The Mathematical Association of Victoria

# Trial Examination 2016

# MATHEMATICAL METHODS

## Written Examination 2 - SOLUTIONS

## SECTION A

**1. A 2. C 3. E 4. C 5. A 6. D 7. E 8. B 9. B 10. A**

**11. E 12. D 13. A 14. E 15. B 16. D 17. D 18. C 19. C 20. B**

### SECTION A

### Question 1

$f : R \to R, f(x) = -2 - 3\cos\left(\frac{x}{4} + 1\right)$

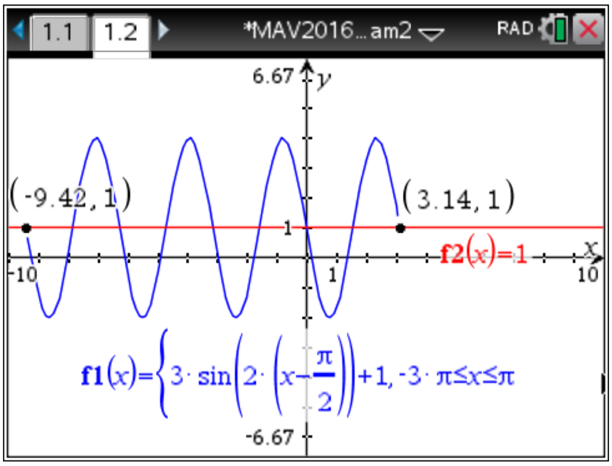
The range is $[-3-2, 3-2] = [-5, 1]$

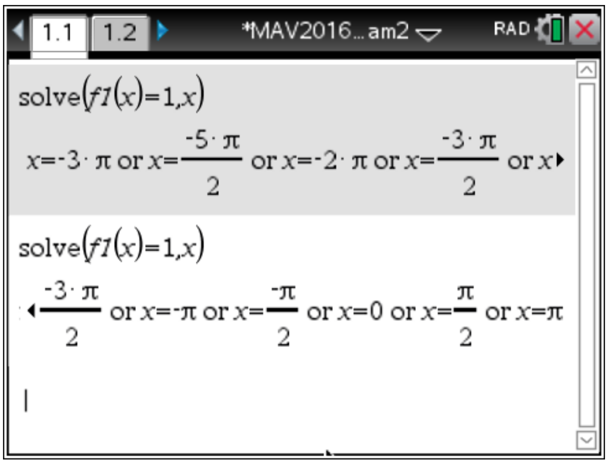
The Period is $\frac{2\pi}{\frac{1}{4}} = 8\pi$ **A**

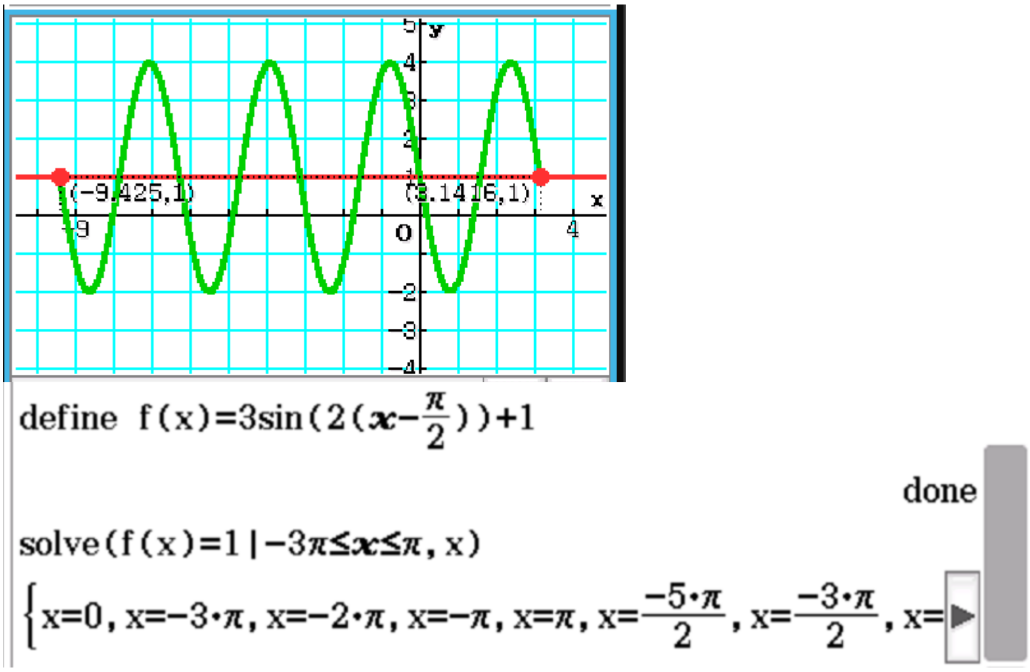
### Question 2

$g : [-3\pi, \pi] \to R, g(x) = 3\sin\left(2\left(x - \frac{\pi}{2}\right)\right) + 1$

There are 9 intersections when $k = 1$. **C**






**Question 3**

The domain of $f\left(g(x)\right)$ is the same as the domain of $g$ where $g(x)=\tan(\pi x)$.

The asymptotes of $g$ have equations $y=\frac{1}{2}+k, k\in Z$.

Hence the domain of $f\left(g(x)\right)$ is $R\setminus\left\{\frac{1}{2}+k\right\}, k\in Z$ **E**

**Question 4**

$$\sin\left(\frac{3\pi}{2}-x\right)=-\cos(x)$$

$$\sqrt{15^2-2^2}=\sqrt{221}$$

$$\sin\left(\frac{3\pi}{2}-x\right)=-\frac{\sqrt{221}}{15}$$ **C**

**Question 5**

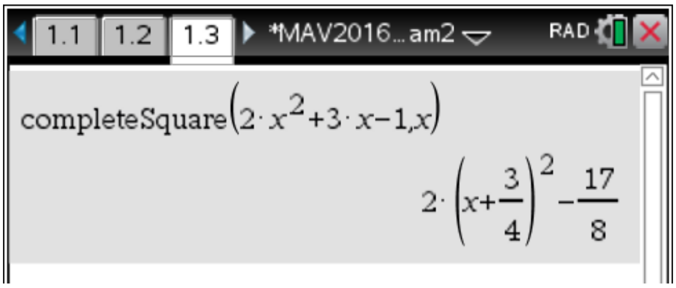
$$y=2x^2+3x-1=2\left(x+\frac{3}{4}\right)^2-\frac{17}{8}$$

The graph of $y=x^2$ has been dilated by a factor of 2 from the $x$-axis and

Then translated $\frac{3}{4}$ units to the left and $\frac{17}{8}$ units down.

A possible rule for the transformation $T:R^2\to R^2$ is

$$T\left(\begin{bmatrix}x\\y\end{bmatrix}\right)=\begin{bmatrix}1&0\\0&2\end{bmatrix}\begin{bmatrix}x\\y\end{bmatrix}-\begin{bmatrix}\frac{3}{4}\\\frac{17}{8}\end{bmatrix}$$ **A**

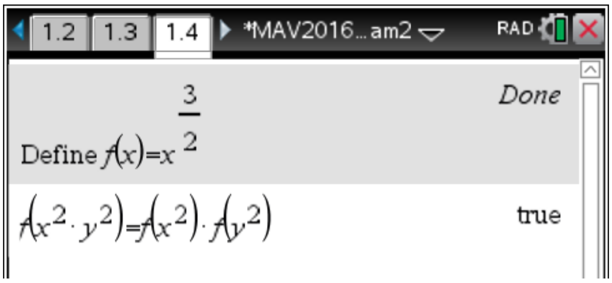


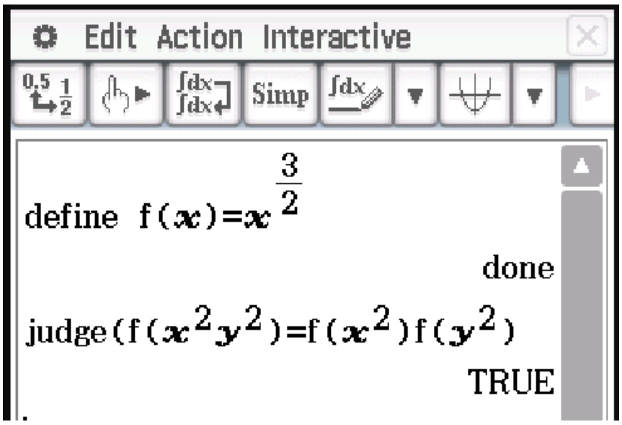

**Question 6**

$f(x) = x^{\frac{3}{2}}$

$f(x^2y^2) = \left(x^2y^2\right)^{\frac{3}{2}} = x^3y^3$

$f(x^2)f\left(y^2\right) = \left(x^2\right)^{\frac{3}{2}}\left(y^2\right)^{\frac{3}{2}} = x^3y^3$ **D**

