Year 2017

VCE

Mathematical Methods

Trial Examination 2

Solutions

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- While every care has been taken, no guarantee is given that these answers are free from error. Please contact us if you believe you have found an error.

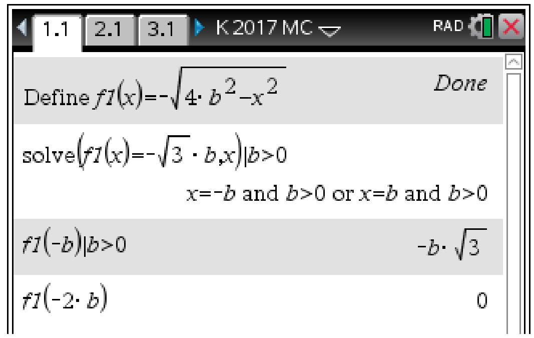
SECTION A

ANSWERS

1	A	B	C	**D**	E
2	A	**B**	C	D	E
3	**A**	B	C	D	E
4	A	B	C	D	**E**
5	A	B	**C**	D	E
6	A	**B**	C	D	E
7	A	B	**C**	D	E
8	**A**	B	C	D	E
9	A	B	C	D	**E**
10	A	B	C	**D**	E
11	A	B	C	**D**	E
12	A	B	**C**	D	E
13	**A**	B	C	D	E
14	A	B	C	D	**E**
15	A	B	C	**D**	E
16	A	**B**	C	D	E
17	A	B	**C**	D	E
18	A	**B**	C	D	E
19	**A**	B	C	D	E
20	A	B	C	D	**E**

SECTION A

Question 1 Answer D



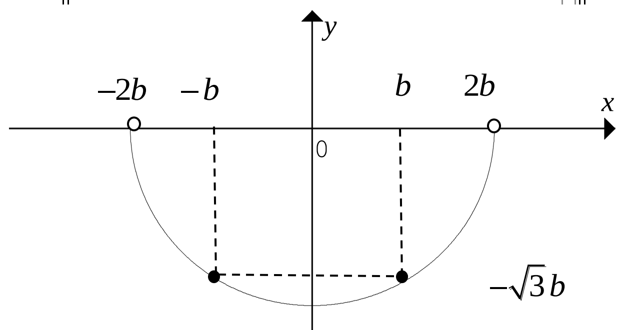
The graph is the bottom half of a circle with centre at the origin and radius $2b$.

$f:D\to R\,,\ f(x)=-\sqrt{4b^2-x^2}$

solving $f(x)=0 \ \Rightarrow\ x=\pm 2b$

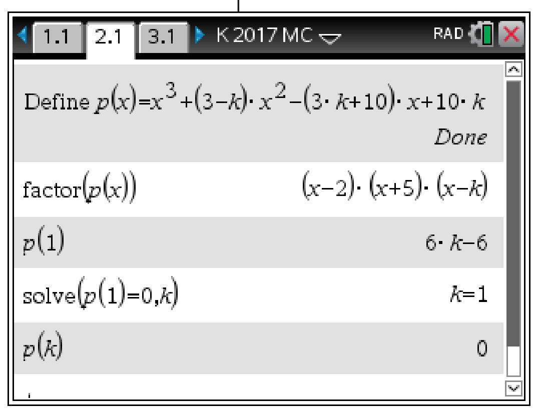
solving $f(x)=-\sqrt{3}\,b$

$$-\sqrt{4b^2-x^2}=-\sqrt{3}b$$
$$4b^2-x^2=3b^2$$
$$x^2=b^2$$
$$x=\pm b$$



since the range is $\left[-\sqrt{3}\,b,0\right)$ the domain could be $(-2b,-b]$ or $[b,2b)$

Question 2 Answer B



$$p(x)=x^3+(3-k)x^2-(3k+10)x+10k$$
$$=(x-2)(x+5)(x-k)$$

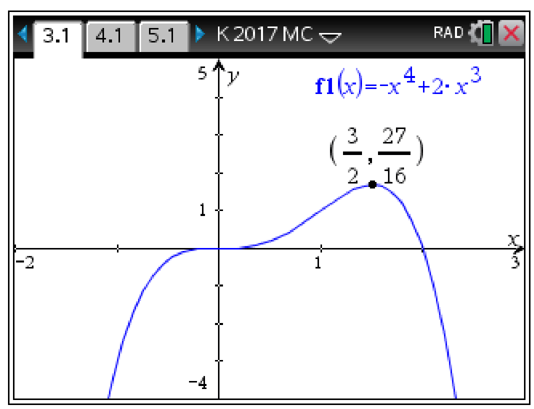
$p(1)=6k-6,\quad p(1)=0\ \Rightarrow\ k=1$

Since $(x-k)$ is a factor, then $p(k)=0$

$(x-2)$ and $(x+5)$ are both factors

$(x+k)$ is not a factor.

Question 3 Answer A

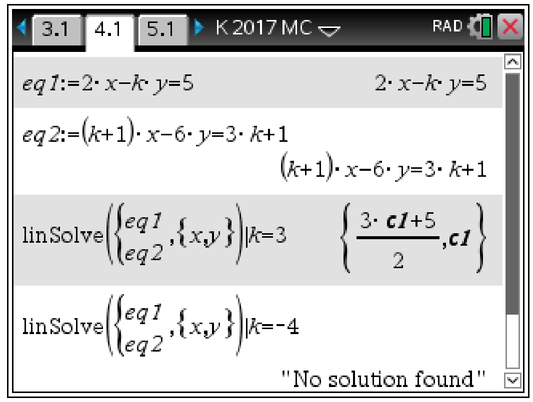


$f:(-\infty,b)\to R\,,\ f(x)=-x^4+2x^3$

$f'(x)=-4x^3+6x^2=2x^2(3-2x)=0$

The graph has an inflexion point at $x=0$ the origin, and a maximum turning point at $\left(\frac{3}{2},\frac{27}{16}\right)$. The function is only a one-one function, when its domain is restricted to $(-\infty,b)$ where $b<\frac{3}{2}$.

Question 4 **Answer E**



$(1)\ \ 2x-ky=5$

$ky=2x-5 \Rightarrow y=\dfrac{2x}{k}-\dfrac{5}{k}$

$(2)\ \ (k+1)x-6y=3k+1$

$6y=(k+1)x-(3k+1) \Rightarrow y=\dfrac{(k+1)x}{6}-\dfrac{3k+1}{6}$

equating gradients, when the lines are parallel

$\dfrac{2}{k}=\dfrac{k+1}{6} \Rightarrow k(k+1)=12 \Rightarrow k^2+k-12=(k+4)(k-3)=0$

There is a unique solution when $k\in R\setminus\{-4,3\}$

When $k=3$ the equations become $\begin{aligned}2x-3y&=5\\4x-6y&=10\end{aligned}$

these lines are coincident, that is the same line, so there is infinite number of solutions when $k=3$

When $k=-4$ the equations become $\begin{aligned}2x+4y&=5\\-3x-6y&=-9\end{aligned}$

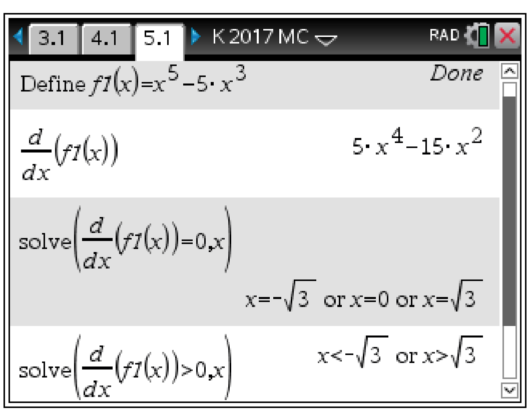
these lines are parallel so there is no solution when $k=-4$

