Neap:

Trial Examination 2015

VCE Mathematical Methods (CAS) Units 3&4

Written Examination 1

Suggested Solutions

Question 1 (3 marks)

a. $h(x) = x\sin(x^2) = u(x)v(x)$, where $u(x) = x$ and $v(x) = \sin(x^2)$.

Hence $u'(x) = 1$ and $v'(x) = 2x\cos(x^2)$.

$h'(x) = v(x)u'(x) + u(x)v'(x)$ M1

$= \sin(x^2) \times 1 + x \times 2x\cos(x^2)$

$= \sin(x^2) + 2x^2\cos(x^2)$ A1

b. $h'\left(\sqrt{\frac{\pi}{2}}\right) = \sin\left(\frac{\pi}{2}\right) + 2 \times \frac{\pi}{2}\cos\left(\frac{\pi}{2}\right)$

$= 1$ A1

Question 2 (3 marks)

$$f(x) = e^{5x}(e^{x} + e^{-x})$$
$$= e^{5x} \times e^{x} + e^{5x} \times e^{-x}$$
$$= e^{5x+x} + e^{5x-x}$$
$$= e^{6x} + e^{4x}$$

Hence $\int f(x)dx = \int (e^{6x} + e^{4x})dx$

$$= \int e^{6x}dx + \int e^{4x}dx$$

$= \frac{e^{6x}}{6} + \frac{e^{4x}}{4} + C$, where C is a constant. M1

$g(x) = \frac{e^{6x}}{6} + \frac{e^{4x}}{4} + C$ for a particular value of the constant C.

Since we are given that $g(0) = 0$, we can find the value of C.

$g(0) = \frac{e^{6 \times 0}}{6} + \frac{e^{4 \times 0}}{4} + C$ M1

$$= \frac{e^{0}}{6} + \frac{e^{0}}{4} + C$$
$$= \frac{1}{6} + \frac{1}{4} + C$$
$$= \frac{5}{12} + C$$

Thus $0 = \frac{5}{12} + C$

$$C = -\frac{5}{12}$$

$g(x) = \frac{e^{6x}}{6} + \frac{e^{4x}}{4} - \frac{5}{12}$ A1

Question 3 (4 marks)

a. $\sin\left(2x-\frac{\pi}{6}\right)=\frac{\sqrt{3}}{2}$ $\qquad -\pi<x<\pi$

$$2x-\frac{\pi}{6}=\sin^{-1}\left(\frac{\sqrt{3}}{2}\right)$$

$$=\frac{\pi}{3}$$

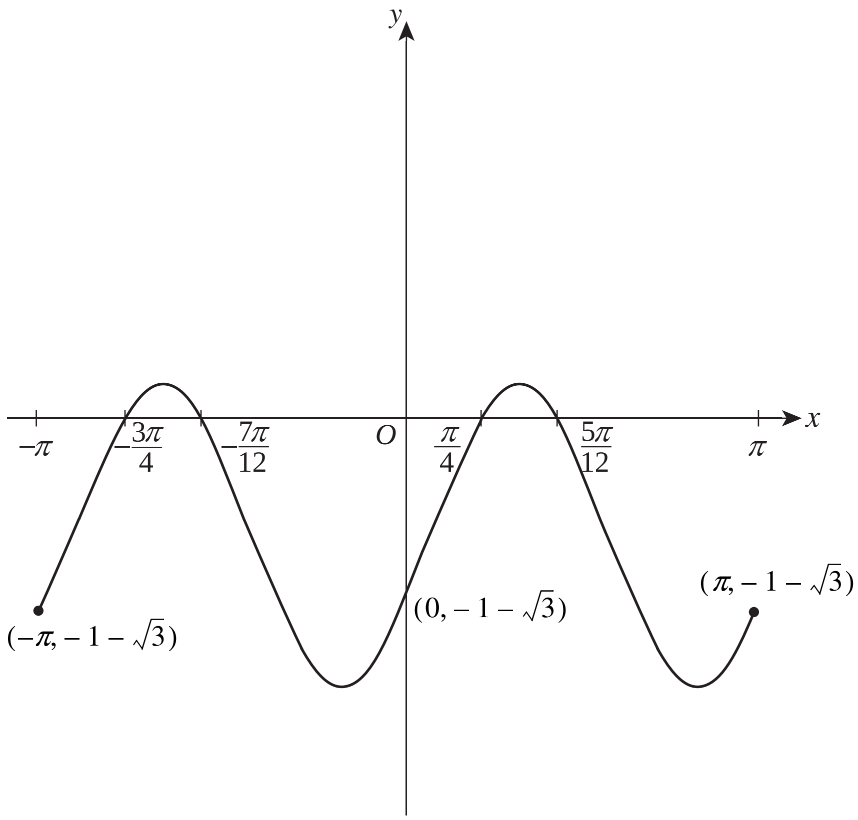
$2x-\frac{\pi}{6}=\frac{\pi}{3}$ $\qquad -\frac{13\pi}{6}<2x-\frac{\pi}{6}<\frac{11\pi}{6}$ M1

