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Student Name……………………………………

MATHEMATICAL METHODS UNITS 3 & 4

TRIAL EXAMINATION 2

2016

Reading Time: 15 minutes
Writing time: 2 hours

Instructions to students

This exam consists of Section A and Section B.
Section A consists of 20 multiple-choice questions, which should be answered on the detachable answer sheet which can be found on page 22 of this exam.
Section B consists of 5 extended-answer questions.
Section A begins on page 2 of this exam and is worth 20 marks.
Section B begins on page 8 of this exam and is worth 60 marks.
There is a total of 80 marks available.
All questions in Section A and Section B should be answered.
In Section B, where more than one mark is allocated to a question, appropriate working must be shown.
Where a numerical answer is required, an exact value must be given unless otherwise directed.
Diagrams in this exam are not to scale except where otherwise stated.
Students may bring one bound reference into the exam.
Students may bring into the exam one approved technology (calculator or software) and, if desired, one scientific calculator. Calculator memory does not need to be cleared. For approved computer-based CAS, full functionality may be used.
A formula sheet can be found on pages 20 and 21 of this exam.

SECTION A – Multiple-choice questions

Question 1

The period of the function $y = 2\tan\left(\frac{3\pi x}{4}\right)$ is

A. $\frac{3}{4}$

B. $\frac{3\pi}{4}$

C. $\frac{4}{3}$

D. $\frac{3\pi}{2}$

E. $\frac{8}{3}$

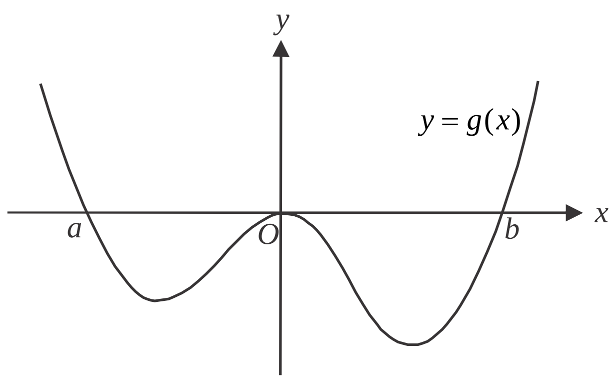
Question 2

The function with rule $f(x) = (x-1)^2$ has a range of $[1,\infty)$.
The domain of *f* could be

A. $x \in (-\infty, 0]$
B. $x \in (-\infty, 0)$
C. $x \in (2, \infty)$
D. $x \in [0, 2]$
E. $x \in (-\infty, 0) \cup (2, \infty)$

Question 3

The graph of the function *g* is shown below.



The rule for *g* could be

A. $g(x) = -x(x+a)(x+b)$
B. $g(x) = -x^2(x-a)(x-b)$
C. $g(x) = x(x-a)(x-b)$
D. $g(x) = x^2(x-a)(x-b)$
E. $g(x) = x^2(x+a)(x-b)$

Question 4

Let $f:[0,a]\rightarrow R,\ f(x)=\cos\left(2\left(x+\frac{\pi}{3}\right)\right)$.

If the inverse function f^{-1} exists, then the maximum possible value of a is

A. $\frac{\pi}{6}$

B. $\frac{\pi}{4}$

C. $\frac{\pi}{3}$

D. $\frac{\pi}{2}$

E. $\frac{2\pi}{3}$

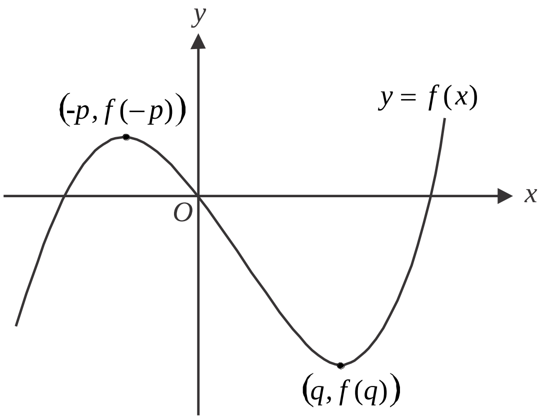
Question 5

The tangent to the graph of $y=\log_e(ax),\ a>0$ at the point where $x=\frac{1}{a}$, has a y-intercept of

A. -1
B. $-a$
C. 0
D. 1
E. a

Question 6

The graph of $y=f(x)$ is shown below.



