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# MATHS METHODS 3 & 4 TRIAL EXAMINATION 1 SOLUTIONS 2017

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**Question 1** (4 marks)

**a.**
$$\frac{d}{dx}(2x\log_e(x))$$
$$= 2\log_e(x) + 2x \times \frac{1}{x} \qquad \text{(product rule)}$$
$$= 2\log_e(x) + 2 \qquad \text{(stop here!)}$$

**(1 mark)**

**b.**
$$f(x) = \frac{\tan(x)}{3x}$$
$$f'(x) = \frac{3x \times \sec^2(x) - 3\tan(x)}{9x^2} \qquad \text{(quotient rule)}$$

**(1 mark)**

$$f'\left(\frac{\pi}{3}\right) = \frac{3 \times \frac{\pi}{3} \times \sec^2\left(\frac{\pi}{3}\right) - 3\tan\left(\frac{\pi}{3}\right)}{9 \times \frac{\pi^2}{9}}$$
$$= \frac{\pi \times 4 - 3 \times \sqrt{3}}{\pi^2}$$
$$= \frac{4\pi - 3\sqrt{3}}{\pi^2}$$

**(1 mark)**

since
$$\sec^2\left(\frac{\pi}{3}\right) = \frac{1}{\cos^2\left(\frac{\pi}{3}\right)}$$
$$= 1 \div \left(\frac{1}{2}\right)^2$$
$$= 4$$

**(1 mark)**

**Question 2** (3 marks)

$$g'(x) = 3 - \frac{2}{x}, \quad x > 0$$
$$g(x) = \int \left(3 - \frac{2}{x}\right) dx$$

**(1 mark)**

$$= 3x - 2\log_e(x) + c, \quad \text{since } x > 0$$

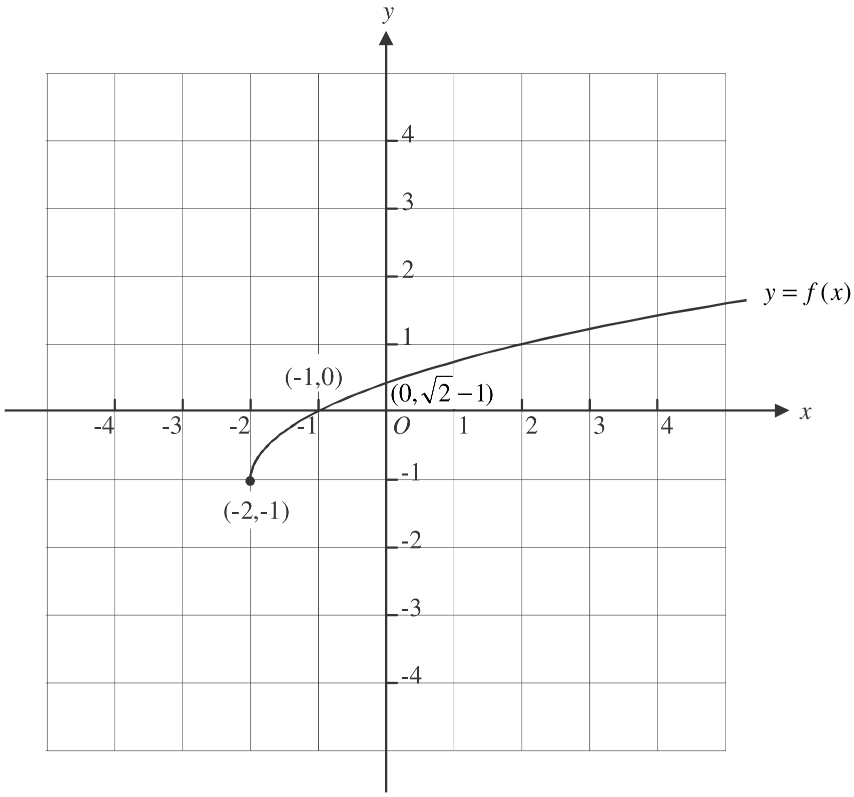
**(1 mark)**

Since $g(1) = 2$,
$$2 = 3 \times 1 - 2\log_e(1) + c$$
$$c = 2 - 3 \qquad (\text{since } \log_e(1) = 0)$$
$$c = -1$$
So $g(x) = 3x - 2\log_e(x) - 1$

**(1 mark)**

**Question 3** (4 marks)

**a.**




The graph of *f* is obtained by translating the graph of $y=\sqrt{x}$ two units left and one unit down. The endpoint is therefore located at $(-2,-1)$.