$$2x-\frac{\pi}{6}=-\frac{5\pi}{3},-\frac{4\pi}{3},\frac{\pi}{3},\frac{2\pi}{3}$$

$$2x=-\frac{3\pi}{2},-\frac{7\pi}{6},\frac{\pi}{2},\frac{5\pi}{6}$$

$x=-\frac{3\pi}{4},-\frac{7\pi}{12},\frac{\pi}{4},\frac{5\pi}{12}$ A1

b.



correct shape A1
correct intercepts and endpoints A1

Question 4 (3 marks)

$$3 \times 9^{x+1} + 1 - 5^2 \times 3^x - 3^{x+1} = 0$$

$$3 \times (3^2)^{x+1} + 1 - 25 \times 3^x - 3^{x+1} = 0$$

$$3 \times 3^{2x} \times 9 + 1 - 25 \times 3^x - 3 \times 3^x = 0$$

$$27 \times 3^{2x} - 28 \times 3^x + 1 = 0$$

$$27 \times 3^{2x} - 28 \times 3^x + 1 = 0$$ *quadratic form* M1

Let $3^x = a$.

$$27a^2 - 28a + 1 = 0$$

$$(27a - 1)(a - 1) = 0$$

$$a = \frac{1}{27}, 1$$ A1

Therefore $3^x = \frac{1}{27}, 1$.

$3^x = 1$ gives $x = 0$ and $3^x = \frac{1}{27}$ gives $x = -3$. A1

Question 5 (5 marks)

a. Method 1:

Complete the square to find the turning point:

$$f(x) = -\left(x - \frac{1}{2}\right)^2 + \frac{25}{4}$$ M1

Coordinates are $\left(\frac{1}{2}, \frac{25}{4}\right)$. A1

Method 2:

$f'(x) = 1 - 2x$

There is a maximum when $f'(x) = 0$. M1

$$1 - 2x = 0$$

$$x = \frac{1}{2}$$

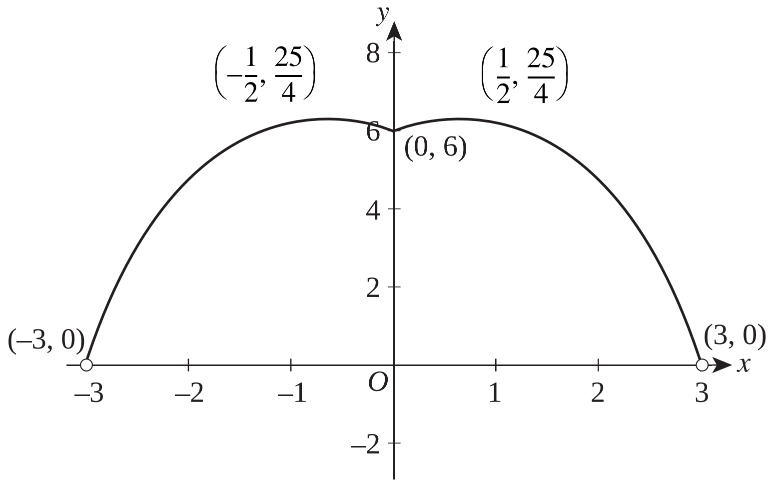
$$f\left(\frac{1}{2}\right) = \frac{25}{4}$$

Coordinates are $\left(\frac{1}{2}, \frac{25}{4}\right)$. A1

b. **i.** $f(g(x)) = 6 + |x| - |x^2|$

domain: $x \in (-3, 3)$ A1

ii.



correct shape A1

correct intercepts and turning points A1

Question 6 (4 marks)

a. Let $u = x^3 + 1$ and then $y = \log_e(u)$.