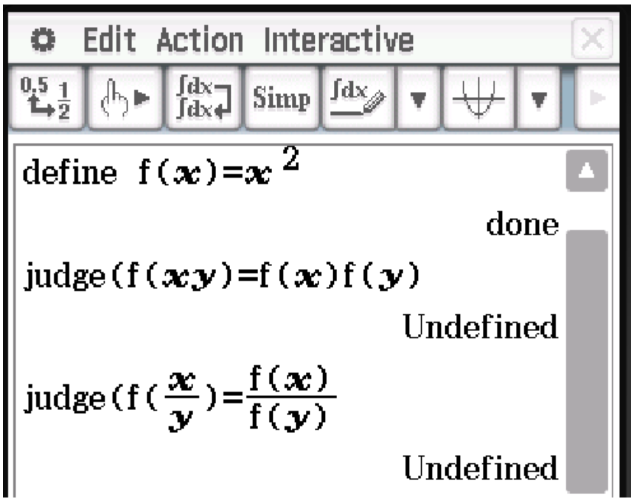



Using counter examples

$f(xy) = f(x)f(y),\ f\left(-1\times -1\right) = 1,\ f\left(-1\right)\times f\left(-1\right)$ has no real solution

$f\left(\frac{x}{y}\right) = \frac{f(x)}{f(y)},\ f\left(\frac{-1}{-1}\right) = 1,\ \frac{f(-1)}{f(-1)}$ has no real solution




$f(x+y)=f(x)+f(y)$, $f(1+1)=2^{\frac{3}{2}}, f(1)+f(1)=2$

$f(x-y)=f(x)-f(y)$, $f(2-1)=1, f(1)-f(1)=0$

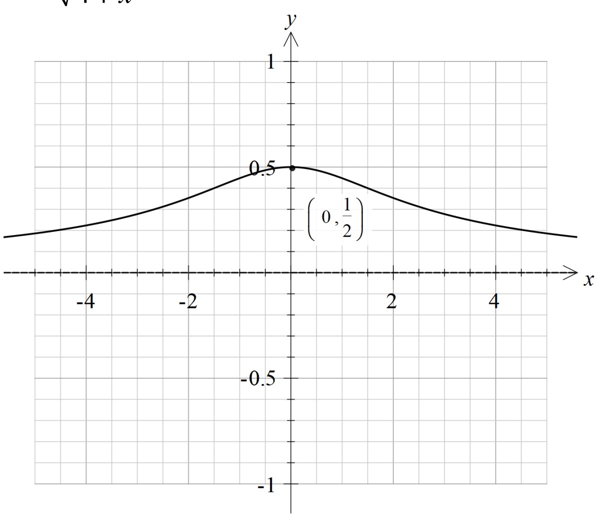
**Question 7**

When $x = 0$, $y = 0.5$.

Either **C** or **E**

When $x = 2$, $y \approx 0.35$

$y=\dfrac{1}{\sqrt{4+x^2}}$ **E**



### Question 8

$f:(1,\infty)\to R, f(x)=\dfrac{3}{(x-1)^2}+2$

Range is $(2,\infty)$ which is the domain of the inverse function.

Let $y=\dfrac{3}{(x-1)^2}+2$

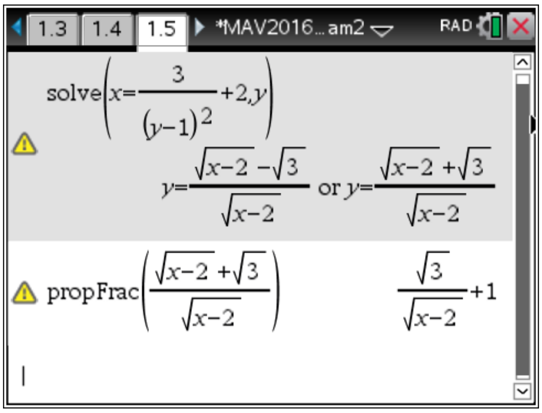
Inverse swap $x$ and $y$.

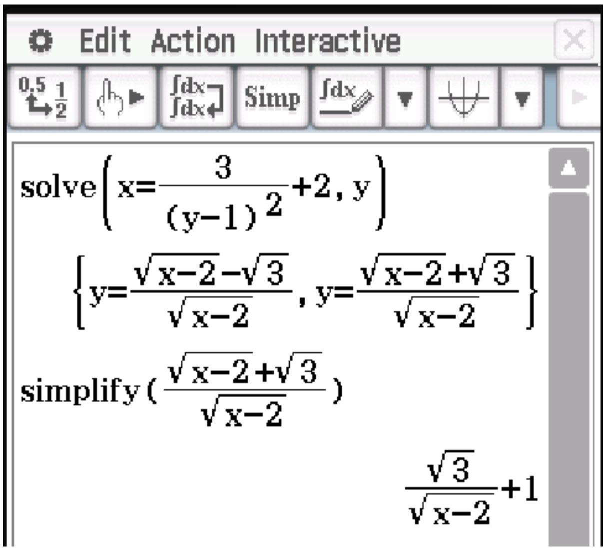
$$x=\frac{3}{(y-1)^2}+2$$

$$y=\pm\sqrt{\frac{3}{x-2}}+1$$

$y=\sqrt{\dfrac{3}{x-2}}+1$, as the domain of $f$ is $(1,\infty)$.

$f^{-1}:(2,\infty)\to R, f^{-1}(x)=\sqrt{\dfrac{3}{x-2}}+1$ **B**

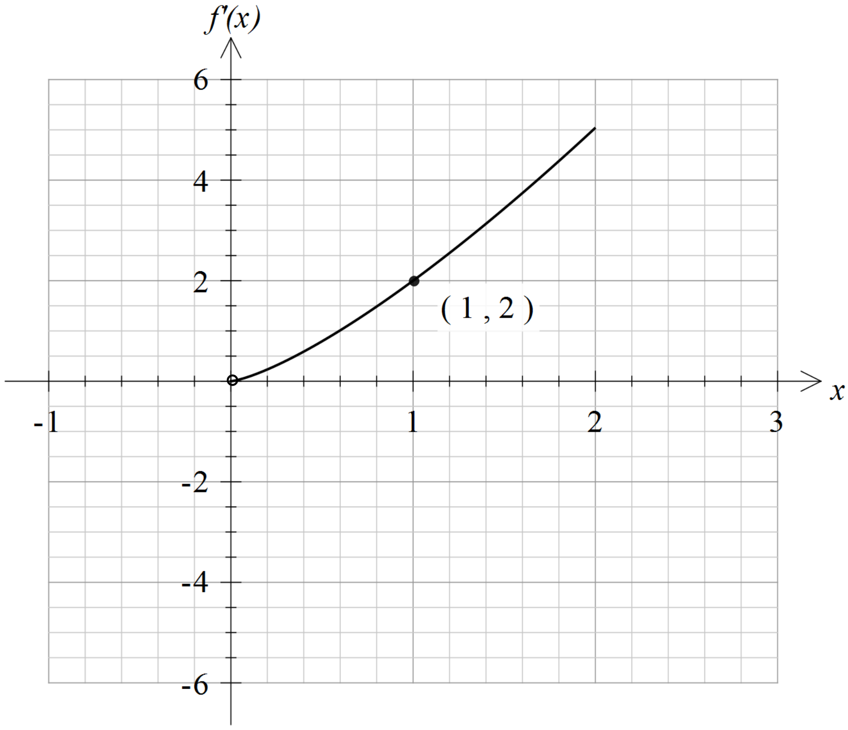


**Question 9**

The point (1, 2) is on the curve of $f\,'$.

Thus the gradient of the graph of $f$ at $x = 1$ is 2. **B**




**Question 10**

If $f(x) = ax^5 + bx^4 + cx^3 + dx^2 + ex + h$ has a stationary point of inflection at $(-3,0)$ and a turning point at (2, 0) then $f(x) = a(x+3)^3(x-2)^2$.