The graph has stationary points at the points where $x=-p$ and $x=q$.
The largest interval for which the function f is strictly decreasing is

A. $x=(-\infty,-p)$
B. $x\in(-p,q)$
C. $x\in[-p,q]$
D. $x\in(0,q)$
E. $x\in[0,q]$

Question 7

The inverse function of $g:[1,\infty)\to R,\ g(x)=\sqrt{x-1}+2$ is

A. $g^{-1}:[-2,1]\to R,\ g^{-1}(x)=(x+2)^2+1$
B. $g^{-1}:[1,\infty)\to R,\ g^{-1}(x)=(x+2)^2-1$
C. $g^{-1}:[2,\infty)\to R,\ g^{-1}(x)=(x+2)^2-1$
D. $g^{-1}:[1,\infty)\to R,\ g^{-1}(x)=(x-2)^2+1$
E. $g^{-1}:[2,\infty)\to R,\ g^{-1}(x)=(x-2)^2+1$

Question 8

A sample space contains two mutually exclusive events *A* and *B* with $\Pr(A)=p$ and $\Pr(B)=\dfrac{p}{2}$.
$\Pr(A\cup B')$ is equal to

A. $\dfrac{3p}{2}$
B. $\dfrac{p^2}{2}$
C. $(1-p)\left(1-\dfrac{p}{2}\right)$
D. $\dfrac{2-p}{2}$
E. $\dfrac{2-3p}{2}$

Question 9

For $f(x)=e^x-2x$, the average rate of change with respect to *x* over the interval [0,1] is

A. $e-1$
B. $e-2$
C. $e-3$
D. $\dfrac{e-1}{2}$
E. $\dfrac{1}{e-2}$

Question 10

The random variable *X* has a normal distribution with mean 14 and standard deviation of 2.5.
If the random variable *Z* has the standard normal distribution, then $\Pr(9<X<16.5)$ is equal to

A. $\Pr(-2.5<Z<2)$
B. $\Pr(-2.5<Z<1)$
C. $\Pr(-2<Z<2.5)$
D. $\Pr(-2<Z<1)$
E. $\Pr(-1<Z<2.5)$

Question 11

Leonie and her Mum talk by phone once each day.
Either Leonie phones her Mum or vice-versa.

If Leonie phones one day the probability she phones the next day is $\frac{1}{3}$.

If Leonie doesn't phone one day the probability she doesn't phone the next day is $\frac{1}{5}$.

On Monday Leonie phoned her Mum.
The probability that Leonie phones her Mum exactly once over the following three days is

A. $\frac{1}{75}$

B. $\frac{2}{5}$

C. $\frac{23}{45}$

D. $\frac{26}{45}$

E. $\frac{38}{75}$

Question 12

Fifty-four percent of employees at a large corporation are parents.
A random sample of thirty of the corporation's employees is taken.
The probability that less than half of them are parents is closest to

A. 0.1048
B. 0.1312
C. 0.2661
D. 0.3129
E. 0.3973

Question 13

If $\int_0^3 f(x)dx = 2$ and $\int_0^3 (a - 4f(x))dx = 7$, then the value of a, where $a \in R$, is

A. $-\frac{1}{3}$

B. 3

C. 5

D. 6

E. $\frac{31}{3}$

Question 14

The simultaneous linear equations $ax+3y=a-3$ and $2x+(a+1)y=-1$ have no solution for

A. $a=2$
B. $a=-3$
C. $a \in R \setminus \{-3,2\}$
D. $a=2$ and $a=-3$
E. $a \in R \setminus \{2\}$

Question 15

Consider the cubic function $g: R \to R, g(x)=ax^3+2bx^2+x+5$, where a and b are positive constants. The graph of g has more than one stationary point when

A. $a<\frac{3b^2}{4}$
B. $a>\frac{3b^2}{4}$
C. $a<\frac{4b^2}{3}$
D. $a>\frac{4b^2}{3}$
E. $a>\frac{2\sqrt{3}b^2}{3}$

Question 16

A randomly selected group of 820 people on the electoral role were asked whether they favoured fixed five year terms for Federal parliament. Fifty-three percent of the group favoured the idea. An approximate 95% confidence interval for the proportion p of people on the electoral role who favour the idea can be found by calculating