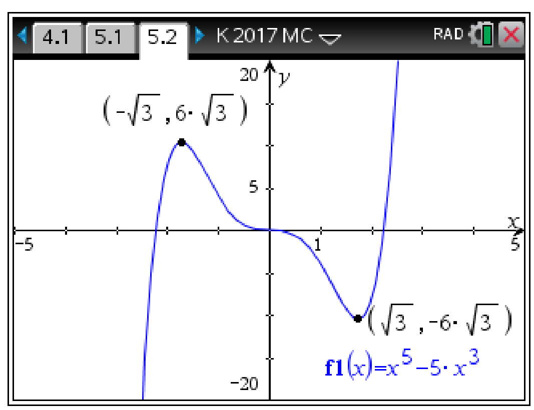
Question 5 **Answer C**

$y=x^5-5x^3 \Rightarrow \dfrac{dy}{dx}=5x^4-15x^2=5x^2\left(x^2-3\right)$, stationary points occur when $\dfrac{dy}{dx}=0$

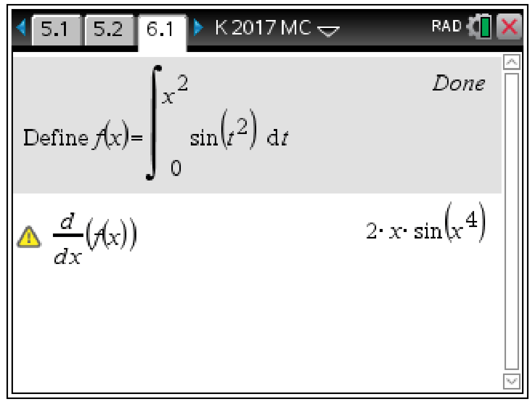
$\Rightarrow x=0$ the point of inflexion $\Rightarrow x=\pm\sqrt{3}$ the turning points.

The graph has a positive gradient $\dfrac{dy}{dx}>0 \Rightarrow x\in\left(-\infty,-\sqrt{3}\right)\cup\left(\sqrt{3},\infty\right)$





Question 6 **Answer B**



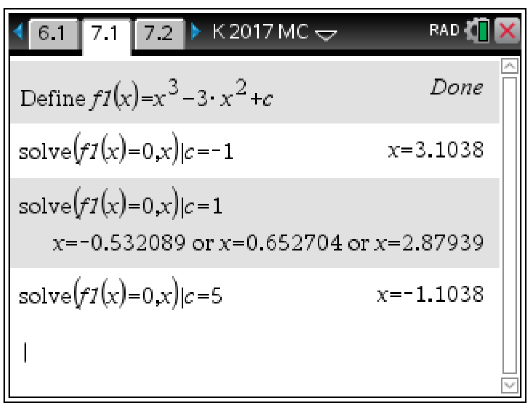
$$f(x)=\int_0^{x^2}\sin\left(t^2\right)dt$$

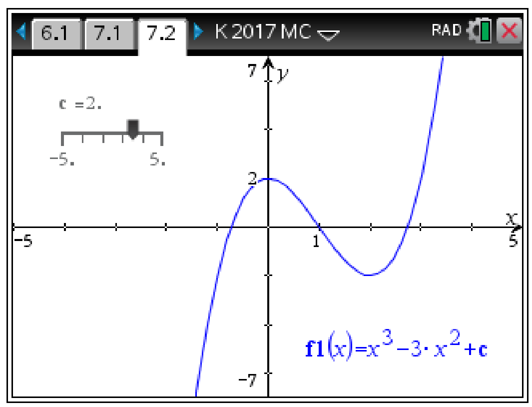
$$f'(x)=2x\sin\left(x^4\right)$$

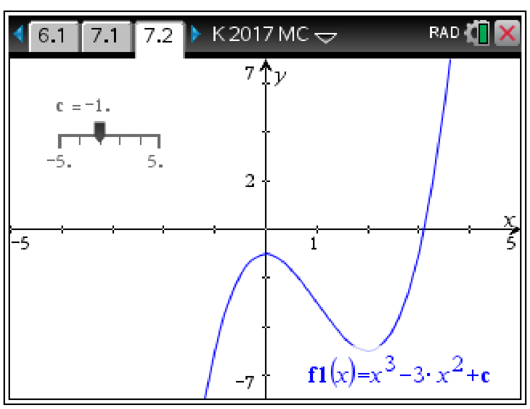
Question 7 **Answer C**

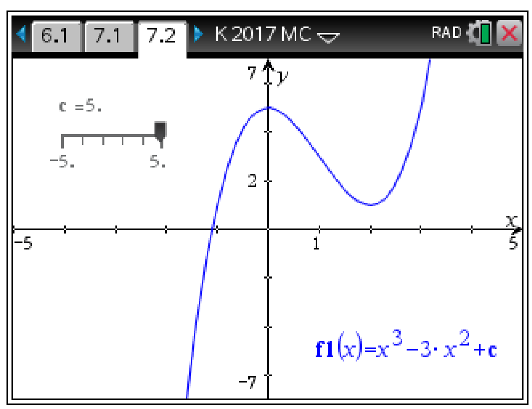
$y=x^3-3x^2+c$ $\quad \dfrac{dy}{dx}=3x^2-6x=3x(x-2)=0$ turning points occur when $x=0,2$

when $x=0$, $y=c$, when $x=2$, $y=8-12+c=c-4$, turning points at $(0,c)$ and $(2,c-4)$. The graph crosses the *x*-axis three times when $c>0$ and $c-4<0$, that is when $0<c<4$.









Question 8 **Answer A**

$f(x)=\dfrac{\log_e(3x)}{g(x)}$ using the quotient rule

$$f'(x)=\frac{g(x)\dfrac{d}{dx}\left[\log_e(3x)\right]-g'(x)\log_e(3x)}{\left[g(x)\right]^2}=\frac{\dfrac{g(x)}{x}-g'(x)\log_e(3x)}{\left[g(x)\right]^2}$$

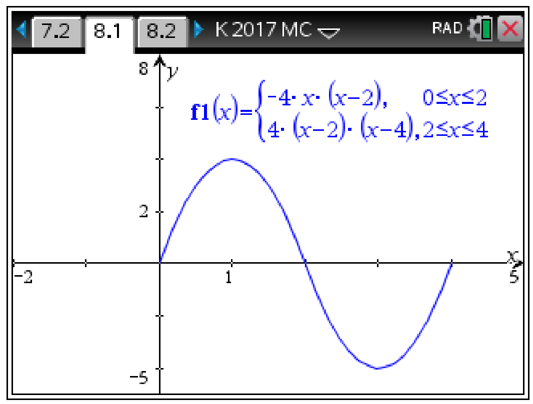
$$f'(2)=\frac{\dfrac{g(2)}{2}-g'(2)\log_e(6)}{\left[g(2)\right]^2}=\frac{\dfrac{4}{2}-3\log_e(6)}{4^2}=\frac{2-3\log_e(6)}{16}$$

Question 9 **Answer E**

$F(x)=\int_0^x f(t)\,dt$

Note that $y=f(t)$ may also be modelled by

$y=4\sin\left(\dfrac{\pi t}{2}\right)$



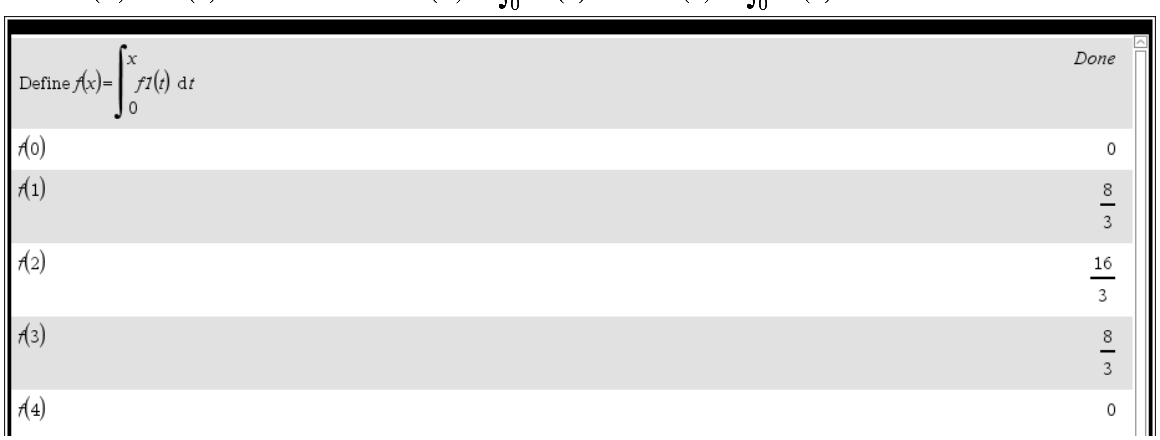
A. $F(0)=0=\int_0^0 f(t)\,dt$, is true, by properties of definite integrals

B. $F(4)=0=\int_0^4 f(t)\,dt$, is true, the value of the definite integral is zero.

C. $F(2)=\int_0^2 f(t)\,dt=2F(1)=2\int_0^1 f(t)\,dt$ is true, by symmetry, area between 0 to 2 is double the area from 0 to 1.