$h = a \times 3^3 \times (-2)^2 = 108a$

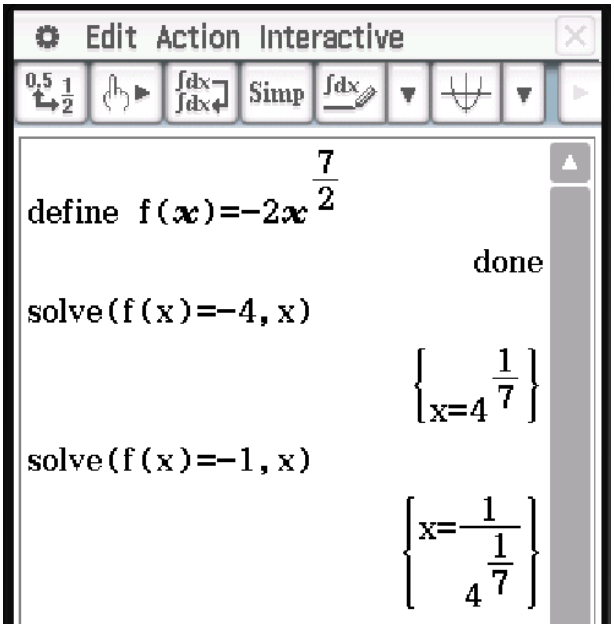
$\frac{h}{a} = 108$ **A**

**Question 11**

$f(x) = -2x^{\frac{7}{2}}$ , $y = -4$ to $y = -1$

Solve $f(x) = -4, x = 2^{\frac{2}{7}}$

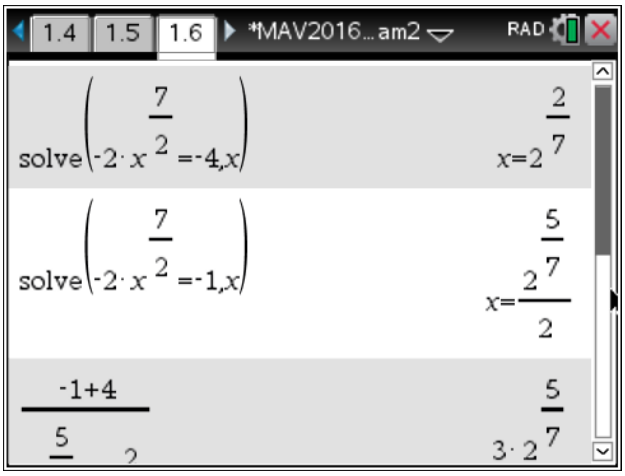
and $f(x) = -1, x = 2^{-\frac{2}{7}}$

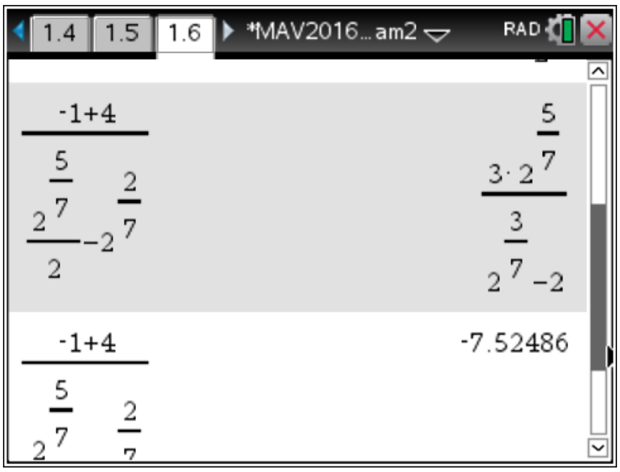


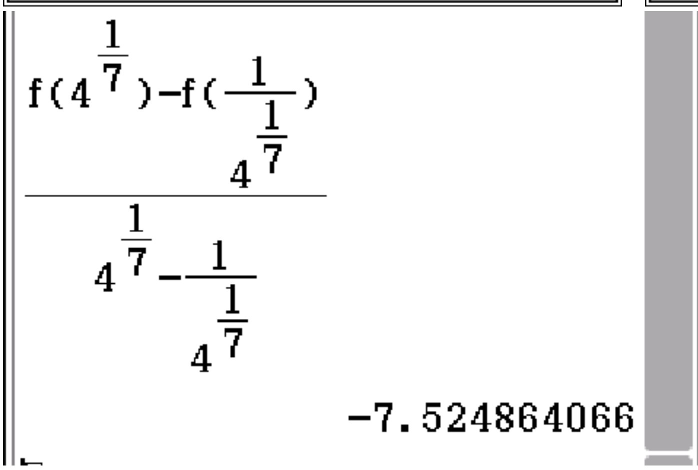

Average rate of change $= \dfrac{y_2 - y_1}{x_2 - x_1}$

$$= \frac{-1+4}{2^{-\frac{2}{7}} - 2^{\frac{2}{7}}}$$

$$\approx -7.5$$ **E**



## Question 12

$\log_a(b) = c$ and $\log_c(a) = b$

$\log_a(b) + \log_c(a) = b + c$

$\log_a(b) + \dfrac{\log_a(a)}{\log_a(c)} = b + c$

$\log_a(b) + \dfrac{1}{\log_a(c)} = b + c$

$\log_a\left(\dfrac{b}{c}\right) \neq b + c$ **D**

## Question 13

$f'(x) = 2x^3 - 2x$

$f(x) = \int \left(2x^3 - 2x\right) dx$

$= \dfrac{x^4}{2} - x^2 + c$

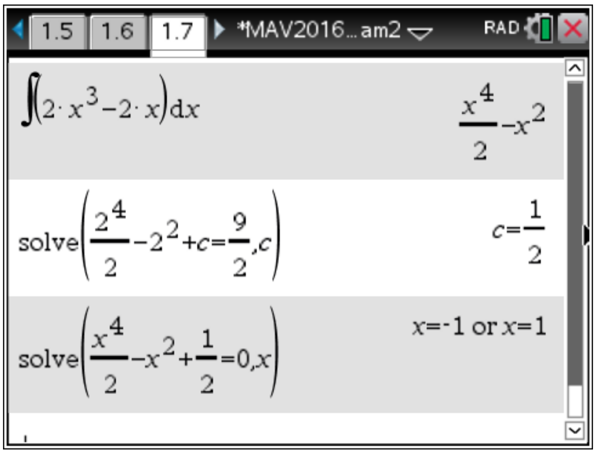
$f(2) = \dfrac{9}{2}$

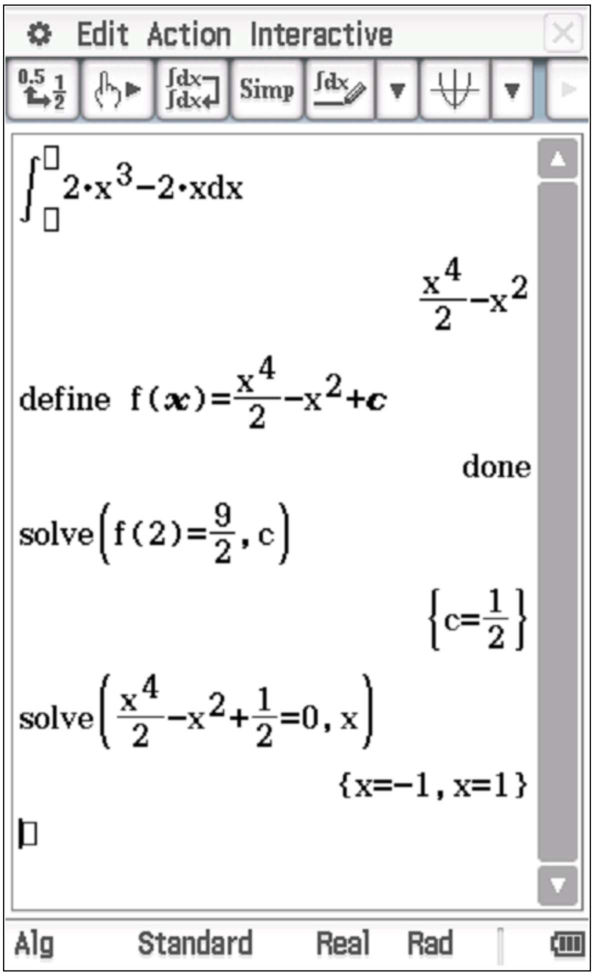
$\dfrac{9}{2} = \dfrac{2^4}{2} - 2^2 + c$

$c = \dfrac{1}{2}$

Solve $\dfrac{x^4}{2} - x^2 + \dfrac{1}{2} = 0$ for $x$.