<u>*x*-intercept</u> occurs when $y=0$

$$y=\sqrt{x+2}-1$$

So, $0=\sqrt{x+2}-1$

$$1=\sqrt{x+2}$$

$$1=x+2 \qquad \text{(ie square both sides)}$$

$$x=-1$$

*x*-intercept occurs at $(-1,0)$

<u>*y*-intercept</u> occurs when $x=0$

$$y=\sqrt{0+2}-1$$

$$y=\sqrt{2}-1$$

*y*-intercept occurs at $\left(0,\sqrt{2}-1\right)$

**(1 mark)** – correct endpoint **(1 mark)** – correct intercepts **(1 mark)** – correct shape

**b.**

$$\begin{aligned}\text{average rate of change} &= \frac{f(2)-f(-1)}{2-{}^{-}1}\\ &= \frac{\left(\sqrt{4}-1\right)-\left(\sqrt{1}-1\right)}{3}\\ &= \frac{1-0}{3}\\ &= \frac{1}{3}\end{aligned}$$

**(1 mark)**

**Question 4** (3 marks)

**a.** Let $X$ represent the number of girls selected in a week.

$$X \sim Bi\left(3, \frac{2}{5}\right)$$

$$\Pr(X = 0) = {}^3C_0\left(\frac{2}{5}\right)^0\left(\frac{3}{5}\right)^3$$
$$= \frac{27}{125}$$

**(1 mark)**

**b.** $\Pr(X \geq 2) = \Pr(X = 2) + \Pr(X = 3)$

$$= {}^3C_2\left(\frac{2}{5}\right)^2\left(\frac{3}{5}\right)^1 + {}^3C_3\left(\frac{2}{5}\right)^3\left(\frac{3}{5}\right)^0$$
$$= 3\times\frac{4}{25}\times\frac{3}{5} + 1\times\frac{8}{125}$$
$$= \frac{36}{125} + \frac{8}{125}$$
$$= \frac{44}{125}$$

**(1 mark)**

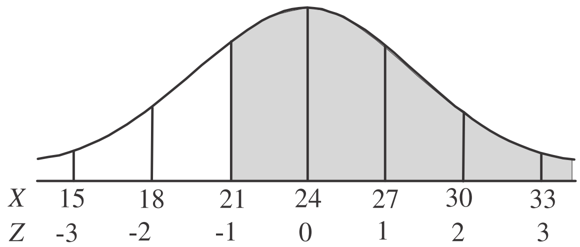
**c.** We have a binomial distribution, so $\text{mean} = E(X)$

$$= np$$
$$= 3\times\frac{2}{5}$$
$$= \frac{6}{5}$$

**(1 mark)**

**Question 5** (4 marks)

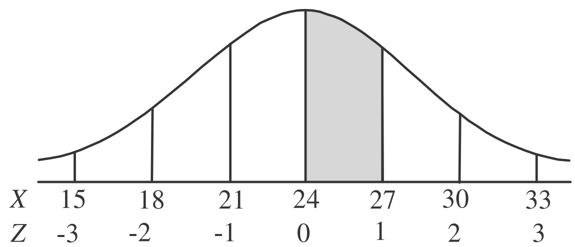
**a.** Draw a diagram.
$\Pr(X > 21)$ is shaded.
Since $\Pr(Z > 1) = 0.16$,
$\Pr(Z < -1) = 0.16$
because of the symmetry of the normal curve.
Now, $\Pr(Z < -1) = \Pr(X < 21)$
So $\Pr(X > 21) = 1 - \Pr(X < 21)$
$= 1 - 0.16$
$= 0.84$