D. $F(2)=\int_0^2 f(t)\,dt=2F(3)=2\int_0^3 f(t)\,dt$, is true, similar to **C.**

E. $F(3)+F(1)=0$, is false, $F(3)=\int_0^3 f(t)\,dt=F(1)=\int_0^1 f(t)\,dt$.

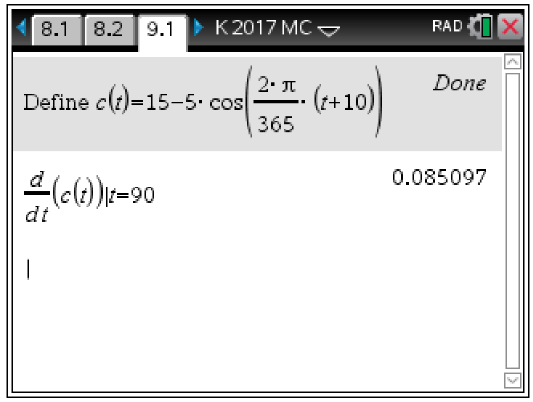


Question 10 **Answer D**

$$C(t)=15-5\cos\left(\frac{2\pi}{365}(t+10)\right)$$

$$\frac{dC}{dt}=\frac{5\times 2\pi}{365}\sin\left(\frac{2\pi}{365}(t+10)\right)$$

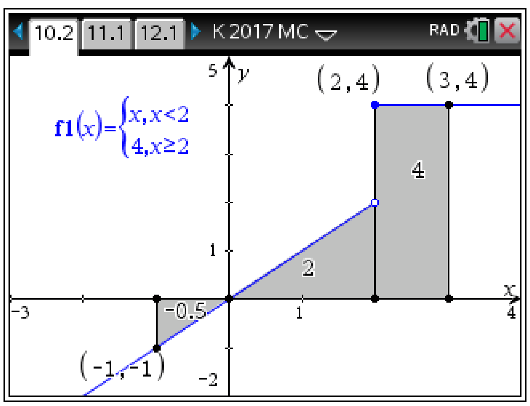
$$\left.\frac{dC}{dt}\right|_{t=90}=\frac{10\pi}{365}\sin\left(\frac{200\pi}{365}\right)\approx 0.085$$

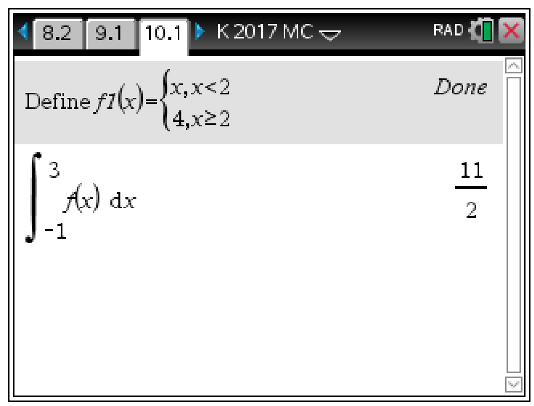


Question 11 **Answer D**

$$f(x)=\begin{cases} x & \text{for } x<2 \\ 4 & \text{for } x\ge 2\end{cases} \qquad \text{Then } \int_{-1}^{3} f(x)\,dx=5\frac{1}{2}$$

Although the function is not continuous at $x=2$, the definite integral can still be calculated as the area of the rectangle and the area of two triangles. The smaller triangle below the *x*-axis has a value of -0.5, it is not an area, the value of the definite integral is $4+2-0.5=5\frac{1}{2}=\frac{11}{2}$





Question 12 **Answer C**

Let $y=f^{-1}(x)=g(x)$ then $x=f(y)$ now differentiate with respect to *y*,

$$\frac{dx}{dy}=f'(y), \text{ inverting } \frac{dy}{dx}=\frac{d}{dx}\left[g(x)\right]=g'(x)=\frac{1}{f'(y)}=\frac{1}{f'(g(x))}$$

Now $f(1)=2$ so that $g(2)=1$, and $g'(2)=\frac{1}{f'(g(2))}=\frac{1}{f'(1)}=\frac{1}{-4}=-\frac{1}{4}$

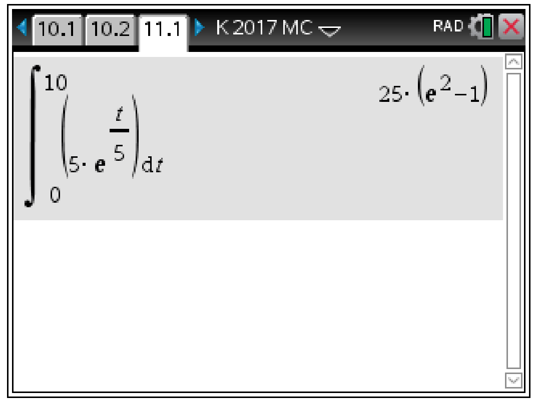
Question 13 **Answer A**

From the graphs $f(1)=1$, $f(2)=0$, $g(1)=2$

$$f(g(1))=f(2)=0$$

$$g(f(1))=g(1)=2$$

Question 14 **Answer E**



$$\frac{dN}{dt} = 5e^{0.2t} = 5e^{\frac{t}{5}}$$

$$N = \int_0^{10} 5e^{\frac{t}{5}}\,dt = \left[25e^{\frac{t}{5}}\right]_0^{10}$$

$$= 25\left(e^{\frac{10}{5}} - e^0\right) = 25\left(e^2 - 1\right)$$

Question 15 **Answer D**

$$\Pr(RBW) + \Pr(RWB) + \Pr(WRB) + \Pr(WBR) + \Pr(BWR) + \Pr(BRW)$$

There are 6 different ways, the total number of balls is $(r+b+w)$ when one ball is drawn there is $(r+b+w-1)$ and when two balls are drawn there are $(r+b+w-2)$ balls remaining.

The total number of ways of drawing balls of different colours is

$$\frac{6rbw}{(r+b+w)(r+b+w-1)(r+b+w-2)}$$

Question 16 **Answer B**

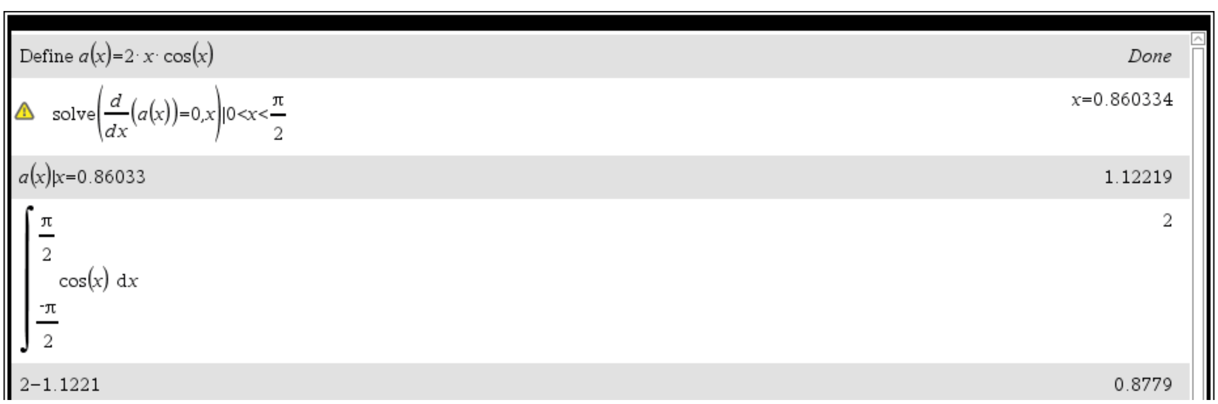
The graph of $y = \cos(x)$ crosses the x-axis at $x = \pm\frac{\pi}{2}$.