$x = \pm 1$ **A**






**Question 14**

$$\int_{2}^{3}(g(x))dx = -3$$

$$= 2\int_{3}^{2}(1-2g(x))dx$$

$$= 2\int_{3}^{2}(1)dx - 4\int_{3}^{2}(g(x))$$

$$= 2[x]_3^2 - 4\times 3$$

$$= -14$$ **E**

**Question 15**

Area $= \frac{1}{2}\left(e^0 + e + e^2 + e^3\right)$

$= \frac{1}{2}\left(1 + e + e^2 + e^3\right)$

The rectangles are below the curve.
Underestimate **B**

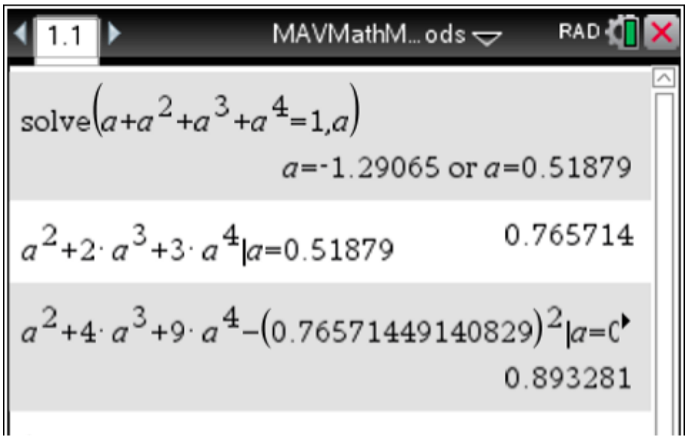
**Question 16**

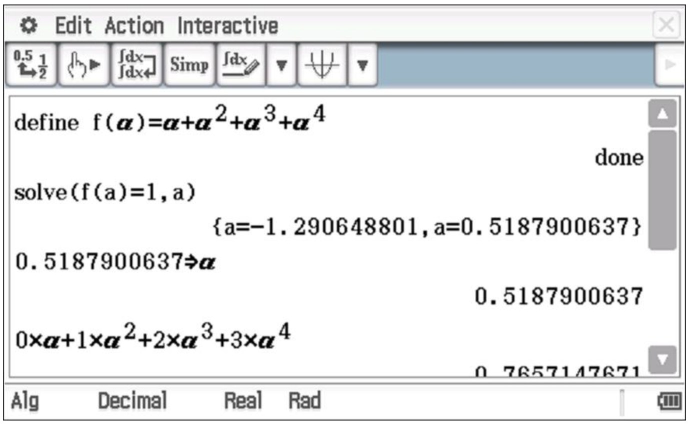
Solve $a + a^2 + a^3 + a^4 = 1$ for $a$

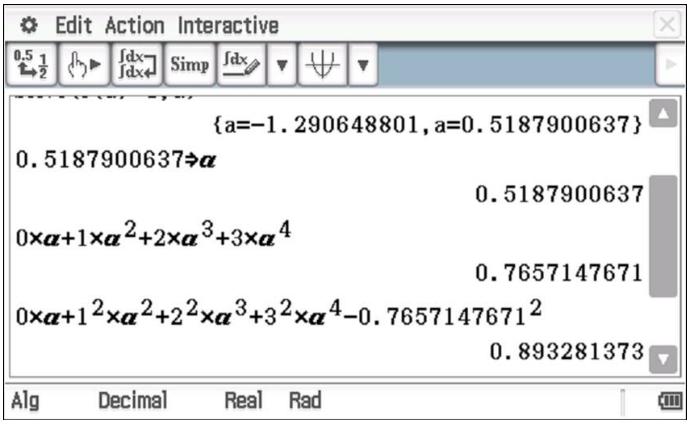
$a \approx 0.51879$

$E(X) = a^2 + 2\times a^3 + 3\times a^4 \approx 0.766$

$Var(X) = a^2 + 4a^3 + 9a^4 - (0.765714\ldots)^2 \approx 0.893$ **D**




## Question 17

| | $J$ | $J'$ | |
|---|---|---|---|
| $R$ | $0.3p$ | $0.7p$ | $p$ |
| $R'$ | $0.3-0.3p$ | $p^2$ | $1-p$ |
| | $0.3$ | $0.7$ | $1$ |

Independent events $\Pr(J \cap R) = \Pr(J) \times \Pr(R)$

Let $a = \Pr(J \cap R) = 0.3p$

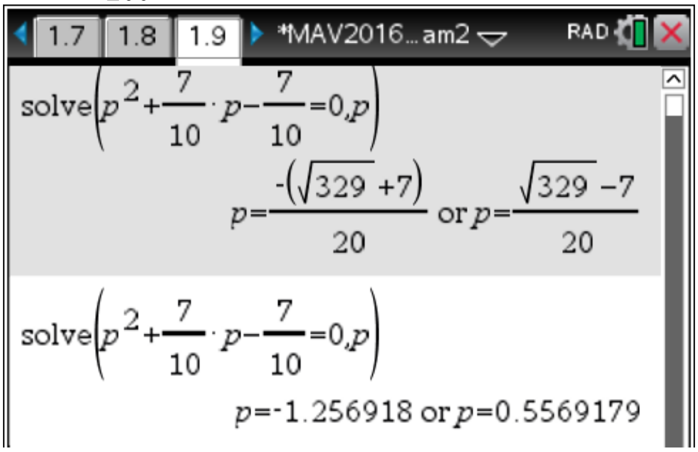
$0.3 + p - a + p^2 = 1$

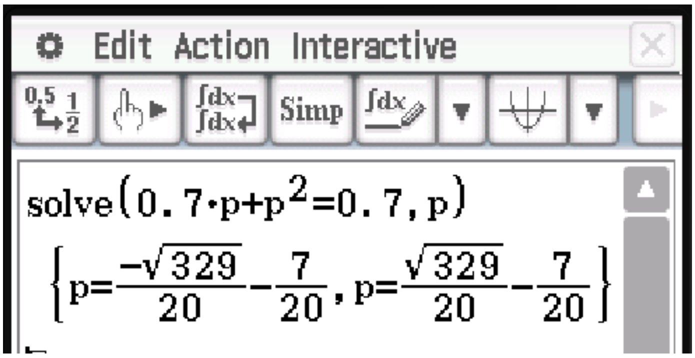
Solve $p^2 + 0.7p - 0.7 = 0$ for $p$.

$$p = \frac{\sqrt{329} - 7}{20}$$

$$a = 0.3p$$

$$a = \frac{3\left(\sqrt{329} - 7\right)}{200}$$ **D**




## Question 18

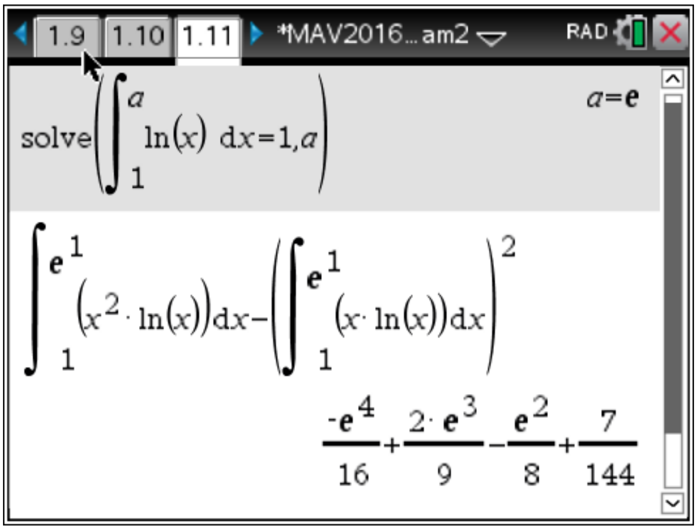
Solve $\int_1^a (\log_e(x))\,dx = 1$ for $a$.

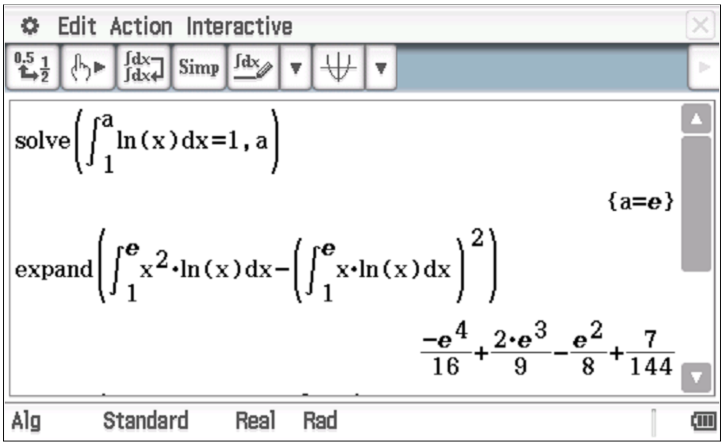
$a = e$

$$\text{Var}(X) = \text{E}(X^2) - (\text{E}(X))^2$$

$$\text{Var}(X) = \int_1^e (x^2 \log_e(x))\,dx - \left(\int_1^e (x \log_e(x))\,dx\right)^2$$

