2}{\pi}}^{2+\frac{2}{\pi}} \sin\left(\dfrac{\pi}{2}x-1\right)dx = 4$

$\therefore m = \pi$



1M, 1A

QUESTION 3

a. $h = 2r \quad \therefore r = \dfrac{h}{2}$ **1A**

b. (i) Sketch the graph of $\dfrac{dV}{dt} = -0.5\, m^3/s$ **1A**

(ii) Find an expression for V in terms of h:

$$V = \frac{1}{3}\pi r^2 h$$

As $r = \dfrac{h}{2}$ then $V = \dfrac{1}{3}\pi\left(\dfrac{h}{2}\right)^2 (h) = \dfrac{1}{3}\pi\dfrac{h^3}{4} = \dfrac{1}{12}\pi h^3$ **1M**

$$\therefore \frac{dV}{dh} = \frac{\pi}{4}h^2$$

When $h = 5$ then $\dfrac{dV}{dh} = \dfrac{25\pi}{4}$. **1A**

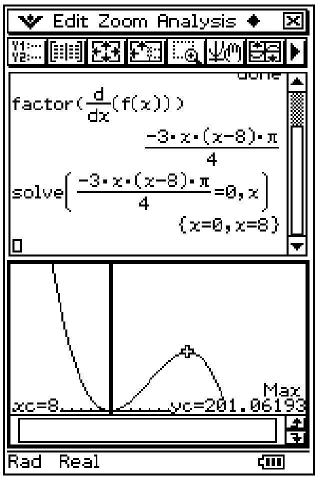
c. Volume of cone: $V = \dfrac{1}{3}\pi r^2 h$

When $h = 10$ and $r = 5$ then $V = \dfrac{1}{3}\pi 250$

$= \dfrac{250}{3}\pi\ cm^3$ as required **1M**

d. Cylinder with diameter of x cm and a height of $12 - x$ cm.

To find the maximum volume, let $\dfrac{dV}{dx} = 0$, solve for x and then calculate the required values.



$$V = \pi r^2 h \quad \text{where} \quad r = \left(\frac{x}{2}\right)$$

$$\therefore V = \pi\left(\frac{x}{2}\right)^2 (12 - x)$$

$$\therefore \frac{dV}{dx} = \pi\left(\frac{-3x(x-8)}{4}\right) = 0$$

$$\therefore x = 0, 8$$

From the graph, a local maximum occurs at $x = 8$. Therefore, the diameter is 8 cm and the height is $12 - 8 = 4$ cm.

3M, 1A

e. Volume of contents of cone $=\frac{250}{3}\pi = 83\frac{1}{3}\pi\ cm^3$

Volume of contents of cylinder $=\pi r^2 h = \pi 4^2 \times 4 = 64\pi\ cm^3$

As the cylinder holds a smaller volume than the cone, Lurch will **NOT** be able to catch all contents in the bowl. **1A**

f. (i) $\Pr(X < a) = 0.9332$

$\Pr(z < z_a) = 0.9332$

Using inverse normal: $z_a = 1.5$

$\Pr(X > b) = 0.841345$

$\Pr(z > z_b) = 0.841345$

$\therefore \Pr(z < z_b) = 0.158655$

Using inverse normal: $z_b = -1$

Substitute into $Z = \frac{X-\mu}{\sigma}$: $1.5 = \frac{a-\mu}{\sigma}$ **1M**

$\therefore \sigma = \frac{a-\mu}{1.5}$ …. (1)

$-1 = \frac{b-\mu}{\sigma}$ **1M**

$\therefore -\sigma = b-\mu$ …. (2)

Add equations (1) and (2) together:

$$0 = \frac{a-\mu}{1.5} + b - \mu$$

$$0 = a - \mu + 1.5b - 1.5\mu$$

$$\frac{5}{2}\mu = a + \frac{3b}{2} = \frac{2a+3b}{2}$$

$$\therefore \mu = \frac{2a+3b}{5}$$ **1A**

Substitute $\mu = \frac{2a+3b}{5}$ into equation (2): $\sigma = \frac{2a-2b}{5}$. **1A**

(ii) When $\mu = 100$ and $\sigma = 20$:

$z_a = 1.5$ i.e. a is 1.5 standard deviation units above the mean.