The area of the rectangle is $A(x) = 2x\cos(x)$, solving $\frac{dA}{dx} = 0$ for x, with $0 < x < \frac{\pi}{2}$,

gives $x = 0.8603$. The maximum area of the rectangle is $A(0.8603) = 1.222$.

The area under the cosine wave is $\int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} \cos(x)\,dx = 2$, so the minimum shaded area is

$$2 - 1.122 = 0.878$$



Question 17 **Answer C**

$p=0.35$, $n=200$, 95% $z=1.96$, $\sigma_{\hat{p}}=\sqrt{\dfrac{p(1-p)}{n}}=\sqrt{\dfrac{0.35\times 0.65}{200}}=0.0334$

The width of a 95% confidence interval is $2z\sqrt{\dfrac{p(1-p)}{n}}=2\times 1.96\times 0.0334=0.132$

Question 18 **Answer B**

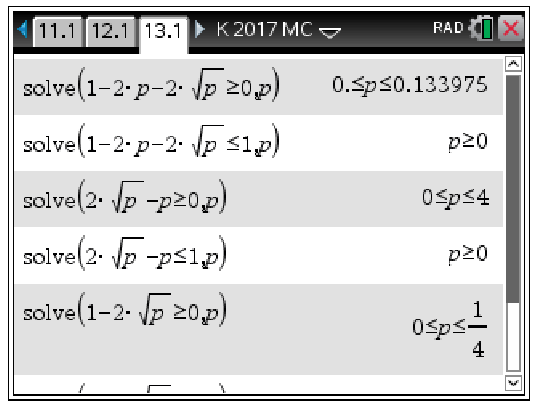
$Z\overset{d}{=}N\left(\mu=0,\sigma^2=1\right)$, given that $\Pr(-c<Z<c)=C$ by symmetry $\Rightarrow$ $\Pr(0<Z<c)=\dfrac{C}{2}$

$$\Pr(Z<c\mid Z>0)=\frac{\Pr(0<Z<c)}{\Pr(Z>0)}=\frac{\frac{C}{2}}{0.5}=C$$

Question 19 **Answer A**

$\Pr(A)=3p,\Pr(B)=2\sqrt{p}$ and $\Pr(A\cap B)=p$

	A	A'	
B	p	$2\sqrt{p}-p$	$2\sqrt{p}$
B'	$2p$	$1-2p-2\sqrt{p}$	$1-2\sqrt{p}$
	$3p$	$1-3p$	



For valid probabilities, all probabilities must be between 0 and 1. That is $0\le 3p\le 1$, $0\le 1-2\sqrt{p}\le 1, 0\le 2\sqrt{p}-p\le 1$, etc

solving $0\le 1-2p-2\sqrt{p}\le 1$ gives $p=0.13398$, any value greater than this will result in negative probabilities, which is not possible.

Question 20 **Answer E**

$X\overset{d}{=}\text{Bi}(n=?,p=?)$

$\Pr(\text{more than one})=\Pr(X>1)=1-\left[\Pr(X=0)+\Pr(X=1)\right]=1-\left(0.65^8+8(0.35)(0.65)^7\right)$

Now $\Pr(X=0)=q^n$ and $\Pr(X=1)=npq^{n-1}$

$n=8$, $q=0.65$ and $p=0.35$

8 trials and $p=\Pr(\text{success})=0.35$

END OF SECTION A SUGGESTED ANSWERS

SECTION B

Question 1

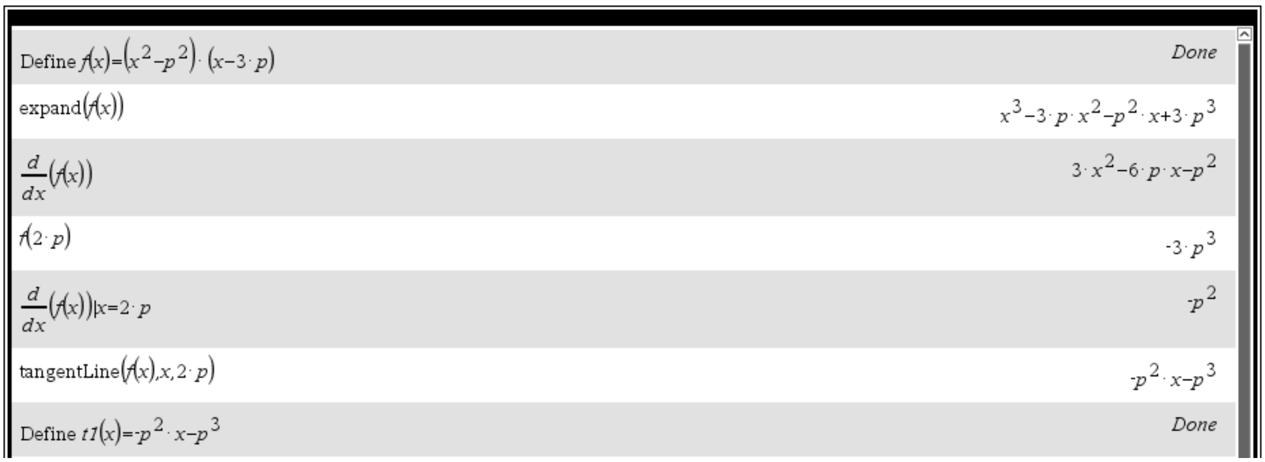
a. $f(x)=\left(x^2-p^2\right)(x-3p)=x^3-3px^2-p^2x+3p^3$

$f'(x)=3x^2-6px-p^2$

At P, $x=2p$, $f(2p)=-3p^3$ $\quad f'(2p)=-p^2$ A1

$T:\quad y+3p^3=-p^2(x-2p)=-p^2x+2p^3$ A1

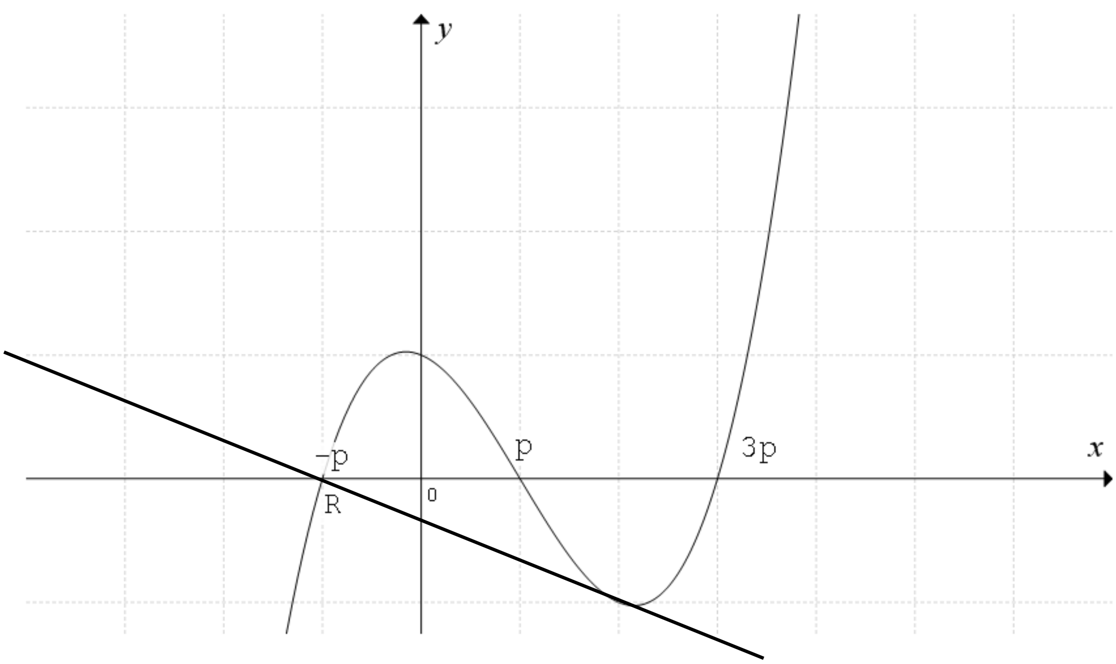
$t1(x)=y=-p^2x-p^3 \qquad m_1=-p^2\ ,\ c_1=-p^3$