Pr($X$ >21) is shaded

**(1 mark)**

**b.** Again, draw a diagram.
$\Pr(24 < X < 27)$ is shaded.
$\Pr(24 < X < 27)$
$= \Pr(X < 27) - \Pr(X < 24)$
$= \Pr(Z < 1) - 0.5$
$= 1 - 0.16 - 0.5$
$= 0.34$



Pr(24<$X$ <27) is shaded

**(1 mark)**

**c.** $\Pr(X<21 \mid X<24)$

$=\dfrac{\Pr(X<21\cap X<24)}{\Pr(X<24)}$ (conditional probability formula) **(1 mark)**

$=\dfrac{\Pr(X<21)}{\Pr(X<24)}$

$=\dfrac{0.16}{0.5}$

$=\dfrac{16}{50}$

$=\dfrac{8}{25}$

Alternatively, $\dfrac{16}{50}=\dfrac{32}{100}$
$=0.32$

**(1 mark)**

**Question 6** (5 marks)

**a.** $L(x)=g(x)-f(x)$

$=5-x-\dfrac{4}{x}$

**(1 mark)**

**b.** $L(x)=5-x-4x^{-1}$

$L'(x)=-1+\dfrac{4}{x^2}$

$L'(x)=0$ for max/min.

$-1+\dfrac{4}{x^2}=0$ **(1 mark)**

$\dfrac{4}{x^2}=1$

$x^2=4$

$x=\pm 2$ reject $x=-2$ since $x>0$

So $x=2$

$L(2)=5-2-\dfrac{4}{2}$

$=1$

Maximum length is 1 unit.

**(1 mark)**

**c.** gradient $=\tan(\theta)$

So $f'(2)=\tan(\theta)$ **(1 mark)**

Now $f'(x)=\dfrac{-4}{x^2}$

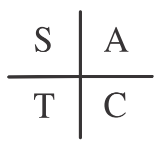
So $f'(2)=-1$

$\tan(\theta)=-1$

$\theta=\dfrac{3\pi^c}{4}$

or $\theta=135^\circ$

$\left(\theta=-\dfrac{\pi}{4}\right.$ or $-45^\circ$ also acceptable)

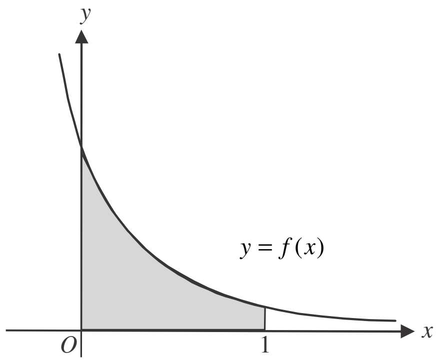



**(1 mark)**

**Question 7** (5 marks)

**a.** The area required is shaded in the diagram below.

$$\text{area} = \int_0^1 f(x)dx$$
$$= \int_0^1 e^{-2x}dx \qquad \textbf{(1 mark)}$$
$$= \left[-\frac{1}{2}e^{-2x}\right]_0^1$$
$$= -\frac{1}{2}(e^{-2} - e^0)$$
$$= \frac{1}{2}\left(1 - \frac{1}{e^2}\right)$$



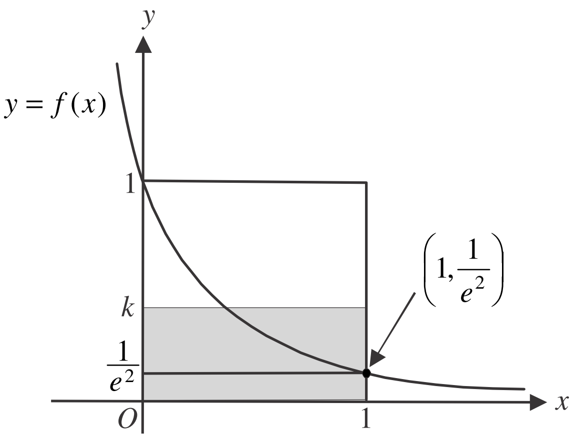

**(1 mark)**

**b.** $$k = \frac{1}{1-0}\int_0^1 f(x)dx$$ (using the average value formula which you must memorise because it isn't on the formula sheet)

$$= \int_0^1 f(x)dx$$
$$= \frac{1}{2}\left(1 - \frac{1}{e^2}\right)$$ using the answer from part **a.**