$\therefore a = \mu + 1.5\sigma = 100 + 1.5(20) = 130$

$z_b = -1$

$\therefore b = \mu - 1\sigma = 100 - 20 = 80$

Answer: $a = 130,\ b = 80$ **(Both correct) 1A**

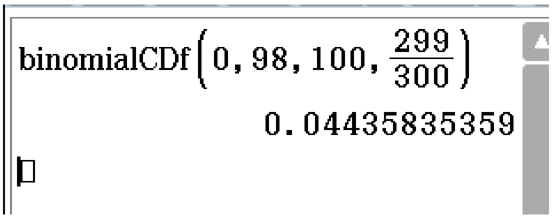
QUESTION 4

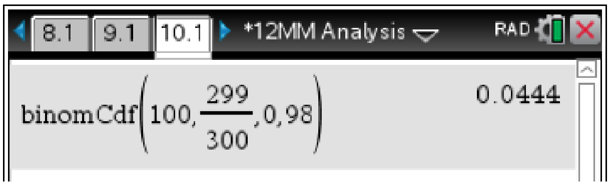
a. Binomial, $p=\frac{299}{300}, q=\frac{1}{300}$, $n=1,000,000$ **M1**

Where $\mu=np=1,000,000\times\frac{299}{300}=996666\frac{2}{3}$ **A1**

b. $p=\frac{299}{300}$, $n=100$

$\Pr(X<99)=\Pr(X\le 98)=0.044358=0.0444$ **A1**





c. (i)

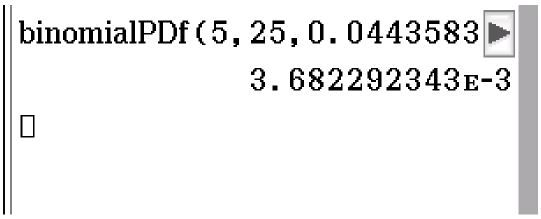
$x	Profit $1.20	Loss $0.50
Pr(X=x)	0.9556	0.0444

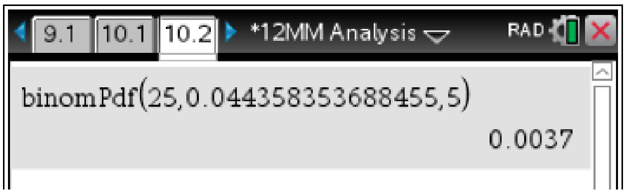
E(P) = ($1.20 x 0.9556) - ($0.50 x 0.0444) = $1.12 **M1 A1**

(ii) Binomial distribution: $X\sim Bi(25, 0.044358)$ **M1**

Let X = Number of packets

$\Pr(X=5)=0.0037$ **A1**





d. (i) $\mu_p = p = 0.9200$ **A1**

$$sd\left(P\right) = \sigma_p = \sqrt{\frac{p(1-p)}{n}} = \sqrt{\frac{0.92(1-0.92)}{160}} = 0.0215$$ **A1**

(ii) The Normal distribution can be used when $np \geq 10$ and $n(1-p) \geq 10$. **M1**

$np \geq 10$ i.e. Smallest possible value of np is 10
$0.9200n = 10$
$\therefore n = 10.8696 = 11$

$n(1-p) \geq 10$ i.e. Smallest possible value of $n(1-p)$ is 10
$0.08n = 10$
$\therefore n = 125$

The smallest value of n that satisfies both conditions is 125. **A1**

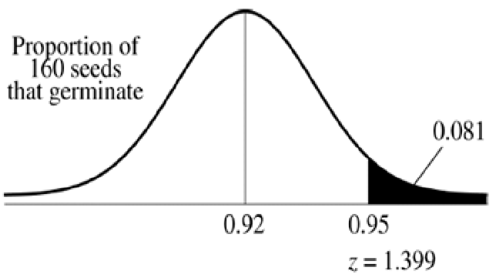
(iii) Let $\sigma = \dfrac{0.0215}{2} = 0.01075$

$$\sqrt{\frac{0.92(1-0.92)}{n}} = 0.01075$$

$n = 636.885 = 637$ **A1**

(iv) $$z = \frac{\hat{p} - \mu_{\hat{p}}}{\sqrt{\frac{p(1-p)}{n}}} = \frac{0.95 - 0.92}{\sqrt{\frac{(0.92)(0.08)}{160}}} = 1.399$$ **M1**

$\Pr(z > 1.399) = 0.0810$ **A1**



e. $$n = p^*(1-p^*)\left(\frac{Z_{\alpha/2}}{M}\right)^2 = 0.5(1-0.5)\left(\frac{1.645}{0.05}\right)^2 = 270.6025 = 271$$ **M1 A1**

271 seeds