b. $t1(x)=y=-p^2x-p^3=-p^2(x+p)\ ,\ f(x)=(x+p)(x-p)(x-3p)$

solving $t1(x)=f(x)\quad \Rightarrow x=-p \qquad \text{R}\ (-p,0)$ A1

tangent must pass through $(-p,0)$ G1



c. At R, $x=-p$, $f(-p)=0$ $f'(-p)=8p^2$ A1

$T: \quad y-0=8p^2(x+p)=8p^2x+8p^3$

$t2(x)=y=8p^2x+8p^3 \qquad m_2=8p^2 \;,\; c_2=8p^3$ A1

solve(t1(x)=f(x),x)	x=2·p or x=-p
f(-p)	0
$\frac{d}{dx}(f(x))\|x=-p$	8·p^2
tangentLine(f(x),x,-p)	8·p^2·x+8·p^3
Define t2(x)=8·p^2·x+8·p^3	Done

d. solving $f(x)=t2(x)$

$x^3-3px^2-p^2x+3p^3=8p^2x+8p^3$

$\Rightarrow x=-p$ or $x=5p$, $f(5p)=48p^3$ M1

S $(5p, 48p^3)$ A1

solve(t2(x)=f(x),x)	x=5·p or x=-p
f(5·p)	48·p^3

e. $A_1=\int_{-p}^{2p}\left(f(x)-t1(x)\right)dx$ M1

$A_1=\int_{-p}^{2p}\left(x^3-3px^2+4p^3\right)dx$ A1

f. $A_2=\int_{-p}^{5p}\left(t2(x)-f(x)\right)dx$ M1

$A_2=\int_{-p}^{5p}\left(-x^3+3px^2+9p^2x+5p^3\right)dx$ A1

g. $A_1=\dfrac{27p^4}{4}$, $A_2=108p^4$, $\dfrac{A_2}{A_1}=16$ A1

f(x)−t1(x)	$x^3-3\cdot p\cdot x^2+4\cdot p^3$
$a1:=\int_{-p}^{2\cdot p}(x^3-3\cdot p\cdot x^2+4\cdot p^3)dx$	$\frac{27\cdot p^4}{4}$
t2(x)−f(x)	$-x^3+3\cdot p\cdot x^2+9\cdot p^2\cdot x+5\cdot p^3$
$a2:=\int_{-p}^{5\cdot p}(-x^3+3\cdot p\cdot x^2+9\cdot p^2\cdot x+5\cdot p^3)dx$	$108\cdot p^4$
⚠ $\frac{a2}{a1}$	16

Question 2

a.i. $H \overset{d}{=} N\left(\mu = 58, \sigma^2 = 6^2\right)$

$\Pr(H > 50) = 0.9088$ A1

ii. $Y \overset{d}{=} \text{Bi}\left(n = 12, p = 1 - 0.9088\right)$

$\Pr(Y \geq 3) = 0.0894$ A1

iii. $X \overset{d}{=} \text{Bi}\left(n = ?, p = 0.9088\right)$

$\Pr(X \geq 5) \geq 0.95$ to find the value of *n*, using trial and error

$n = 7$ A1

normCdf(50,∞,58,6)	0.908789
p:=0.90878871813013	0.908789
binomCdf(12,1−p,3,12)	0.089416
binomCdf(n,p,5,n)|n=5	0.61989
binomCdf(n,p,5,n)|n=6	0.902595
binomCdf(n,p,5,n)|n=7	0.979952

b.i. $E \overset{d}{=} N\left(\mu = ?, \sigma^2 = ?^2\right)$

jumbo eggs, $J = \Pr(E > 66.7) = 0.023 \quad \Rightarrow \quad \Pr(E \leq 66.7) = 0.977$ M1

$(1) \quad \dfrac{66.7 - \mu}{\sigma} = 1.9954$

medium eggs, $M = \Pr(E < 50) = 0.09$ A1

$(2) \quad \dfrac{50 - \mu}{\sigma} = -1.34076$

solving (1) and (2) $\mu = 56.7$, $\sigma = 5.0$ A1

invNorm(1−0.023)	1.99539
invNorm(0.09)	-1.34076
$\frac{66.7-m}{s}=1.995393311222$	$\frac{66.7-m}{s}=1.99539$
$\frac{50-m}{s}=-1.3407550347445$	$\frac{50-m}{s}=-1.34076$
solve$\left(\frac{66.7-m}{s}=1.9954 \text{ and } \frac{50-m}{s}=-1.34076,\{m,s\}\right)$	s=5.00576 and m=56.7115

b.ii. $\Pr(E \geq w) = \dfrac{35.2 + 2.3}{100} = 0.375$, $\Pr(E \leq w) = 0.625$

minimum weight for extra large egg is $w = 58.3$ A1

```
invNorm((53.5+9)/100, 56.7, 5)                                    58.2932
```

iii. completing the table A1

	medium	large	extra large	jumbo
probability	0.09	0.535	0.352	0.023
egg weight grams	<50	50 to <58.3	58.3 to ≤ 66.7	>66.7
revenue cents	5	15	20	25

expected revenue $E(R) = 0.09 \times 5 + 0.535 \times 15 + 0.352 \times 20 + 0.023 \times 25 = 16.09$

$= 16$ cents A1

iv. $\Pr(50 \leq E \leq 66.7 \mid E \leq 66.7) = \dfrac{\Pr(50 \leq E \leq 66.7)}{\Pr(E \leq 66.7)} = \dfrac{0.535 + 0.352}{1 - 0.023}$

$= 0.908$ A1

```
(0.535+0.352)/(1-0.023)                                           0.907881
0.09·5+0.535·15+0.352·20+0.023·25                                 16.09
```

c.i. $E(\hat{P}) = \dfrac{x}{n} = \dfrac{5}{95} = 0.0526$

$\text{Var}(\hat{P}) = \dfrac{\hat{p}(1 - \hat{p})}{n} = \dfrac{0.0526 \times (1 - 0.0526)}{95} = 0.000525$ A1

ii. $\text{Sd}\left(\hat{P}\right)=\sqrt{0.000525}=0.0229$

$\Pr\left(0.0526-2\times0.0229\le P\le 0.0526+2\times0.0229\right)=\Pr\left(0.0068\le P\le 0.098\right)$

for 95 eggs, we require

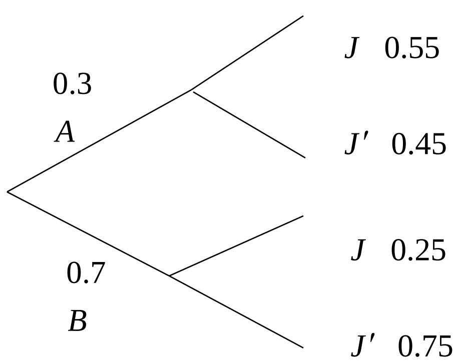
$\Pr\left(0.0068\times95\le J\le 0.098\times95\right)=\Pr\left(0.64\le J\le 9.35\right)$ M1

Since it is discrete, $J\overset{d}{=}\text{Bi}\left(n=95,\ p=\dfrac{5}{95}\right)$

$\Pr\left(1\le J\le 9\right)=0.966$ A1

$\frac{5}{95}$	0.052632
$\dfrac{\frac{5}{95}\cdot\frac{90}{95}}{95}$	0.000525
$\sqrt{5.2485785099868\text{E-}4}$	0.02291
0.0526+2· 0.02291	0.09842
0.0526−2· 0.02291	0.00678
0.00678· 95	0.6441
0.09842· 95	9.3499
$\text{binomCdf}\left(95,\frac{5}{95},1,9\right)$	0.966125

d. tree diagram



$\Pr\left(J'\right)=\Pr\left(J'\cap A\right)+\Pr\left(J'\cap B\right)=0.3\times0.45+0.7\times0.75=0.66$ M1