A. $0.53-\sqrt{\frac{0.53\times0.47}{820}}<p<0.53+\sqrt{\frac{0.53\times0.47}{820}}$
B. $0.53-0.95\sqrt{\frac{0.53\times0.47}{820}}<p<0.53+0.95\sqrt{\frac{0.53\times0.47}{820}}$
C. $0.53-1.64\sqrt{\frac{0.53\times0.47}{820}}<p<0.53+1.64\sqrt{\frac{0.53\times0.47}{820}}$
D. $0.53-1.96\sqrt{\frac{0.53\times0.47}{820}}<p<0.53+1.96\sqrt{\frac{0.53\times0.47}{820}}$
E. $0.53-2.58\sqrt{\frac{0.53\times0.47}{820}}<p<0.53+2.58\sqrt{\frac{0.53\times0.47}{820}}$

Question 17

In a population of tropical fish, 15% have a disease. A random sample of 300 of the fish is taken.
A normal approximation is used to find the approximate probability that less than 20% of the fish in the sample have the disease. That approximate probability is closest to

A. 0.97938
B. 0.98326
C. 0.98723
D. 0.99225
E. 0.99235

Question 18

The transformation $T: R^2 \to R^2,\ T\left(\begin{bmatrix} x \\ y \end{bmatrix}\right) = \begin{bmatrix} -1 & 0 \\ 0 & 1 \end{bmatrix}\begin{bmatrix} x \\ y \end{bmatrix} + \begin{bmatrix} 0 \\ -1 \end{bmatrix}$, maps the graph of the function *f* to the graph of the function $y = \sqrt{x}$.
The rule of *f* is

A. $f(x) = -\sqrt{x} - 1$
B. $f(x) = \sqrt{-x} + 1$
C. $f(x) = \sqrt{x-1} - 1$
D. $f(x) = -\sqrt{x} + 1$
E. $f(x) = -\sqrt{x-1}$

Question 19

If $g(x+2) = x^2 + 3x + 7$, then $g(x)$ is equal to

A. $x^2 - 2x$
B. $x^2 + x$
C. $x^2 - x - 3$
D. $x^2 - x + 5$
E. $x^2 + x - 1$

Question 20

The length of service of volunteers in a large community organisation is represented by the random variable *L* which is normally distributed.
The mean of *L* is 6.7 years and the standard deviation is σ.
For the current population of 8 000 volunteers, the length of service of 1 400 of them is expected to exceed 8 years. The value of σ, in years, is closest to

A. 1.30
B. 1.39
C. 1.67
D. 1.89
E. 2.31

SECTION B

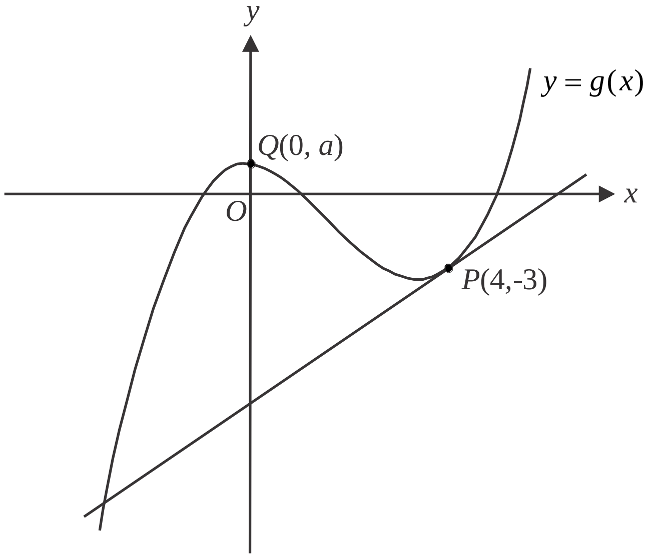
Answer all questions in this section.

Question 1 (8 marks)

Let $g : R \to R,\ g(x) = \frac{1}{5}(x^2 - 1)(x - 5)$.

The points $P(4,-3)$ and $Q(0, a)$ lie on the graph of g where a is a positive constant.

