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Mathematical Methods(CAS)

2015

Trial Examination 2
(2 hours)

SECTION 1 Multiple-choice questions

Instructions for Section 1

Answer **all** questions.
Choose the response that is **correct** for the question.
A correct answer scores 1, an incorrect answer scores 0.
Marks will **not** be deducted for incorrect answers.
No marks will be given if more than one answer is completed for any question.

Question 1 Which one of the following points ***cannot*** be an intersection of $y = f(x)$ and $x = f(y)$?

A. (a, a), $a \in R$

B. $(\sqrt{a}, \sqrt{a})$, $a \geq 0$

C. (a^{-1}, a^{-1}), $a \in R \setminus \{0\}$

D. $(-a, a)$, $a > 0$

E. (a^2, a^2), $a \in R^-$

Question 2 The equation $2^{\frac{x+b}{2}} = 2^x + 2^{-1}$ has two distinct solutions in x if

A. $b = 0.5$

B. $0.99 < b \leq 2$

C. $1 \leq b < 10$

D. $1 \leq b < 1.1$

E. $1 < b < 1.01$

Question 3 Given $a > 0$, $f(x) = \sqrt{x}$ and $g(x) = \dfrac{x+2a}{x+a}$, $f(g(x))$ is defined for $x \in$

A. $(-2a, -a]$

B. $[-2a, -a)$

C. $R \setminus (-2a, -a]$

D. $R \setminus [-2a, -a)$

E. $(-\infty, -2a] \cup [-a, \infty)$

Question 4 A tangent to the curve $y = f(x)$ at $x = 1$ has a gradient of m. The curve is transformed by a dilation of factor of 2 from the *x*-axis followed by a dilation of factor of 0.5 from the *y*-axis.
The tangent to the transformed curve at $x = 0.5$ has a gradient of

A. $0.25m$

B. $0.5m$

C. m

D. $2m$

E. $4m$

Question 5 A line is perpendicular to the graph of $y = \log_e |2x|$ at $x = a$ where $a < 0$.
The gradient of the line is

A. $-2a$

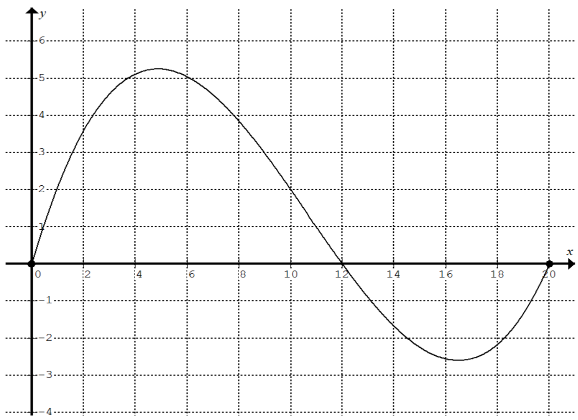
B. $2a$

C. $-a$

D. a

E. $-0.5a$

Question 6 The average value of the function shown in the graph below is closest to



A. 0.8

B. 1.3

C. 1.8

D. 2.3

E. 2.8

Question 7 The number of points of inflection in the graph of a fourth degree polynomial can ***only*** be

A. $0, 1$ or 2

B. 0 or 2

C. 0 or 1

D. 1 or 2

E. 2

Question 8 The amplitude and period of $f(x)=\left(\frac{\sin(nx)}{2}\right)^2$ are respectively

A. $\frac{1}{8}, \frac{\pi}{n}$

B. $\frac{1}{4}, \frac{\pi}{2n}$

C. $\frac{1}{2}, \frac{\pi}{n}$

D. $\frac{1}{2}, \frac{\pi}{2n}$

E. $\frac{1}{2}, 2n\pi$

Question 9 In the graph of $y=\tan\left(nx-\frac{\pi}{2}\right)$ where n is a positive integer, the number of asymptotes in the interval $(0, 2\pi)$ is

A. n

B. $n+1$

C. $n+2$

D. $2n-1$

E. $2n+1$

Question 10 Which one of the following functions does **not** satisfy $f(-x)+f(x)=0$?

A. $f(x)=x^3-x$

B. $f(x)=-2\sin(2x)$

C. $f(x)=(x+1)(x-1)^{-1}$

D. $f(x)=\sec(x+0.5\pi)$

E. $f(x)=\begin{cases}-a, & x\geq 0\\ a, & x<0\end{cases}$ where a is a positive real constant

Question 11 Given $\int_1^0 f(x-0.5)dx=2$, the value of $\int_{-0.5}^0 f(1-2x)dx$ is

A. -0.5

B. -1

C. -1.5

D. 1.5

E. 1

Question 12 $f(x)=b+(x+1)(x+\sqrt{a})(x-\sqrt{a})$ is defined for some real values of a and b.
The graph of $y=f(x)$ crosses the ***negative*** x-axis only once when