$$= -\frac{e^4}{16} + \frac{2e^3}{9} - \frac{e^2}{8} + \frac{7}{144}$$ **C**



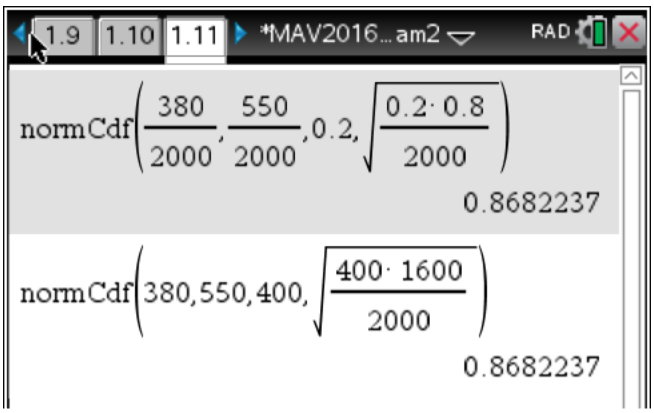



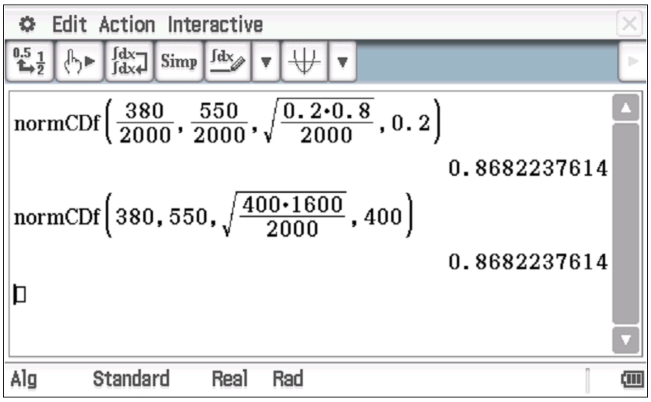

## Question 19

$n = 2000$ and $p = 0.2$. Since $n$ is large, the distribution of $\hat{P}$ is approximately normal with $\mu = 0.2$ and $\sigma = \sqrt{\dfrac{0.2 \times 0.8}{2000}}$.

$$\Pr\left(\frac{380}{2000} < X < \frac{550}{2000}\right) \approx 0.868$$

**C**






## Question 20

If 200 such samples were taken, $0.9 \times 200 = 180$ of the confidence intervals would be expected to contain *p*. **B**

## SECTION B
## EXTENDED RESPONSE QUESTIONS

**Question 1** (14 marks)

**a.** $f(x) = ax^{\frac{2}{3}}$

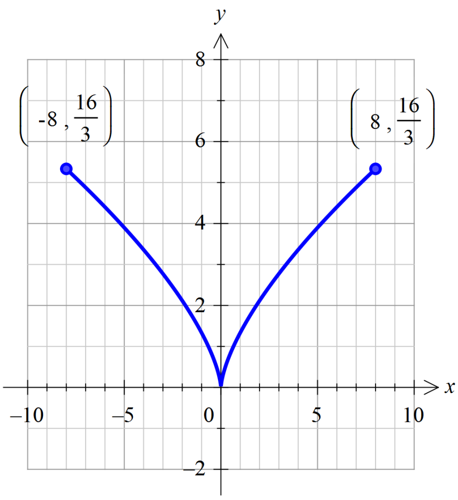
Substitute the point $\left(-8, \frac{16}{3}\right)$

$\frac{16}{3} = a(-8)^{\frac{2}{3}}$

$\frac{16}{3} = 4a$ **1M Show that**

$a = \frac{4}{3}$

**b.** Shape **1A**
Endpoints **1A**




**c.** $f_1(x) = f(-x)$ reflects the graph over the $y$-axis.

As the function $f(x) = \frac{4}{3}x^{\frac{2}{3}}$ is an even function the transformation described makes no difference. **1A**

**d.**

- a dilation of a factor of 2 units from the $x$-axis gives $y = \frac{8}{3}x^{\frac{2}{3}}$

  then

- a translation of 3 units up gives $f_2(x) = \frac{8}{3}x^{\frac{2}{3}} + 3$ **1A**

**e**.

- a reflection in the $x$-axis gives $y_1 = -\frac{4}{3}x^{\frac{2}{3}}$
- a reflection in the $y$-axis gives $y_2 = -\frac{4}{3}(-x)^{\frac{2}{3}} = -\frac{4}{3}x^{\frac{2}{3}}$
- a translation of 1 unit in the positive direction of the $y$-axis gives $y_3 = -\frac{4}{3}x^{\frac{2}{3}} + 1$
- a translation of 3 units in the positive direction of the $x$-axis gives $y_4 = -\frac{4}{3}(x-3)^{\frac{2}{3}} + 1$

The image equation is $f_3(x) = -\frac{4}{3}(x-3)^{\frac{2}{3}} + 1$ **2A**

**f.** $T: R^2 \to R^2, T\left(\begin{bmatrix} x \\ y \end{bmatrix}\right) = \begin{bmatrix} 2 & 0 \\ 0 & -\frac{1}{3} \end{bmatrix}\begin{bmatrix} x \\ y \end{bmatrix} + \begin{bmatrix} 1 \\ 2 \end{bmatrix}$

gives $\begin{bmatrix} 2 & 0 \\ 0 & -\frac{1}{3} \end{bmatrix}\begin{bmatrix} x \\ y \end{bmatrix} + \begin{bmatrix} 1 \\ 2 \end{bmatrix} = \begin{bmatrix} 2x+1 \\ -\frac{1}{3}y+2 \end{bmatrix} = \begin{bmatrix} x_1 \\ y_1 \end{bmatrix}$

Rearrange to get

$x = \frac{x_1 - 1}{2}$

$y = \frac{y_1 - 2}{-\frac{1}{3}} = -3(y_1 - 2)$ **1M**

Susbtitute to find the image equation for $f(x) = \frac{4}{3}x^{\frac{2}{3}}$.

$-3(y_1 - 2) = \frac{4}{3}\left(\frac{x_1 - 1}{2}\right)^{\frac{2}{3}}$ **1M**

Image equation is

$y_1 = -\frac{4}{9}\left(\frac{x_1 - 1}{2}\right)^{\frac{2}{3}} + 2$

Giving $y = 2 - \frac{4}{9}\left(\frac{x-1}{2}\right)^{\frac{2}{3}}$ **1A**

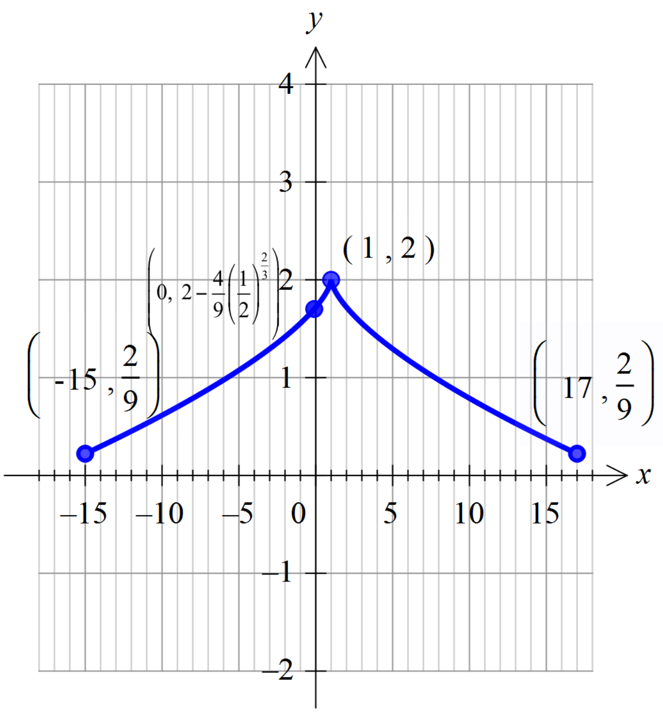
**g**. $y = 2 - \frac{4}{9}\left(\frac{x-1}{2}\right)^{\frac{2}{3}}$ gives a dilation of a factor of 2 from $y$-axis and a translation of 1 unit in the positive direction of the $x$-axis from the original function $f(x) = \frac{4}{3}x^{\frac{2}{3}}$ where the domain was $[-8, 8]$.

A dilation of a factor of 2 from $y$-axis gives the domain $[-16, 16]$.