$\frac{dy}{du} = \frac{1}{u}$ and $\frac{du}{dx} = 3x^2$. M1

The chain rule states that $\frac{dy}{dx} = \frac{dy}{du} \times \frac{du}{dx}$.

Hence $\frac{dy}{dx} = \frac{1}{u} \times 3x^2$

$= \frac{1}{x^3 + 1} \times 3x^2$

$= \frac{3x^2}{x^3 + 1}$ A1

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b. By the fundamental theorem of calculus, $\int \frac{3x^2}{x^3+1}dx = \log_e(x^3+1) + C$, where C is a constant.

However, $(x + x^{-2})^{-1} = \frac{1}{x + x^{-2}}$

$= \frac{1}{x + x^{-2}}$

$= \frac{x^2}{x^3+1}$ M1

We can find the required antiderivative as follows:

$\int (x + x^{-2})^{-1} dx = \int \frac{x^2}{x^3+1}dx$

$= \frac{1}{3} \times \int \frac{3x^2}{x^3+1}dx$

$= \frac{1}{3}\log_e(x^3+1) + K$, where K is a constant, since $K = \frac{C}{3}$. A1

Question 7 (4 marks)

a. C = floor + roof + walls + foundations

$= \frac{80}{9}s^2 + \frac{40}{9}s^2 + 40sh + 40s$

$= \frac{40}{3}s^2 + 40sh + 40s$ A1

b. $4800 = \frac{40}{3}s^2 + 40sh + 40s$

$120 = s\left(\frac{s}{3} + h + 1\right)$

$\frac{120}{s} - \frac{s}{3} - 1 = h$

$h = \frac{360 - s^2 - 3s}{3s}$ A1

c. $V = s^2h$

$= s^2\left(\frac{360 - s^2 - 3s}{3s}\right)$

$= 120s - \frac{s^3}{3} - s^2$ A1

d. $\frac{dV}{ds} = 120 - s^2 - 2s$

$\frac{dV}{ds} = 0$ when $0 = 120 - s^2 - 2s$

$= s^2 + 2s - 120$

$= (s - 10)(s + 12)$

$s = 10$ and $s = -12$

Since $s > 0$, $s = 10$.

When $s = 10$, $h = \frac{360 - 10^2 - 3(10)}{3(10)}$

$= \frac{230}{30}$

$= \frac{23}{3}$ m A1

Question 8 (3 marks)

Let $y = g(x)$.

$y - 2 = \frac{1}{(3x - 1)^2}$

From the transformation, $x' = a(x + b)$ gives $a(x + b) = 3x + 1$

$ax + ab = 3x + 1$

$a = 3$ A1

$b = \frac{1}{3}$ A1

$y' = y + c$ gives $y + c = y - 2$

$c = -2$ A1

Question 9 (5 marks)

a. $\Sigma p(x) = 1$

$3k^2 - 1 + 3k + 4k + 2k + k = 7$ M1

$3k^2 + 10k - 8 = 0$

$(3k - 2)(k + 4) = 0$

$k = \frac{2}{3}, -4$ A1

Since $k > 0$, $k = \frac{2}{3}$. *one solution only* A1

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b. $\Pr(X \geq 1) = 1 - \Pr(X = 0)$ M1

$$\Pr(X = 0) = \frac{3\left(\frac{2}{3}\right)^2 - 1}{7}$$

$$= \frac{1}{7}\left(\frac{4}{3} - 1\right)$$

$$= \frac{1}{21}$$

$$\Pr(X \geq 1) = 1 - \frac{1}{21}$$

$$= \frac{20}{21}$$ A1

Question 10 (6 marks)

a. Since $f(x)$ is a probability density function, $\int_{-\infty}^{\infty} f(x)dx = 1$.