The graph of g and the tangent to the graph of g at the point $P(4,-3)$ are shown below.



a. Find the value of a. 1 mark

b. For the tangent to the graph of g at the point $P(4,-3)$, find

i. its gradient. 1 mark

ii. its equation. 1 mark

c. Find the distance PQ. Give your answer in the form $b\sqrt{c}$, where b and c are positive integers. 2 marks

d. Find the area enclosed by the graph of g and the tangent to the graph of g at the point $P(4,-3)$. 3 marks

Question 2 (12 marks)

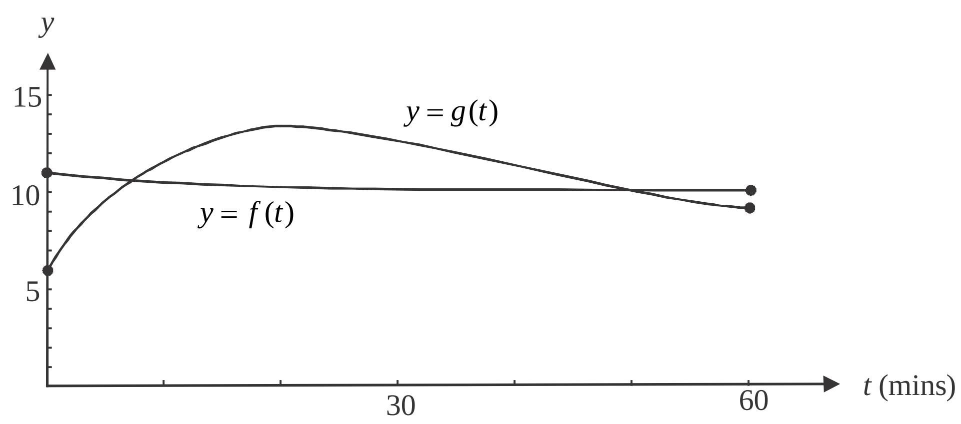
The stock market value of two stocks, Foolsgold and Gold Inc., are modelled respectively by the functions

$$f:[0,60]\to R, f(t)=e^{-\frac{t}{20}}+10$$

$$\text{and } g:[0,60]\to R, g(t)=te^{-\frac{t}{20}}+6$$

where *f* and *g* represent the value of the respective stocks, in dollars, *t* minutes after the opening of trade on a particular day.

The graphs of the functions are shown below.



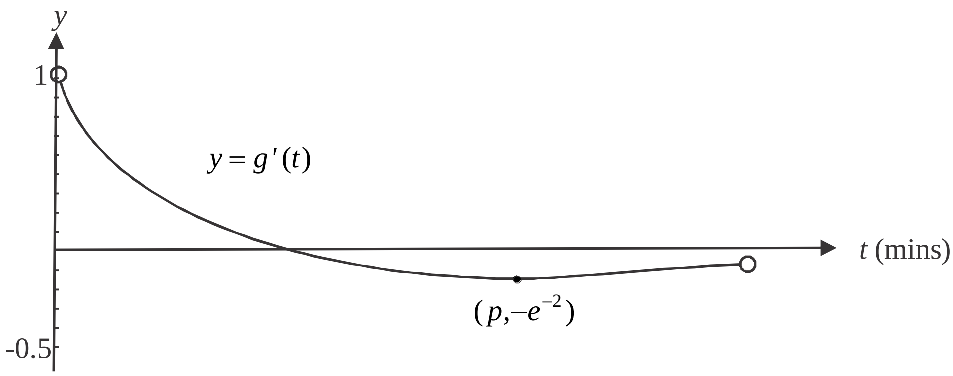
a. Find the values of *t* when the values of the two stocks were equal. Give your answers correct to three decimal places. 2 marks

b. Find the maximum value of the Gold Inc. stock during the first hour of trade. Give your answer to the nearest cent. 2 marks

c. Find the average value of the Gold Inc. stock during the first hour of trade. Give your answer to the nearest cent. 2 marks

d. During the period when the value of the Gold Inc. stock was greater than the value of the Foolsgold stock, find the value of t when the difference in the values was a maximum. 3 marks

The graph of the derivative function $g'(t) = \left(1 - \frac{t}{20}\right)e^{-\frac{t}{20}}, \quad t \in (0,60)$ is shown below. The graph has a minimum turning point at the point $(p, -e^{-2})$ where p is a positive integer.



e. **i.** Find the value of p. 2 marks

ii. Hence find the value of the Gold Inc. stock when the rate at which it was decreasing was a maximum. Give your answer to the nearest cent. 1 mark

Question 3 (17 marks)