**(1 mark)**

**c.**




The area of the shaded rectangle (shown above) gives the average value of *f* between $x = 0$ and $x = 1$.
This rectangle has a height of *k* units and a width of 1 unit. **(1 mark)**

The larger rectangle with top left corner point at $(0,1)$ and bottom left corner at the origin has an area of 1 square unit.
The smaller rectangle with top left corner point at $\left(0, \frac{1}{e^2}\right)$ and bottom left corner at the origin has an area of $\frac{1}{e^2}$ square units.

Since all three rectangles have a width of one, then $\frac{1}{e^2} < k < 1$. **(1 mark)**

**Question 8** (4 marks)

**a.** $$\frac{d}{dx}\left(x\cos\left(\frac{x}{2}\right)\right)=1\times\cos\left(\frac{x}{2}\right)+x\times-\frac{1}{2}\sin\left(\frac{x}{2}\right)$$ (product rule)

$$=\cos\left(\frac{x}{2}\right)-\frac{x}{2}\sin\left(\frac{x}{2}\right)$$

**(1 mark)**

**b.** $$E(X)=\int_0^{\frac{2\pi}{3}} x\sin\left(\frac{x}{2}\right)dx.$$ **(1 mark)**

From part **a.**, we have

$$\frac{d}{dx}\left(x\cos\left(\frac{x}{2}\right)\right)=\cos\left(\frac{x}{2}\right)-\frac{x}{2}\sin\left(\frac{x}{2}\right)$$

Rearranging this gives,

$$\frac{x}{2}\sin\left(\frac{x}{2}\right)=\cos\left(\frac{x}{2}\right)-\frac{d}{dx}\left(x\cos\left(\frac{x}{2}\right)\right)$$

So $x\sin\left(\frac{x}{2}\right)=2\cos\left(\frac{x}{2}\right)-2\times\frac{d}{dx}\left(x\cos\left(\frac{x}{2}\right)\right)$ (multiply each and every term by 2)

So $$E(X)=\int_0^{\frac{2\pi}{3}} 2\cos\left(\frac{x}{2}\right)dx-\int_0^{\frac{2\pi}{3}} 2\times\frac{d}{dx}\left(x\cos\left(\frac{x}{2}\right)\right)dx$$

$$=2\left[2\sin\left(\frac{x}{2}\right)\right]_0^{\frac{2\pi}{3}}-2\left[x\cos\left(\frac{x}{2}\right)\right]_0^{\frac{2\pi}{3}}$$ (ie the antiderivative "undoes" the derivative) **(1 mark)**

$$=4\left(\sin\left(\frac{\pi}{3}\right)-\sin(0)\right)-2\left(\frac{2\pi}{3}\cos\left(\frac{\pi}{3}\right)-0\right)$$

$$=4\left(\frac{\sqrt{3}}{2}-0\right)-\frac{4\pi}{3}\times\frac{1}{2}$$

$$=2\sqrt{3}-\frac{2\pi}{3}$$

**(1 mark)**

**Question 9** (8 marks)

**a.** Stationary points occur when $f'(x)=0$

$f(x)=x^3-ax^2$

$f'(x)=3x^2-2ax=0$ (remember $a$ is a constant) **(1 mark)**

$x(3x-2a)=0$

$x=0$ or $3x-2a=0$

$3x=2a$

$x=\frac{2a}{3}$ **(1 mark)** correct $x$ coordinates

Now, $f(0)=0$, so one stationary point occurs at $(0,0)$.