A. $b=a$ only

B. $b=a+1$ only

C. $b=a-1$ only

D. $b\geq\sqrt{1+3a}-1$

E. $b\geq a$

Question 13 The complete set of solutions to the equation $1-2^{ax+1}-2\times 4^{ax}+4\times 8^{ax}=0$ is

A. $\left\{\frac{1}{2a},\frac{1}{a}\right\}$

B. $R\setminus\left\{\frac{1}{2a},\frac{1}{a}\right\}$

C. $\left\{-\frac{1}{a},-\frac{1}{2a}\right\}$

D. $R\setminus\left\{-\frac{1}{a},-\frac{1}{2a}\right\}$

E. $\left\{\frac{1}{a},\frac{1}{2a},\frac{1}{3a},\frac{1}{4a},\frac{1}{5a},\ldots\ldots\right\}$

Question 14 For $b>a>1$, $\log_a\left(\frac{x}{b}\right)>\log_a(x+a-ab)$ in the interval

A. $(a,b]$

B. (b,a)

C. $(0,ab]$

D. $((b-a),ab]$

E. $((ab-a),ab)$

Question 15 For $a,b\in R^+$ and e the base of natural logarithms, the graphs of $y=a\log_e(bx)$ and $y=\frac{1}{b}e^{\frac{x}{a}}$ have exactly one common point when

A. $a+b=3.7$

B. $a+b=3.4$

C. $a=be$

D. $b=ea$

E. $e=ab$

Question 16 A general solution to the equation $2\sin\left(\frac{3\pi}{2}-kx\right)+\sqrt{3}=0$ for $k \in R$ is

A. $x=\left(\frac{1}{6}-2n\right)\pi$ where n is an integer

B. $x=\left(2n\pm\frac{1}{6}\right)\pi$ where n is an integer

C. $x=\left(\frac{1-12n}{6k}\right)\pi$ where n is an integer

D. $x=\left(\frac{12n\pm1}{6k}\right)\pi$ where n is an integer

E. $x=\left(\frac{1\pm12n}{6k}\right)\pi$ where n is an integer

Question 17 T transforms point P of coordinates $(a,-1)$ to point P' of coordinates $(1-a,\, a)$.
A possible transformation T is

A. $T: R^2 \to R^2$, $T\begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} -1 & 0 \\ 0 & a \end{bmatrix}\begin{bmatrix} x \\ y \end{bmatrix} + \begin{bmatrix} 1 \\ 0 \end{bmatrix}$

B. $T: R^2 \to R^2$, $T\begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} -1 & 0 \\ 0 & a \end{bmatrix}\left(\begin{bmatrix} x \\ y \end{bmatrix} + \begin{bmatrix} 1 \\ 0 \end{bmatrix}\right)$

C. $T: R^2 \to R^2$, $T\begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} -1 & 0 \\ 0 & a \end{bmatrix}\left(\begin{bmatrix} x \\ y \end{bmatrix} + \begin{bmatrix} -1 \\ 1 \end{bmatrix}\right)$

D. $T: R^2 \to R^2$, $T\begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} -1 & 0 \\ 0 & a \end{bmatrix}\left(\begin{bmatrix} x \\ y \end{bmatrix} + \begin{bmatrix} -1 \\ 2 \end{bmatrix}\right)$

E. $T: R^2 \to R^2$, $T\begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} -1 & 0 \\ 0 & a \end{bmatrix}\begin{bmatrix} x \\ y \end{bmatrix} + \begin{bmatrix} 1 \\ -1 \end{bmatrix}$

Question 18 $\Pr(X=x)=a\sin\left(\frac{x}{2}\right)$ for $x=0, \frac{\pi}{2}, \pi, \frac{3\pi}{2}, 2\pi$ is the probability distribution of random variable X. The value of $\Pr\left(X=\frac{\pi}{2}\right)$ is

A. $1-\frac{1}{\sqrt{2}}$

B. $\frac{\sqrt{2}-1}{2}$

C. $\frac{\sqrt{2}-1}{\sqrt{2}+1}$

D. $\frac{1}{\sqrt{2}}$

E. $\frac{1}{2\sqrt{2}}$

Question 19 Random variable X has a binomial probability distribution. The probability of $X=0$ and the probability of $X=1$ are shown in the following table. The other probabilities are not shown.

x	0	1
$\Pr(X=x)$	0.00001	$5(0.9)(0.0001)$

The mean and standard deviation of X are respectively closest to

A. 0.5 and 0.6

B. 0.5 and 0.7

C. 0.6 and 0.8

D. 0.6 and 0.9

E. 0.7 and 1.0

Question 20 Random variable X has a normal probability distribution. The mean of X is μ and its standard deviation is σ.