A bank has over one million customers. The proportion of these customers who don't use online banking is $\frac{2}{9}$.
The bank takes a number of random samples of its customers. Each sample contains 15 customers. Let X be the random variable that represents the number of customers in a sample who don't use online banking.

a. Find $\Pr(X \leq 5)$. Give your answer correct to three decimal places. 2 marks

Let $\hat{P}$ be the random variable of the distribution of sample proportions of customers who don't use online banking.

b. Find

i. the expected value of $\hat{P}$. 1 mark

ii. the standard deviation of $\hat{P}$. 2 marks

c. Suppose the bank wanted a standard deviation of $\hat{P}$ of less than 0.1.
What is the minimum number of customers that the bank would need to have in each sample in order to achieve this? 2 marks

d. The bank decides to retain its original sample size of 15 customers.
Find the probability that a sample proportion would lie within one standard deviation of the population proportion. Do not use a normal approximation. Give your answer correct to three decimal places. 3 marks

The duration of calls, in minutes, made by customers to the bank's telephone banking service is a continuous random variable *T* with a probability density function *f*, given by

$$f(t)=\begin{cases}\dfrac{3t}{3263}(20-t), & 2\le t\le 15\\ 0, & \text{elsewhere}\end{cases}$$

e. **i.** Find the mean duration of a call. Give your answer in minutes, correct to three decimal places. 2 marks

ii. Find the probability that a call took more than ten minutes. Give your answer correct to three decimal places. 2 marks

iii. Find the probability that a call took more than twelve minutes given that it took more than ten minutes. Give your answer correct to three decimal places. 2 marks

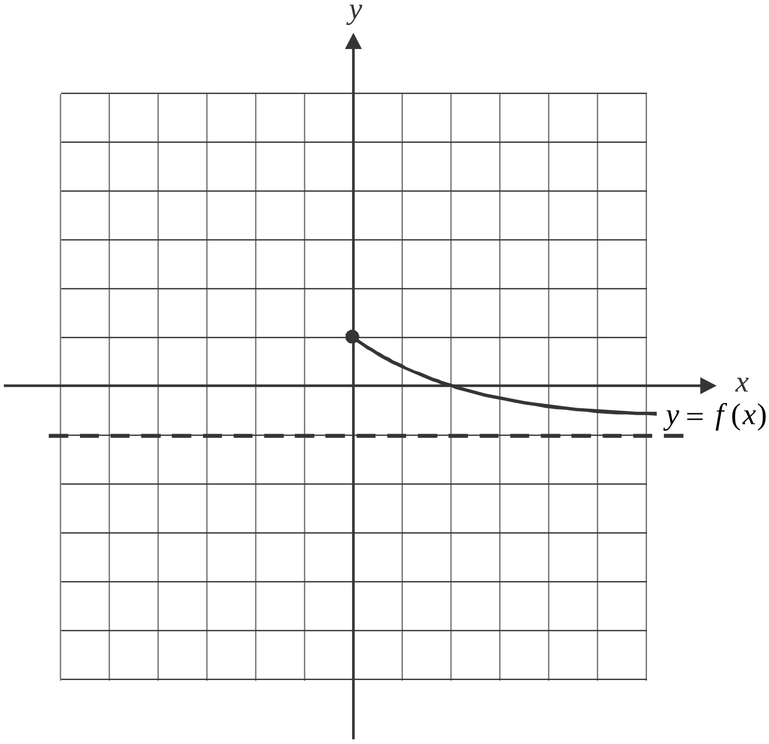
The bank sampled 2000 of its customers across its entire customer base. It was found that 1860 of these customers were satisfied with their last interaction with the bank.

f. Use this sample to find an approximate 90% confidence interval for the proportion of the population of the bank's customers who were satisfied with their last interaction with the bank. Give values correct to three decimal places. 1 mark

Question 4 (8 marks)

Let $f:[0,\infty)\to R,\ f(x)=\dfrac{4}{x+2}-1$.