$$f\left(\frac{2a}{3}\right)=\left(\frac{2a}{3}\right)^3-a\left(\frac{2a}{3}\right)^2$$

$$=\frac{8a^3}{27}-a\times\frac{4a^2}{9}$$

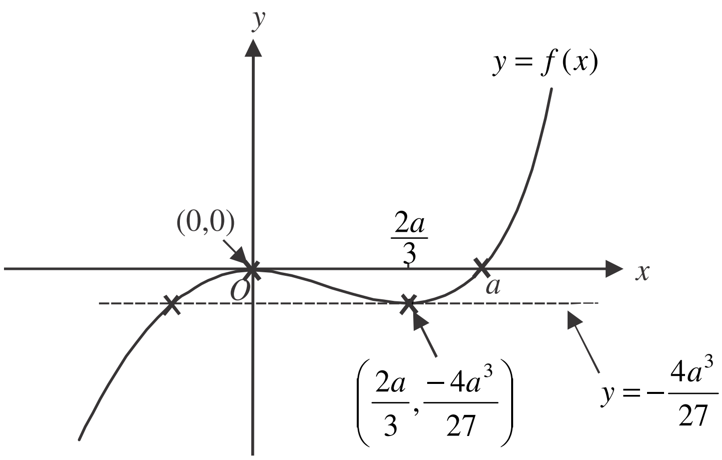
$$=\frac{8a^3}{27}-\frac{4a^3}{9}$$

$$=\frac{8a^3}{27}-\frac{12a^3}{27}$$

$$=\frac{-4a^3}{27}$$

The other stationary point occurs at $\left(\frac{2a}{3},-\frac{4a^3}{27}\right)$. **(1 mark)**

**b.** Look at the graph.




The equation $f(x)=n$ has two solutions when the graph of $y=f(x)$ intersects with the graph of $y=n$ twice. Note that the graph of $y=n$ is a horizontal straight line. The graph of $y=f(x)$ intersects with the graph of $y=0$ (which is of course is the $x$-axis) just twice. So one of the values of $n$ is zero.

The graph of $y=f(x)$ also intersects with the graph of $y=n$ twice when

$n=-\frac{4a^3}{27}$ (using part **a.**).

So $n=0$ or $n=-\frac{4a^3}{27}$. **(1 mark)**

**c.** Method 1

Point $U$ lies between the two stationary points and occurs when $f''(x)=0$. Point $U$ is a point of inflection. **(1 mark)**

$$f''(x)=6x-2a=0$$
$$6x=2a$$
$$x=\frac{a}{3}$$

So $u=\frac{a}{3}$. **(1 mark)**

Method 2

Gradient of tangent given by $f'(x)=3x^2-2ax$ (from part **a.**).

Let the gradient of $f$ at the point $U$ be $m$.

So $3x^2-2ax=m$

i.e. $3x^2-2ax-m=0$

We want one solution to this equation. This is a quadratic equation in the variable $x$ so one solution occurs when

$(-2a)^2-4\times3\times^{-}m=0$ (i.e. the discriminant $b^2-4ac$)

$$4a^2+12m=0$$
$$12m=-4a^2$$
$$m=\frac{-4a^2}{12}$$
$$=\frac{-a^2}{3}$$

So the gradient of the tangent is $\frac{-a^2}{3}$.

**(1 mark)**

Since $f'(x)=3x^2-2ax$

then $3x^2-2ax=\frac{-a^2}{3}$

$$3x^2-2ax+\frac{a^2}{3}=0$$
$$9x^2-6ax+a^2=0$$
$$(3x-a)(3x-a)=0$$
$$3x=a$$
$$x=\frac{a}{3}$$

So $u=\frac{a}{3}$.

**(1 mark)**

**d.** $f'(x)=3x^2-2ax$ from part **a**.

At $V(v,f(v))$, the gradient of the tangent is $3v^2-2av$.

At $W(w,f(w))$, the gradient of the tangent is $3w^2-2aw$.

We are told that the tangents at $V$ and $W$ have the same gradient so

$$3v^2-2av=3w^2-2aw$$ **(1 mark)**

$$3v^2-3w^2=2av-2aw$$
$$3(v^2-w^2)=2a(v-w)$$
$$3(v-w)(v+w)=2a(v-w)$$
$$3(v+w)=2a \quad \text{since } v-w\neq0 \text{ i.e. } v\neq w$$

So $v+w=\frac{2a}{3}$.

**(1 mark)**