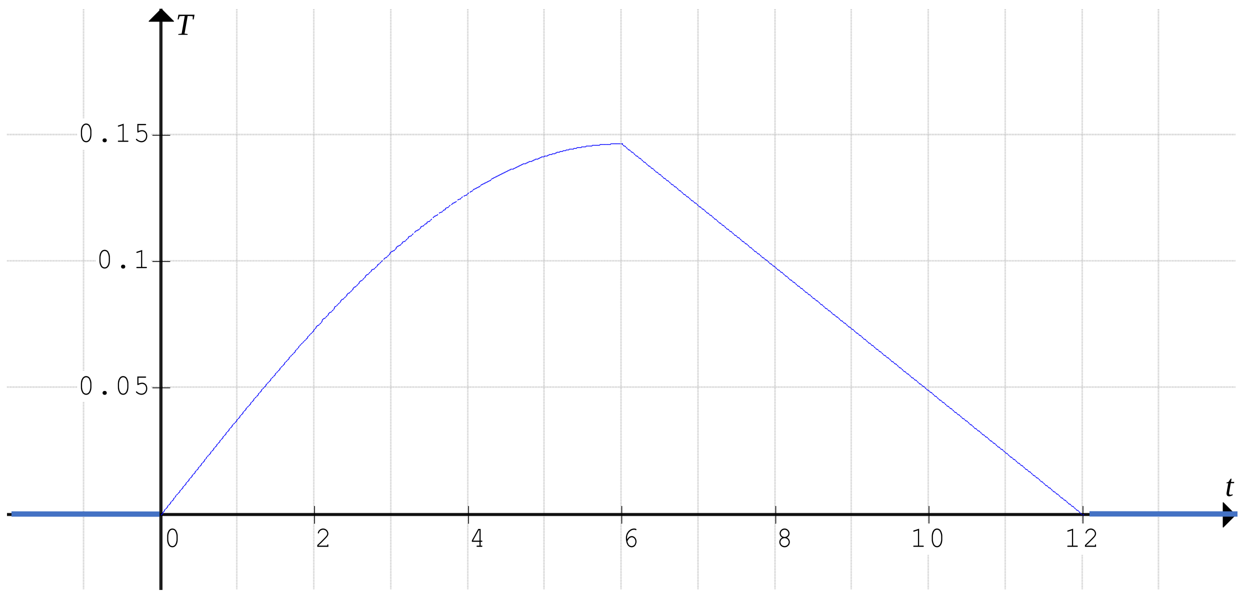
$\Pr\left(B\mid J'\right)=\dfrac{0.7\times0.75}{0.66}=0.795$ A1

e.i. The function is continuous at $t = 6$, so that

$$\sin\left(\frac{\pi}{2}\right) = 1 = b\left(1 - \frac{6}{12}\right) = \frac{b}{2} \quad \Rightarrow \quad b = 2$$ M1

ii. $\frac{\pi}{3(\pi+4)} \approx 0.147$, sine curve, graph joins up with straight line at $(6, 0.147)$

correct scaling, shape, zero elsewhere G2



iii. $$T(t) = \frac{\pi}{3(\pi+4)}\begin{cases} \sin\left(\frac{\pi t}{12}\right) & 0 \le t \le 6 \\ 2\left(1 - \frac{t}{12}\right) & 6 \le t \le 12 \\ 0 & \text{otherwise} \end{cases}$$

$$E(T) = \frac{\pi}{3(\pi+4)}\left[\int_0^6 t\sin\left(\frac{\pi t}{12}\right)dt + 2\int_6^{12} t\left(1 - \frac{t}{12}\right)dt\right] = 5.6586$$

$$E(T^2) = \frac{\pi}{3(\pi+4)}\left[\int_0^6 t^2\sin\left(\frac{\pi t}{12}\right)dt + 2\int_6^{12} t^2\left(1 - \frac{t}{12}\right)dt\right] = 38.3625$$ M1

$$\text{Var}(T) = 38.3625 - (5.6586)^2 = 6.343$$ A1

iv. Since $\frac{\pi}{3(\pi+4)}\int_0^6 \sin\left(\frac{\pi t}{12}\right)dt = 0.56$, the median *m*, satisfies

$$\frac{\pi}{3(\pi+4)}\int_0^m \sin\left(\frac{\pi t}{12}\right)dt = 0.5\text{, solving gives } m = 5.589$$ A1

Define $f1(x)=\frac{\pi}{3\cdot(\pi+4)}\cdot\left(\begin{cases}\sin\left(\frac{\pi\cdot x}{12}\right), & 0\le x\le 6\\ 2\cdot\left(1-\frac{x}{12}\right), & 6\le x\le 12\end{cases}\right)$	*Done*
$\int_0^{12} f1(x)\,dx$	1.
$\int_0^{12} (x\cdot f1(x))\,dx$	5.65863
$\int_0^{12} (x^2\cdot f1(x))\,dx$	38.3625
$38.3625-(5.6586)^2$	6.34275
$\int_0^{6} f1(x)\,dx$	0.560099
⚠ $\text{solve}\left(\int_0^{m} f1(x)\,dx=0.5,m\right)\mid 0<m<6$	$m=5.58935$

Question 3

a. $LHS = \left[C(x)\right]^2 - \left[S(x)\right]^2$

$= \left[\frac{1}{2}\left(e^x + e^{-x}\right)\right]^2 - \left[\frac{1}{2}\left(e^x - e^{-x}\right)\right]^2$

$= \left[\frac{1}{4}\left(e^{2x} + 2 + e^{-2x}\right)\right] - \left[\frac{1}{4}\left(e^{2x} - 2 + e^{-2x}\right)\right]$ M1

$= \frac{1}{4}\left[\left(e^{2x} + 2 + e^{-2x}\right) - \left(e^{2x} - 2 + e^{-2x}\right)\right]$ M1

$= 1 = RHS$

b. $\frac{d}{dx}\left(S(kx)\right) = \frac{d}{dx}\left[\frac{1}{2}\left(e^{kx} - e^{-kx}\right)\right]$

$= \frac{1}{2}\left[\frac{d}{dx}\left(e^{kx}\right) - \frac{d}{dx}\left(e^{-kx}\right)\right]$ M1

$= \frac{1}{2}\left(ke^{kx} + ke^{-kx}\right)$

$= \frac{k}{2}\left(e^{kx} + e^{-kx}\right) = k\left[\frac{1}{2}\left(e^{kx} + e^{-kx}\right)\right]$ M1

$= k\,C(kx)$

Also $\frac{d}{dx}\left(C(kx)\right) = \frac{d}{dx}\left[\frac{1}{2}\left(e^{kx} + e^{-kx}\right)\right] = \frac{1}{2}\left[\frac{d}{dx}\left(e^{kx}\right) + \frac{d}{dx}\left(e^{-kx}\right)\right]$

$= \frac{1}{2}\left(ke^{kx} - ke^{-kx}\right) = \frac{k}{2}\left(e^{kx} - e^{-kx}\right) = k\left[\frac{1}{2}\left(e^{kx} - e^{-kx}\right)\right] = k\,S(kx)$

c. $y = T(kx) = \frac{S(kx)}{C(kx)} = \frac{u}{v}$

$u = S(kx)$ and $v = C(kx)$ using the quotient rule M1

$\frac{du}{dx} = k\,C(kx)$ and $\frac{dv}{dx} = k\,S(kx)$

$\frac{dy}{dx} = \frac{v\frac{du}{dx} - u\frac{dv}{dx}}{v^2} = \frac{k\left[C(kx)\right]^2 - k\left[S(kx)\right]^2}{\left[C(kx)\right]^2}$ M1

$\frac{dy}{dx} = \frac{d}{dx}\left[T(kx)\right] = \frac{k\left(\left[C(kx)\right]^2 - \left[S(kx)\right]^2\right)}{\left[C(kx)\right]^2} = \frac{k}{\left[C(kx)\right]^2}$ from **Question3a** M1

d. $S(x)=\frac{3}{4}$ that is $\frac{1}{2}\left(e^{x}-e^{-x}\right)=\frac{3}{4}$

$e^{x}-e^{-x}=\frac{3}{2}$ Let $u=e^{x}$ $e^{-x}=\frac{1}{u}$

$u-\frac{1}{u}=\frac{3}{2}$ multiply both sides by u, transposing $u^{2}-\frac{3u}{2}=1$