The graph of *f* is shown below.



a. On the same set of axes, sketch the graph of f^{-1}, the inverse function of *f*.
Indicate clearly the coordinates of any axes intercepts and the equation of any asymptotes. 2 marks

The graph of *f* is

- dilated by a factor of 2 units from the *y*-axis and then
- reflected in the *x*-axis and then
- translated 3 units vertically upwards

to become the graph of $y=h(x)$.

b. Write down the rule for *h*. 2 marks

Let $q:[0,\infty)\rightarrow R,\ q(x)=\dfrac{a}{x+2}-1$ where $a\geq 2$.

c. Find, in terms of *a*, the coordinates of the *x*-intercept of the graph of *q*. 1 mark

d. Find the area enclosed by the graph of *q* and the *x* and *y*-axes. Give your answer in the form $\log_e\left(\dfrac{u}{v}\right)-a+2$, where *u* and *v* are functions of *a*. 3 marks

Question 5 (15 marks)

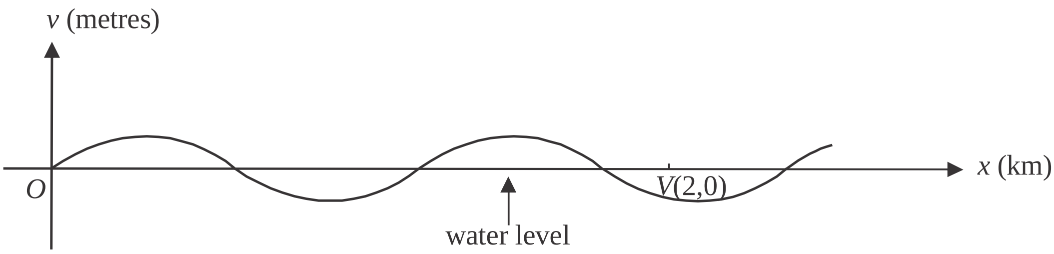
Victoria James is a spy.
She is trapped in a stationary mini-submarine that is being fired on by an enemy ship. The ship is firing 'dolphin' missiles which follow a curved path.
The vertical distance *v*, in metres, of a missile above the surface of the water (or below if $v < 0$) is given by

$$v(x) = 3\sin(\pi ax), \quad x \geq 0, \quad a > 0$$

where *x* kilometres is the horizontal distance of the missile from the ship. The positive constant *a* can be reset for each missile.

a. What is the maximum distance below the surface of the water that a dolphin missile can reach? 1 mark

The enemy ship is stationary and is located at the point *O*(0,0). Victoria is located 2 kilometres away at the point *V*(2,0). The graph below shows the path of the first missile fired at Victoria.



This first missile entered the water for the **second** time at a point that was 0.2 kilometres short of Victoria's position.

b. Show that for this first missile, $a = \frac{5}{3}$. 1 mark

c. Find the acute angle between the horizontal and the tangent to the path of the first missile as it emerges from the water. Give your answer in degrees correct to two decimal places. 2 marks

Further missiles are fired.

d. Find the value of a if a missile was to hit Victoria's submarine as the missile entered the water for the second time. 1 mark

e. Find the values of a for which a missile will pass over Victoria's submarine at $V(2,0)$ before hitting the water for the first time. 2 marks

Victoria leaves her mini-submarine and swims to the enemy ship. She is stationary as she attaches a bomb to the hull of the ship.
Immediately, Victoria starts swimming in a straight line away from the ship.
Victoria's speed, in km/h as she swims away is given by

$$s:[0,d]\rightarrow R,\ \ s(x)=(x+k)^2 - 2^{x+1}+1$$

where x is Victoria's horizontal distance in km from the ship, k is a positive constant and d is Victoria's distance from the ship when she stops swimming.
Victoria's speed is 2km/h when she is 2km from the ship.

f. Show that $k=1$. 1 mark

g. Find the value of *d*. Give your answer correct to two decimal places. 2 marks

h. **i.** Find Victoria's distance from the ship when her speed is a maximum. Give your answer correct to four decimal places. 2 marks

ii. Find the maximum speed Victoria reaches whilst swimming. Give your answer correct to three decimal places. 1 mark

Victoria must detonate the bomb whilst swimming.
To do so, she must be swimming at 1.5km/h or slower and must be at least 3km from the enemy ship.

i. Find the values of x for which Victoria can detonate the bomb. Give your answer as an interval with endpoints expressed correct to two decimal places. 2 marks

Mathematical Methods formulas

Mensuration

area of a trapezium	$\frac{1}{2}(a+b)h$	volume of a pyramid	$\frac{1}{3}Ah$
curved surface area of a cylinder	$2\pi rh$	volume of a sphere	$\frac{4}{3}\pi r^3$
volume of a cylinder	$\pi r^2 h$	area of a triangle	$\frac{1}{2}bc\sin(A)$
volume of a cone	$\frac{1}{3}\pi r^2 h$		