Then a translation of 1 unit in the positive direction of the $x$-axis gives a domain $[-15, 17]$. **1A**

**h**. Shape **1A**
Endpoints **1A**
Coordinates of Cusp and

*y*-intercept $\left(0,\ 2-\frac{4}{9}\left(\frac{1}{2}\right)^{\frac{2}{3}}\right)$ **1A**



**Question 2** (15 marks)

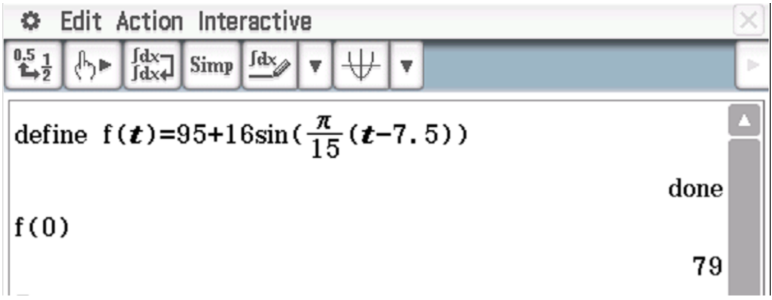
**a.** $P(0) = 95 + 16\sin\left(\frac{\pi}{15}(0 - 7.5)\right)$

$= 95 - 16\sin\left(\frac{\pi}{2}\right)$

$= 95 - 16$

$= 79$

79 beats per minute **1M Show that**




**b.** Period $= \frac{2\pi}{\pi/15} = 30$ **1A**

Amplitude = 16 **1A**

**c.** $P(t) = 95 + 16\sin\left(\frac{\pi}{15}(t - 7.5)\right)$

$P'(t) = 16 \times \frac{\pi}{15}\cos\left(\frac{\pi}{15}(t - 7.5)\right)$ **1A**

$P'(t) = \frac{16\pi}{15}\cos\left(\frac{\pi t}{15} - \frac{\pi}{2}\right)$

$= \frac{16\pi}{15}\cos\left(\frac{\pi}{2} - \frac{\pi t}{15}\right)$

Giving $P'(t) = \frac{16\pi}{15}\sin\left(\frac{\pi t}{15}\right)$. **1M Show that**

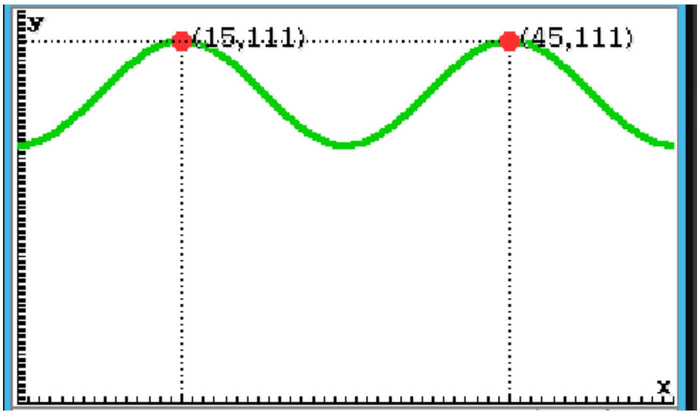
**d.** Solve $\frac{16\pi}{15}\sin\left(\frac{\pi t}{15}\right) = 0$ for $x \in [0, 60]$ 2 cycles **1M**

$\frac{16\pi}{15}\sin\left(\frac{\pi t}{15}\right) = 0$

$\sin\left(\frac{\pi t}{15}\right) = 0$

$\frac{\pi t}{15} = 0, \pi, 2\pi, 3\pi, 4\pi$

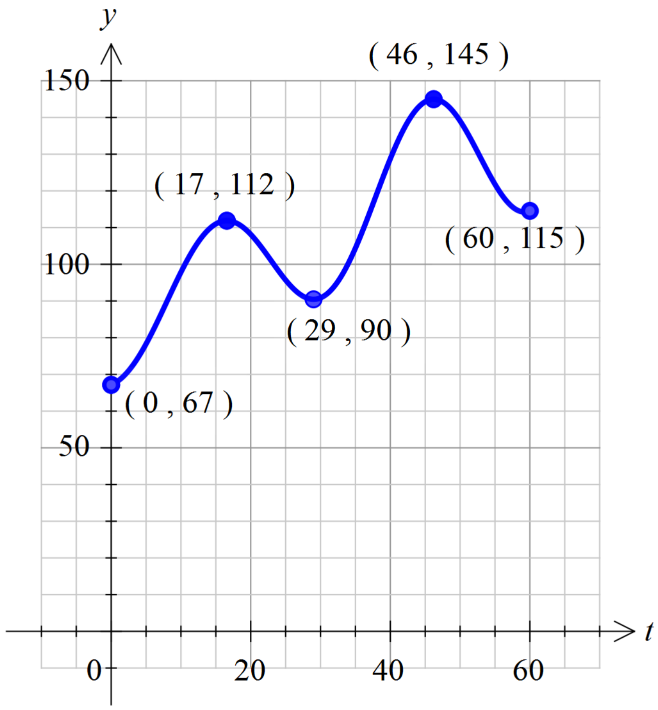
$t = 0, 15, 30, 45, 60$

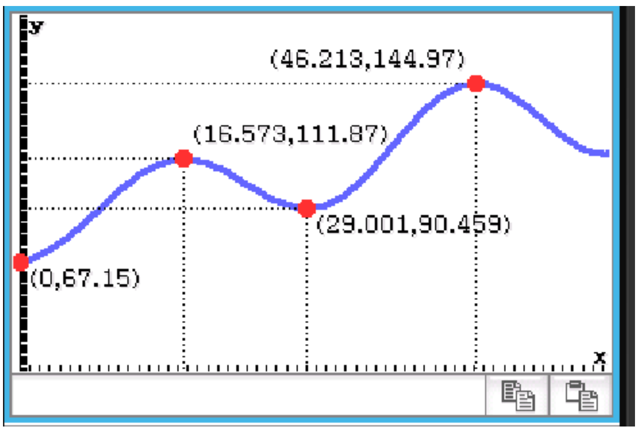



From the graph the local max occurs at $t = 15$ and $t = 45$ **1A**

**e.** $P_N : [0,60] \to R, P_N(t) = 0.01(t+85) \times P(t)$

Shape **1A**
Turning points **1A**
Endpoints **1A**

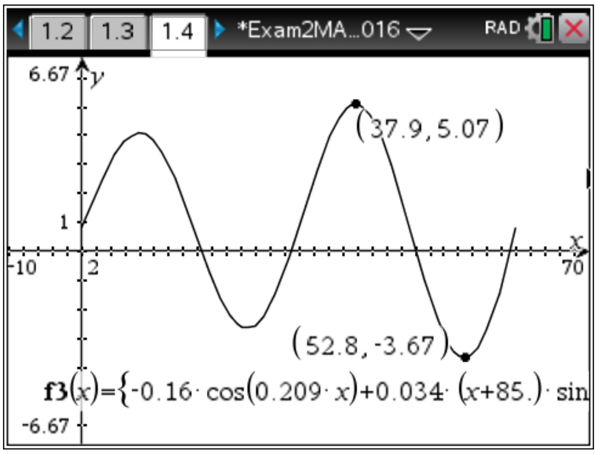


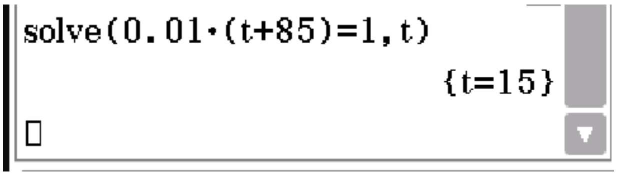
**f.** Maximum heart rate $\approx$ 144.97 beats /minute
Maximum heart rate rounded to the nearest whole number = 145 beats /minute **1A**

**g.** Find when $P_N'(t)$ has its greatest magnitude **1M**

$t = 37.9$ minutes **1A**




**h.** Solve the equation $0.01(t+85)=1$ gives $t = 15$ **1A**

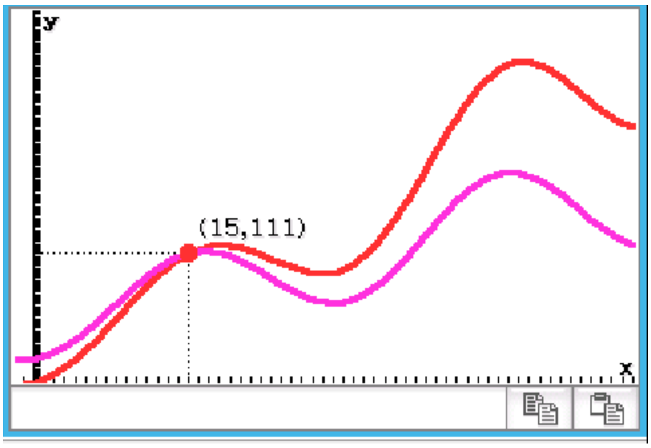



Solve $95+16\sin\left(\frac{\pi}{15}(t-7.5)\right)=0.01(t+85)\times P(t)$

$$95+16\sin\left(\frac{\pi}{15}(t-7.5)\right)=0.01(t+85)\times\left(95+16\sin\left(\frac{\pi}{15}(t-7.5)\right)\right)$$

Gives the equation $1=0.01(t+85)$ where previously found solution $t = 15$

So Sarah's pulse equals Stephen's pulse at $t = 15$.