$\left(u^{2}-\frac{3u}{2}+\frac{9}{16}\right)=1+\frac{9}{16}$ completing the square M1

$\left(u-\frac{3}{4}\right)^{2}=\frac{25}{16}$

$u-\frac{3}{4}=\pm\frac{5}{4}$

$u=e^{x}=2,-\frac{1}{4}$ but $e^{x}>0$ reject the negative

$x=\log_{e}(2)$ A1

e. The graph of $y=S(x)$ is a one-one function, so it has an inverse which is a function. The domain and range are both *R*. A1

If $y=S(x)=\frac{1}{2}\left(e^{x}-e^{-x}\right)$ then the inverse function $x=S^{-1}(y)$

so that $2y=e^{x}-e^{-x}$

$2y-e^{x}+e^{-x}=0$ multiplying by e^{x}

$e^{2x}-2ye^{x}-1=0$ let $u=e^{x}$

$u^{2}-2yu-1=0$ solving for u using the quadratic formulae

$\Delta=4y^{2}+4=4\left(y^{2}+1\right)$ M1

$u=e^{x}=\frac{2y\pm\sqrt{4\left(y^{2}+1\right)}}{2}$ take the positive since $u=e^{x}>0$

$e^{x}=y+\sqrt{\left(y^{2}+1\right)}$ so that $x=S^{-1}(y)=\log_{e}\left(y+\sqrt{y^{2}+1}\right)$

so we have the result

$y=S^{-1}(x)=\log_{e}\left(x+\sqrt{x^{2}+1}\right)$ domain and range both *R*. A1

to check the result from **d.**

$S(x)=\frac{3}{4}$

$x=S^{-1}\left(\frac{3}{4}\right)=\log_{e}\left(\frac{3}{4}+\sqrt{\left(\frac{3}{4}\right)^{2}+1}\right)=\log_{e}\left(\frac{3}{4}+\sqrt{\frac{25}{16}}\right)=\log_{e}(2)$ M1

Question 4

a.i. by similar triangles $\dfrac{1.5}{y}=\dfrac{x}{2} \quad \Rightarrow \quad y=\dfrac{1.5\times 2}{x}=\dfrac{3}{x}$ A1

ii. By Pythagoras's Theorem $AC=\sqrt{x^2+4}$ and $CB=\sqrt{y^2+1.5^2}$

$L(x)=AC+CB=\sqrt{x^2+4}+\sqrt{y^2+1.5^2}$ substitute for y,

$$L(x)=\sqrt{x^2+4}+\sqrt{\frac{9}{x^2}+\frac{9}{4}}$$ M1

$$=\sqrt{x^2+4}+\sqrt{\frac{9\left(x^2+4\right)}{4x^2}} \quad \text{since } x>0$$

$$L(x)=\sqrt{x^2+4}+\frac{3}{2x}\sqrt{x^2+4}$$

$$=\left(\frac{3}{2x}+1\right)\sqrt{x^2+4}$$ M1

iii. $\dfrac{dL}{dx}=\dfrac{x^3-6}{x^2\sqrt{x^2+4}}$ A1

$\dfrac{dL}{dx}=0 \quad \Rightarrow \quad x^3=6$

$x=\sqrt[3]{6}$ A1

iv. $L\left(\sqrt[3]{6}\right)=4.933\,\text{m}$ A1

Define $l(x)=\sqrt{x^2+4}+\sqrt{y^2+\frac{9}{4}}$	*Done*
$l(x)\mid y=\frac{3}{x}$ and $x>0$	$\frac{(2\cdot x+3)\cdot\sqrt{x^2+4}}{2\cdot x}$
Define $l(x)=\frac{(2\cdot x+3)\cdot\sqrt{x^2+4}}{2\cdot x}$	*Done*
$\frac{d}{dx}(l(x))\mid x>0$	$\frac{2\cdot x+3}{2\cdot\sqrt{x^2+4}}-\frac{3\cdot\sqrt{x^2+4}}{2\cdot x^2}$
comDenom$\left(\frac{2\cdot x+3}{2\cdot\sqrt{x^2+4}}-\frac{3\cdot\sqrt{x^2+4}}{2\cdot x^2}\right)$	$\frac{x^3-6}{x^2\cdot\sqrt{x^2+4}}$
solve$\left(\frac{x^3-6}{x^2\cdot\sqrt{x^2+4}}=0,x\right)$	$x=6^{\frac{1}{3}}$
$l(x)\mid x=6^{\frac{1}{3}}$	4.93283

b.i. $\cos(\theta) = \dfrac{2}{AC} \quad \Rightarrow \quad AC = \dfrac{2}{\cos(\theta)}$

$\sin(\theta) = \dfrac{1.5}{CB} \quad \Rightarrow \quad CB = \dfrac{3}{2\sin(\theta)}$

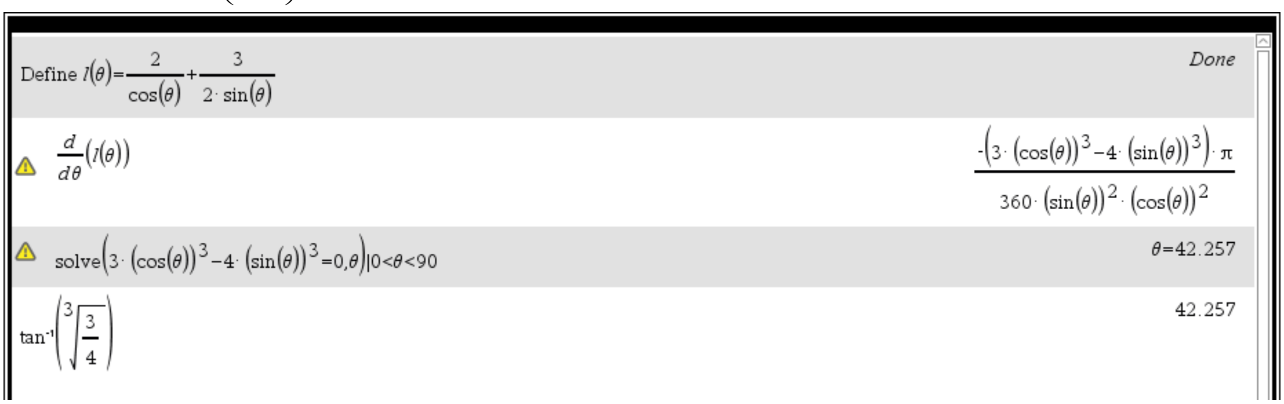
$L(\theta) = AC + BC = \dfrac{2}{\cos(\theta)} + \dfrac{3}{2\sin(\theta)}$ for $0 < \theta < 90^0$ A1

ii. $\dfrac{dL}{d\theta} = \dfrac{\pi\left(4\sin^3(\theta) - 3\cos^3(\theta)\right)}{360\sin^2(\theta)\cos^2(\theta)}$

$\dfrac{dL}{d\theta} = 0 \quad \Rightarrow \quad 4\sin^3(\theta) - 3\cos^3(\theta) = 0$ A1

$\tan^3(\theta) = \dfrac{3}{4} \qquad \tan(\theta) = \sqrt[3]{\dfrac{3}{4}}$

$\theta = \tan^{-1}\left(\sqrt[3]{\dfrac{3}{4}}\right) = 42.26^0$ A1



c.i. **Method 1**

$L(x) = AC + CB = \sqrt{x^2+b^2} + \sqrt{y^2+a^2}$ by similar triangles $\frac{a}{y} = \frac{x}{b} \Rightarrow y = \frac{ab}{x}$

$$L(x) = \sqrt{x^2+b^2} + \sqrt{\frac{a^2b^2}{x^2} + a^2}$$ A1

$$= \sqrt{x^2+b^2} + \sqrt{\frac{a^2\left(x^2+b^2\right)}{x^2}} = \sqrt{x^2+b^2}\left(\frac{a+x}{x}\right)$$ since $x > 0$, $a > 0$