$$\int_{-\infty}^{\infty} f(x)dx = \int_{-\infty}^{0} f(x)dx + \int_{0}^{1} f(x)dx + \int_{1}^{\frac{7}{3}} f(x)dx + \int_{\frac{7}{3}}^{\infty} f(x)dx$$

$$1 = \int_{-\infty}^{0} 0\,dx + \int_{0}^{1} kx^2\,dx \int_{1}^{\frac{7}{3}} \left(\frac{7k - 3x}{4}\right)dx + \int_{\frac{7}{3}}^{\infty} 0\,dx$$ M1

$$1 = 0 + \left[\frac{kx^3}{3}\right]_0^1 + \left[\frac{7k}{4}x - \frac{3x^2}{8}\right]_1^{\frac{7}{3}} + 0$$

$$1 = \frac{k}{3} + \left(\frac{7k}{4} \times \frac{7}{3} - \frac{3}{8}\left(\frac{7}{3}\right)^2\right) - \left(\frac{7k}{4} - \frac{3}{8}\right)$$

$$1 = \frac{k}{3} + \frac{49}{12}k - \frac{49}{24} - \frac{7k}{4} + \frac{3}{8}$$

$$1 = \frac{64k}{24} - \frac{40}{24}$$

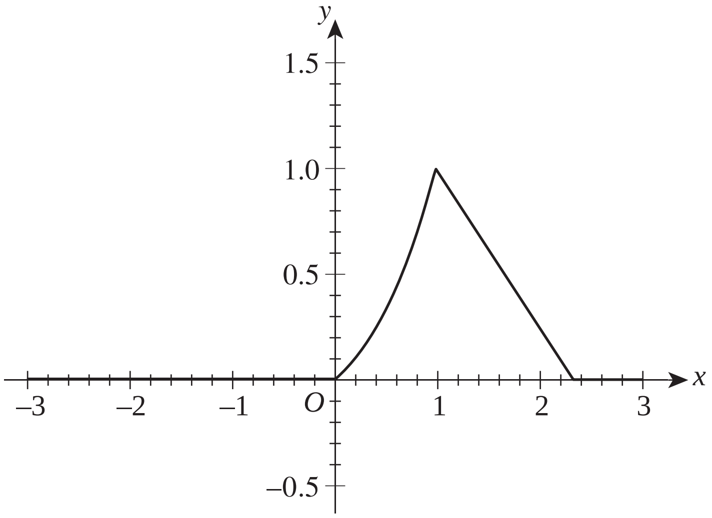
$$\frac{64}{24} = \frac{64k}{24}$$

$$k = 1$$ A1

b. Since $k = 1$:

$$f(x) = \begin{cases} x^2 & 0 \le x \le 1 \\ \dfrac{7-3x}{4} & 1 < x \le \dfrac{7}{3} \\ 0 & \text{elsewhere} \end{cases}$$

The mode is the value of x for which $f(x)$ is a maximum. This can be found by sketching the graph.



M1

maximum value of $f(x)$ occurs when $x = 1$, therefore mode = 1 A1

c. The mean is given by $\int_{-\infty}^{\infty} x f(x) dx$.

$$\int_{-\infty}^{\infty} xf(x)dx = \int_0^1 xf(x)dx + \int_1^{\frac{7}{3}} xf(x)dx$$

$$= \int_0^1 x^3 dx + \int_1^{\frac{7}{3}} \frac{7x - 3x^2}{4} dx \qquad \text{M1}$$

$$= \left[\frac{x^4}{4}\right]_0^1 + \left[\frac{7x^2}{8} - \frac{x^3}{4}\right]_1^{\frac{7}{3}}$$

$$= \frac{1}{4} + \left(\frac{7\left(\frac{7}{3}\right)^2}{8} - \frac{\left(\frac{7}{3}\right)^3}{4}\right) - \left(\frac{7}{8} - \frac{1}{4}\right)$$

$$= \frac{1}{4} + \left(\frac{343}{72} - \frac{343}{108}\right) - \frac{5}{8}$$

$$= \frac{1}{4} + \left(\frac{1029}{216} - \frac{686}{216}\right) - \frac{5}{8}$$

$$= \frac{1}{4} + \frac{343}{216} - \frac{5}{8}$$

$$= \frac{54}{216} + \frac{343}{216} - \frac{135}{216}$$

$$= \frac{262}{216}$$

$$\text{mean} = \frac{131}{108} \qquad \text{A1}$$