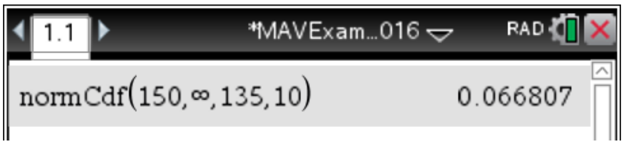



Looking at the graph
Sarah's pulse can be measured to be lower than her twin brother Stephen's
for $t \in (15,60]$ **1A**

**Question 3** (17 marks)

**a.** $X \sim \text{N}(135, 100)$

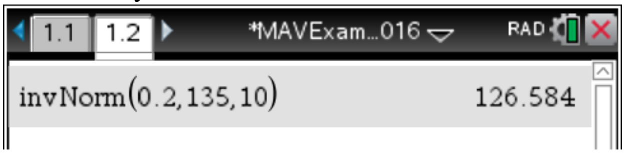
$\Pr(X > 150) = 0.0668... = 7\%$ to the nearest whole percent. **1A**




**b.** $\Pr(X > x) = 0.8$

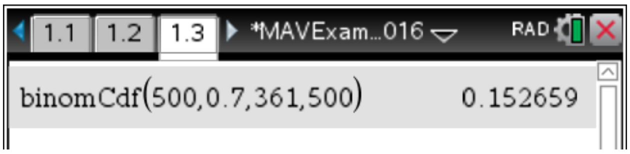
$\Pr(X < x) = 0.2$

$x = 127$ days to the nearest whole number **1A**



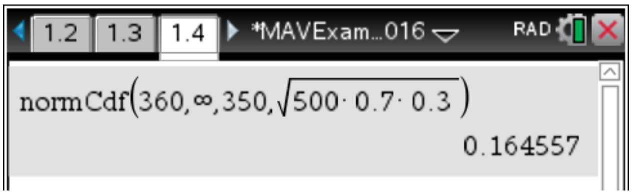

**c.** $Y \sim \text{Bi}(500, 0.7)$ **1A**

$\Pr(Y > 360) = \Pr(Y \geq 361) = 0.1527$ correct to 4 decimal places. **1A**




**d.** $W \sim \text{N}(350, 500 \times 0.7 \times 0.3)$ **1M**

$\Pr(W > 360) = 0.1646$ correct to 4 decimal places. **1A**




**e.** $\mu = 0.7$

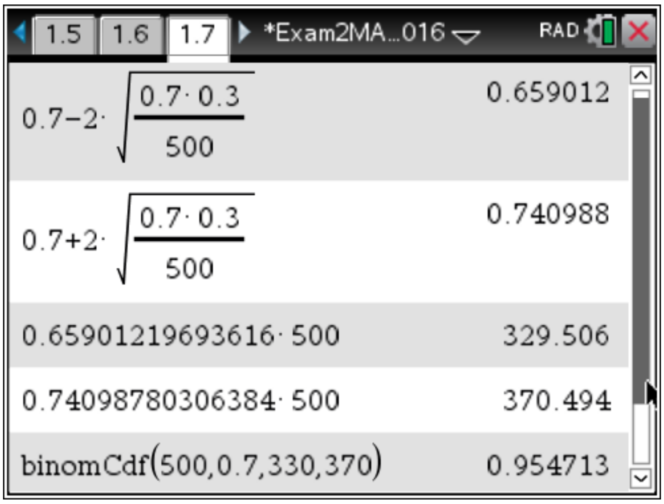
$$\sigma = \sqrt{\frac{0.7 \times 0.3}{500}} = \frac{\sqrt{105}}{500}$$

$$\left(0.7 - 2 \times \frac{\sqrt{105}}{500}, 0.7 + 2 \times \frac{\sqrt{105}}{500}\right)$$ **1M**

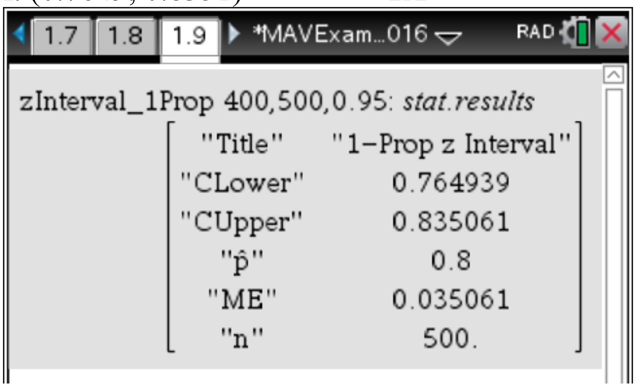
$\Pr\left(0.6590... \leq \hat{P} \leq 0.7409...\right) = \Pr(329.5... \leq X \leq 370.4...)$ **1M**

$= \Pr(330 \leq X \leq 370)$

= 0.9547 correct to four decimal places. **1A**




**f.** (0.7649, 0.8351) **1A**



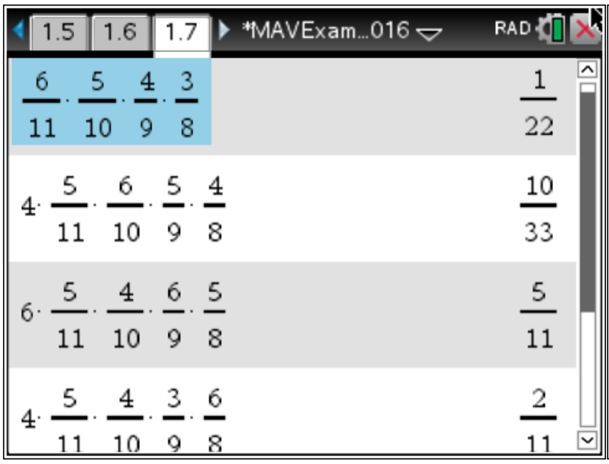

**g.**

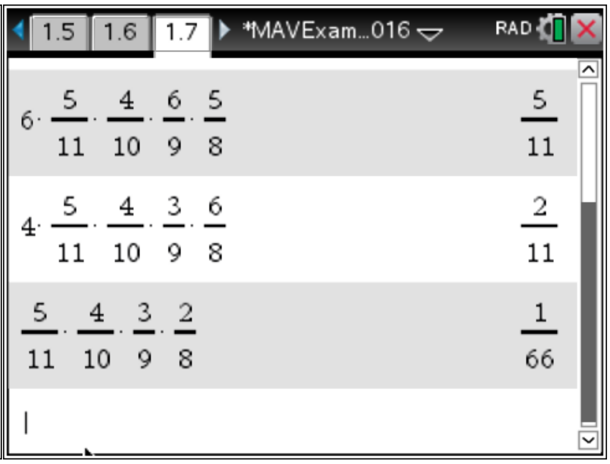
| Proportion of yellow balls $\hat{p}$ | 0 | $\frac{1}{4}$ | $\frac{1}{2}$ | $\frac{3}{4}$ | 1 |
|---|---|---|---|---|---|
| $\Pr\left(\hat{P}=\hat{p}\right)$ | $\frac{1}{22}$ | $\frac{10}{33}$ | $\frac{5}{11}$ | $\frac{2}{11}$ | $\frac{1}{66}$ |

Without replacement

$\frac{6}{11}\times\frac{5}{10}\times\frac{4}{9}\times\frac{3}{8}=\frac{1}{22}$ **1A**

$4\times\frac{5}{11}\times\frac{6}{10}\times\frac{5}{9}\times\frac{4}{8}=\frac{10}{33}$ **1A**




**h.** $\Pr\left(\hat{P}=1 \mid \hat{P}>\frac{1}{2}\right)=\dfrac{\Pr\left(\hat{P}=1 \cap \hat{P}>\frac{1}{2}\right)}{\Pr\left(\hat{P}>\frac{1}{2}\right)}$ **1M**

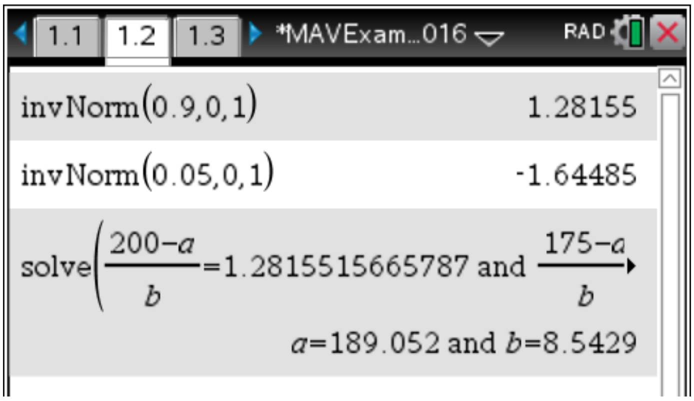
$=\dfrac{\frac{1}{66}}{\frac{13}{66}}=\dfrac{1}{13}$ **1A**

**i.** $\Pr(X>200)=0.1, \Pr(X<185)=0.05$ **1M**

$\Pr\left(Z>\dfrac{200-\mu}{\sigma}\right)=0.1$, $\Pr\left(Z<\dfrac{175-\mu}{\sigma}\right)=0.05$