Calculus

$\frac{d}{dx}\left(x^n\right)=nx^{n-1}$		$\int x^n dx=\frac{1}{n+1}x^{n+1}+c,\ n\neq -1$	
$\frac{d}{dx}\left((ax+b)^n\right)=an(ax+b)^{n-1}$		$\int (ax+b)^n dx=\frac{1}{a(n+1)}(ax+b)^{n+1}+c,\ n\neq -1$	
$\frac{d}{dx}\left(e^{ax}\right)=ae^{ax}$		$\int e^{ax}dx=\frac{1}{a}e^{ax}+c$	
$\frac{d}{dx}\left(\log_e(x)\right)=\frac{1}{x}$		$\int \frac{1}{x}dx=\log_e(x)+c,\ x>0$	
$\frac{d}{dx}\left(\sin(ax)\right)=a\cos(ax)$		$\int \sin(ax)dx=-\frac{1}{a}\cos(ax)+c$	
$\frac{d}{dx}\left(\cos(ax)\right)=-a\sin(ax)$		$\int \cos(ax)\,dx=\frac{1}{a}\sin(ax)+c$	
$\frac{d}{dx}\left(\tan(ax)\right)=\frac{a}{\cos^2(ax)}=a\sec^2(ax)$			
product rule	$\frac{d}{dx}(uv)=u\frac{dv}{dx}+v\frac{du}{dx}$	quotient rule	$\frac{d}{dx}\left(\frac{u}{v}\right)=\frac{v\frac{du}{dx}-u\frac{dv}{dx}}{v^2}$
chain rule	$\frac{dy}{dx}=\frac{dy}{du}\frac{du}{dx}$		

Probability

$\Pr(A) = 1 - \Pr(A')$		$\Pr(A \cup B) = \Pr(A) + \Pr(B) - \Pr(A \cap B)$	
$\Pr(A \mid B) = \dfrac{\Pr(A \cap B)}{\Pr(B)}$			
mean	$\mu = \mathrm{E}(X)$	variance	$\mathrm{var}(X) = \sigma^2 = \mathrm{E}((X-\mu)^2) = \mathrm{E}(X^2) - \mu^2$

Probability distribution		Mean	Variance
discrete	$\Pr(X = x) = p(x)$	$\mu = \Sigma\, x\, p(x)$	$\sigma^2 = \Sigma\,(x-\mu)^2\, p(x)$
continuous	$\Pr(a < X < b) = \int_a^b f(x)dx$	$\mu = \int_{-\infty}^{\infty} x\, f(x)dx$	$\sigma^2 = \int_{-\infty}^{\infty} (x-\mu)^2\, f(x)\,dx$

Sample proportions

$\hat{P} = \dfrac{X}{n}$		mean	$\mathrm{E}(\hat{P}) = p$
standard deviation	$\mathrm{sd}(\hat{P}) = \sqrt{\dfrac{p(1-p)}{n}}$	approximate confidence interval	$\left(\hat{p} - z\sqrt{\dfrac{\hat{p}(1-\hat{p})}{n}},\ \hat{p} + z\sqrt{\dfrac{\hat{p}(1-\hat{p})}{n}}\right)$

MATHEMATICAL METHODS

TRIAL EXAMINATION 2

MULTIPLE - CHOICE ANSWER SHEET

STUDENT NAME:..

INSTRUCTIONS

Fill in the letter that corresponds to your choice. Example: A ▬ C D E

The answer selected is B. Only one answer should be selected.

1.	A	B	C	D	E	11.	A	B	C	D	E
2.	A	B	C	D	E	12.	A	B	C	D	E
3.	A	B	C	D	E	13.	A	B	C	D	E
4.	A	B	C	D	E	14.	A	B	C	D	E
5.	A	B	C	D	E	15.	A	B	C	D	E
6.	A	B	C	D	E	16.	A	B	C	D	E
7.	A	B	C	D	E	17.	A	B	C	D	E
8.	A	B	C	D	E	18.	A	B	C	D	E
9.	A	B	C	D	E	19.	A	B	C	D	E
10.	A	B	C	D	E	20.	A	B	C	D	E