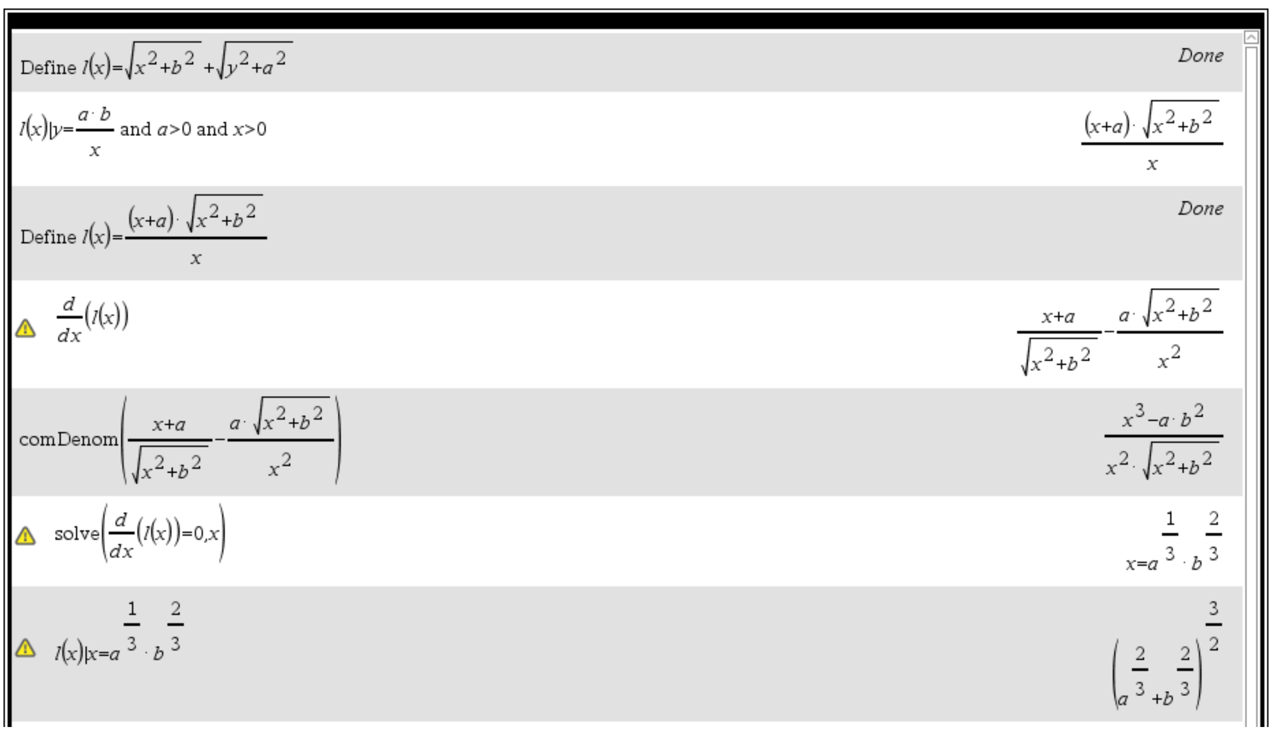
ii. $$\frac{dL}{dx} = \frac{x^3 - ab^2}{x^2\sqrt{x^2+b^2}}$$

$\frac{dL}{dx} = 0$ solving for x, M1

$$\Rightarrow \quad x^3 = ab^2 \Rightarrow x = \sqrt[3]{ab^2} = \left(ab^2\right)^{\frac{1}{3}}$$

$$L_{\min} = L\left(\sqrt[3]{ab^2}\right) = \sqrt{\left(\left(ab^2\right)^{\frac{2}{3}} + b^2\right)}\left(\frac{a+\sqrt[3]{ab^2}}{\sqrt[3]{ab^2}}\right)$$

$L_{\min} = \left(a^{\frac{2}{3}} + b^{\frac{2}{3}}\right)^{\frac{3}{2}}$ $m = \frac{2}{3}$, $n = \frac{3}{2}$ A2



c.i. **Method 2**

The angle θ is at *A*.

$$L(\theta)=\frac{a}{\sin(\theta)}+\frac{b}{\cos(\theta)}$$ A1

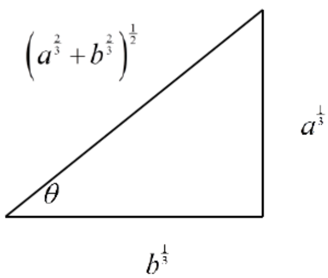
ii. $$\frac{dL}{d\theta}=\frac{\pi\left(-a\cos^3(\theta)+b\sin^3(\theta)\right)}{180\sin^2(\theta)\cos^2(\theta)}$$

$$\frac{dL}{d\theta}=0 \Rightarrow -a\cos^3(\theta)+b\sin^3(\theta)=0$$ M1

$$\tan^3(\theta)=\frac{a}{b} \quad \Rightarrow \quad \tan(\theta)=\left(\frac{a}{b}\right)^{\frac{1}{3}}$$

$$\Rightarrow \quad \theta=\tan^{-1}\left(\frac{a}{b}\right)^{\frac{1}{3}}$$

so that

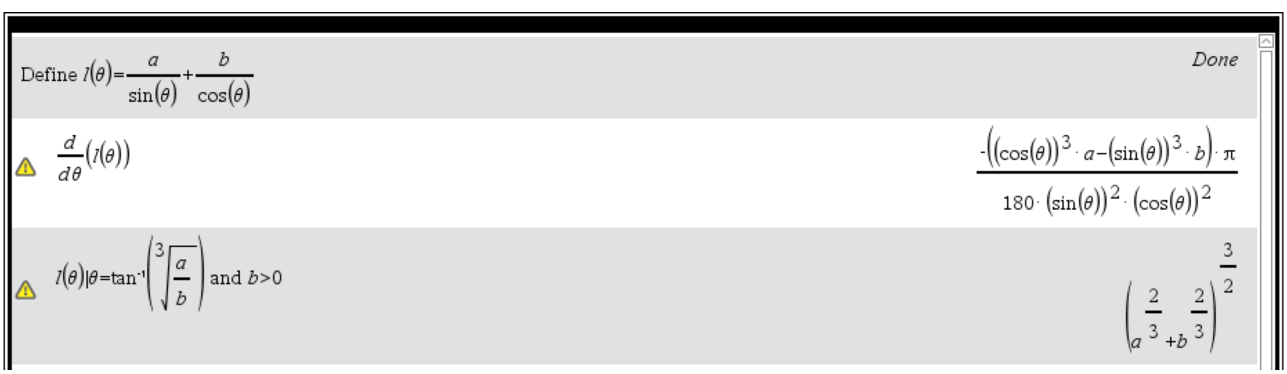


$$\frac{1}{\sin(\theta)}=\frac{\left(a^{\frac{2}{3}}+b^{\frac{2}{3}}\right)^{\frac{1}{2}}}{a^{\frac{1}{3}}} \quad \text{and} \quad \frac{1}{\cos(\theta)}=\frac{\left(a^{\frac{2}{3}}+b^{\frac{2}{3}}\right)^{\frac{1}{2}}}{b^{\frac{1}{3}}}$$

$$L_{\min}=L\left(\tan^{-1}\left(\frac{a}{b}\right)^{\frac{1}{3}}\right)=a\frac{\left(a^{\frac{2}{3}}+b^{\frac{2}{3}}\right)^{\frac{1}{2}}}{a^{\frac{1}{3}}}+b\frac{\left(a^{\frac{2}{3}}+b^{\frac{2}{3}}\right)^{\frac{1}{2}}}{b^{\frac{1}{3}}}$$

$$L_{\min}=\left(a^{\frac{2}{3}}+b^{\frac{2}{3}}\right)^{\frac{1}{2}}\left[a^{\frac{2}{3}}+b^{\frac{2}{3}}\right]$$

$$L_{\min}=\left(a^{\frac{2}{3}}+b^{\frac{2}{3}}\right)^{\frac{3}{2}} \quad m=\frac{2}{3},\ n=\frac{3}{2}$$ A2



END OF SUGGESTED SOLUTIONS