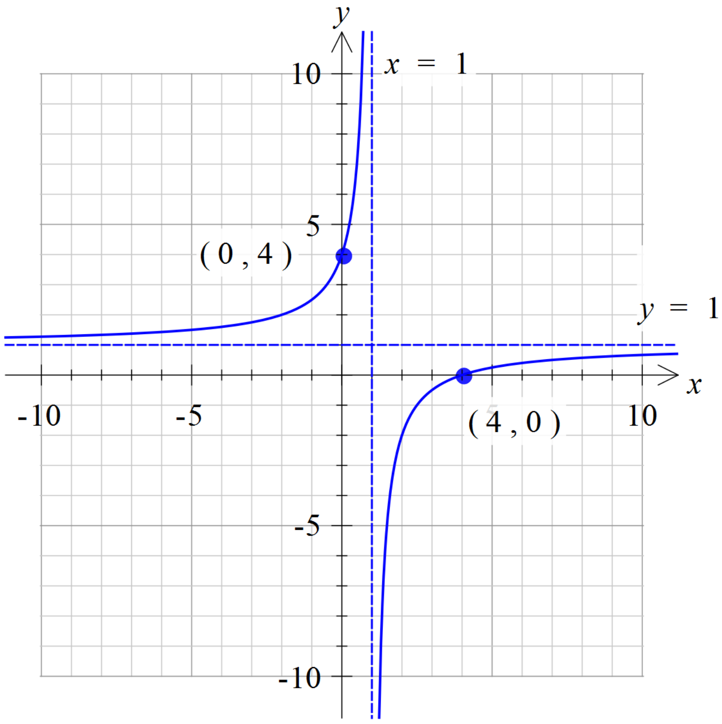
$\dfrac{200-\mu}{\sigma}=1.2815\ldots, \dfrac{175-\mu}{\sigma}=-1.6448\ldots$ **1M**

$\mu=189$ days, $\sigma=9$ days **1A**



**Question 4** (14 marks)

**a.** shape **1A**
asymptotes with equations **1A**
coordinates of axial-intercepts **1A**



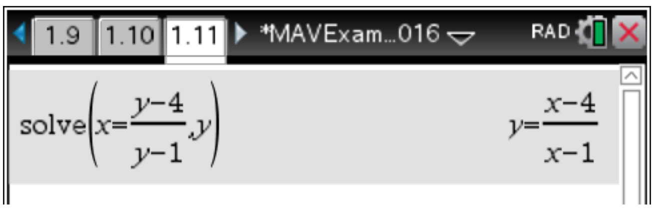

**b.** Let $y = \frac{x-4}{x-1}$

Inverse swap $x$ and $y$.

Solve $x = \frac{y-4}{y-1}$ for $y$.

$f^{-1}(x) = \frac{x-4}{x-1}$

Hence $f(x) = f^{-1}(x)$ **1A**




**c.** $g: R\setminus\{-n\} \to R, g(x) = \frac{x+m}{x+n}$

$g^{-1}(x) = \frac{-nx+m}{x-1}$

$n = -1$ **1A**

$m \in R\setminus\{-1,0\}$ **1A**

**OR**

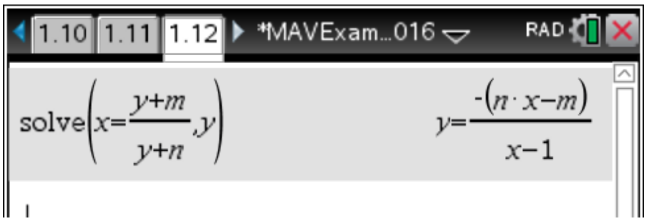
$g(x) = 1 + \dfrac{m+n}{x+n}$

Horizontal asymptote at $y = 1$

Vertical asymptote at $x = 1$

$n = -1$ **1A**

$m \in R \setminus \{-1, 0\}$ **1A**

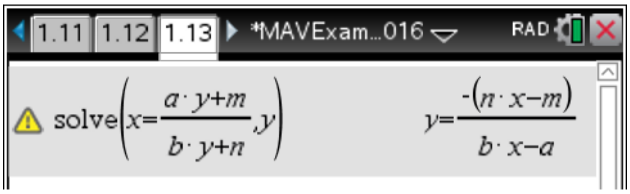



**d.** $h: R \setminus \left\{-\dfrac{b}{n}\right\} \to R, h(x) = \dfrac{ax+m}{bx+n}$

$h^{-1}(x) = \dfrac{-nx+m}{bx-a}$

$ax + m = -nx + m$, $bx - a = bx + n$

$a = -n$ **1M Show that**




**e.** $h_1: R \setminus \left\{-\dfrac{n}{4}\right\} \to R, h_1(x) = \dfrac{3x+2}{4x+n}$

$n = -3$ **1A**

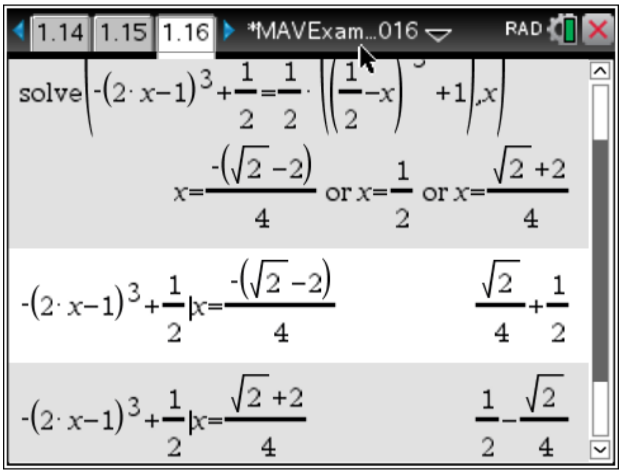
**f.** $w: R \to R, w(x) = -(2x-1)^3 + \dfrac{1}{2}$

$w^{-1}(x) = \dfrac{1}{2}\left(\sqrt[3]{\left(\dfrac{1}{2} - x\right)} + 1\right)$

Solve $w(x) = w^{-1}(x)$ for $x$. **1M**

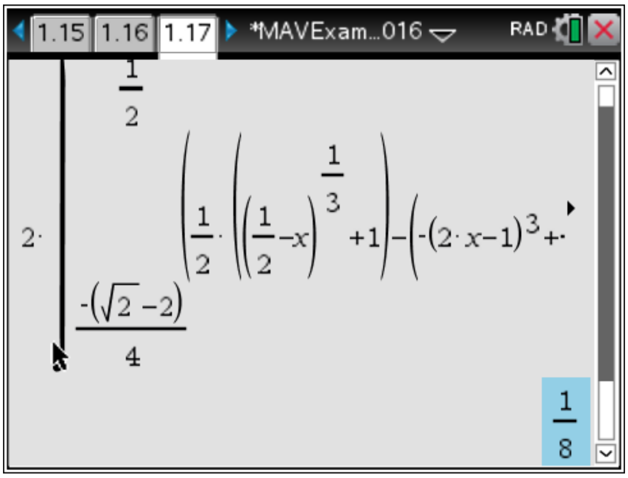
$x = \dfrac{2-\sqrt{2}}{4}, x = \dfrac{1}{2}$ or $x = \dfrac{2+\sqrt{2}}{4}$ **1A**

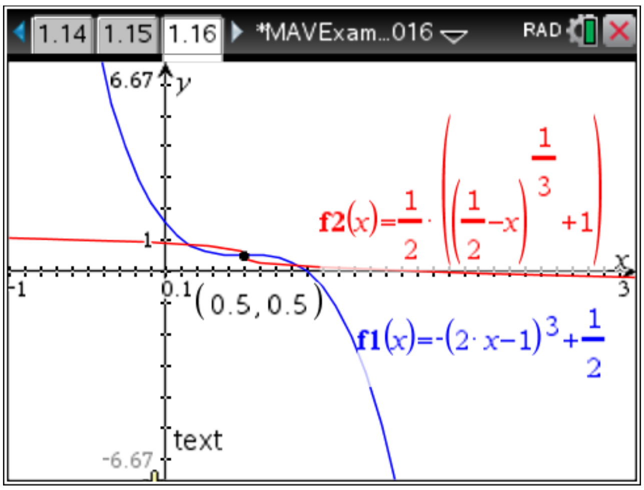
$\left(\dfrac{2-\sqrt{2}}{4}, \dfrac{2+\sqrt{2}}{4}\right), \left(\dfrac{1}{2}, \dfrac{1}{2}\right), \left(\dfrac{2+\sqrt{2}}{4}, \dfrac{2-\sqrt{2}}{4}\right)$ **1H**




**g.** Area $= 2\int_{\frac{2-\sqrt{2}}{4}}^{\frac{1}{2}} \left(\frac{1}{2}\left(\sqrt[3]{\left(\frac{1}{2}-x\right)}+1\right)-\left(-(2x-1)^3+\frac{1}{2}\right)\right)dx$ **1M**

$= \frac{1}{8}$ **1A**




**h.** $w_k : R \to R, w_k(x) = -(rx-1)^3 + s$

$s = \frac{1}{r}$ **1A**