Given $\mu - 2\sigma = 1$ and $\mu + \sigma = 4$, the value of $\Pr\left(\mu - \frac{\sigma}{2} < X < \mu + \frac{3\sigma}{2}\right)$ is closest to

A. 0.52

B. 0.58

C. 0.62

D. 0.68

E. 0.72

Question 21 The probability density function of continuous random variable X is given by

$$f(x) = \begin{cases} p, & 2.5 \le x \le 3.5 \\ 3p, & 3.8 \le x \le 4.3 \\ 0, & \text{elsewhere} \end{cases}, \text{ where } p \text{ is a real constant.}$$

The median of X is closest to

A. 3.91

B. 3.90

C. 3.89

D. 3.88

E. 3.85

Question 22 Given $\Pr(A) = 0.5$, $\Pr(B) = 0.7$ and $\Pr(A \cap B) = x$, then

A. $x < 0.5$

B. $x < 0.7$

C. $0 \le x \le 1$

D. $0 \le x \le 0.2$

E. $0.2 \le x \le 0.5$

Instructions for Section 2

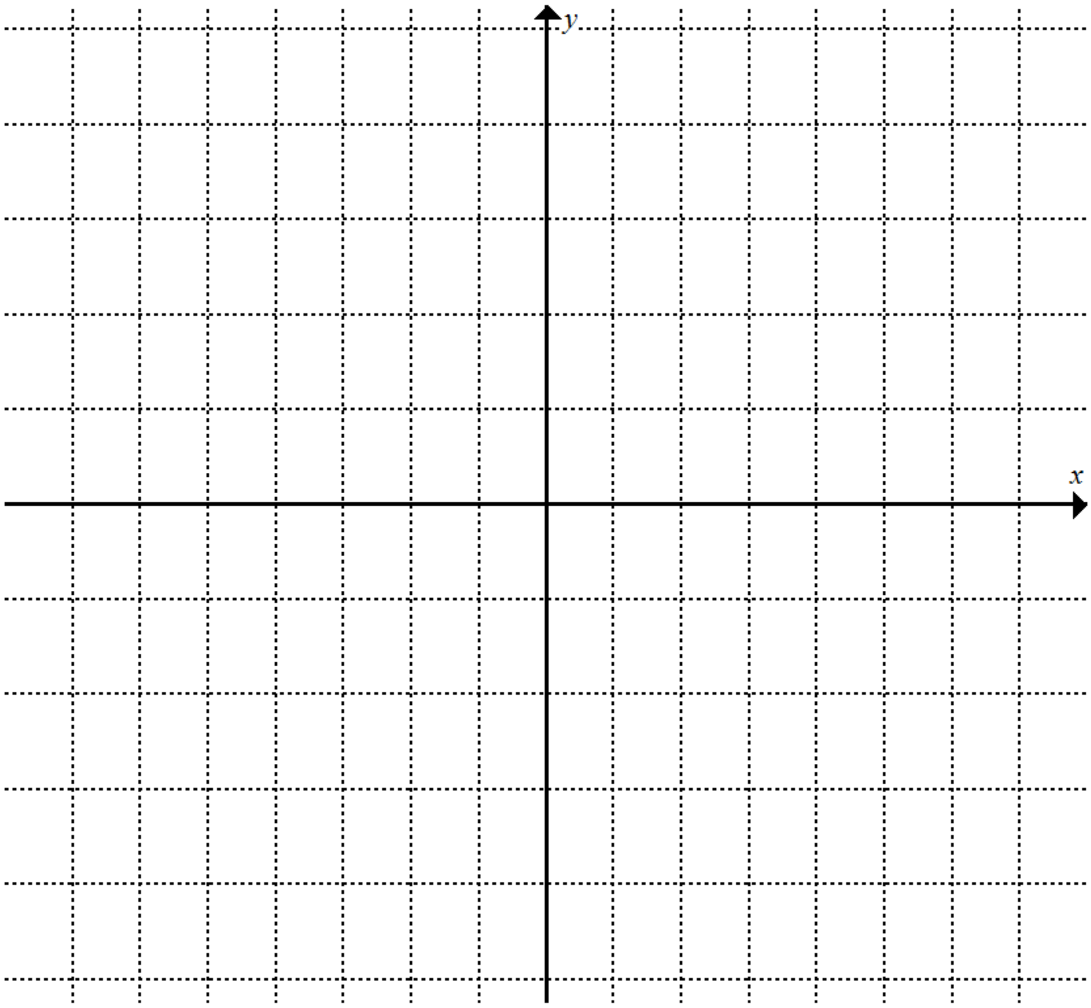
Answer **all** questions.
A decimal approximation will not be accepted if an **exact** answer is required to a question.
In questions where more than one mark is available, appropriate working **must** be shown.
Unless otherwise indicated, the diagrams in this exam are **not** drawn to scale.

Question 1

Consider parabolas $y = f(x)$ and $y = g(x)$, where $f(x) = (x-a)\left(x - \frac{5}{2}\right)$, $g(x) = x^2 - 5x + \frac{25}{4}$, and $-2 < a < -1$.

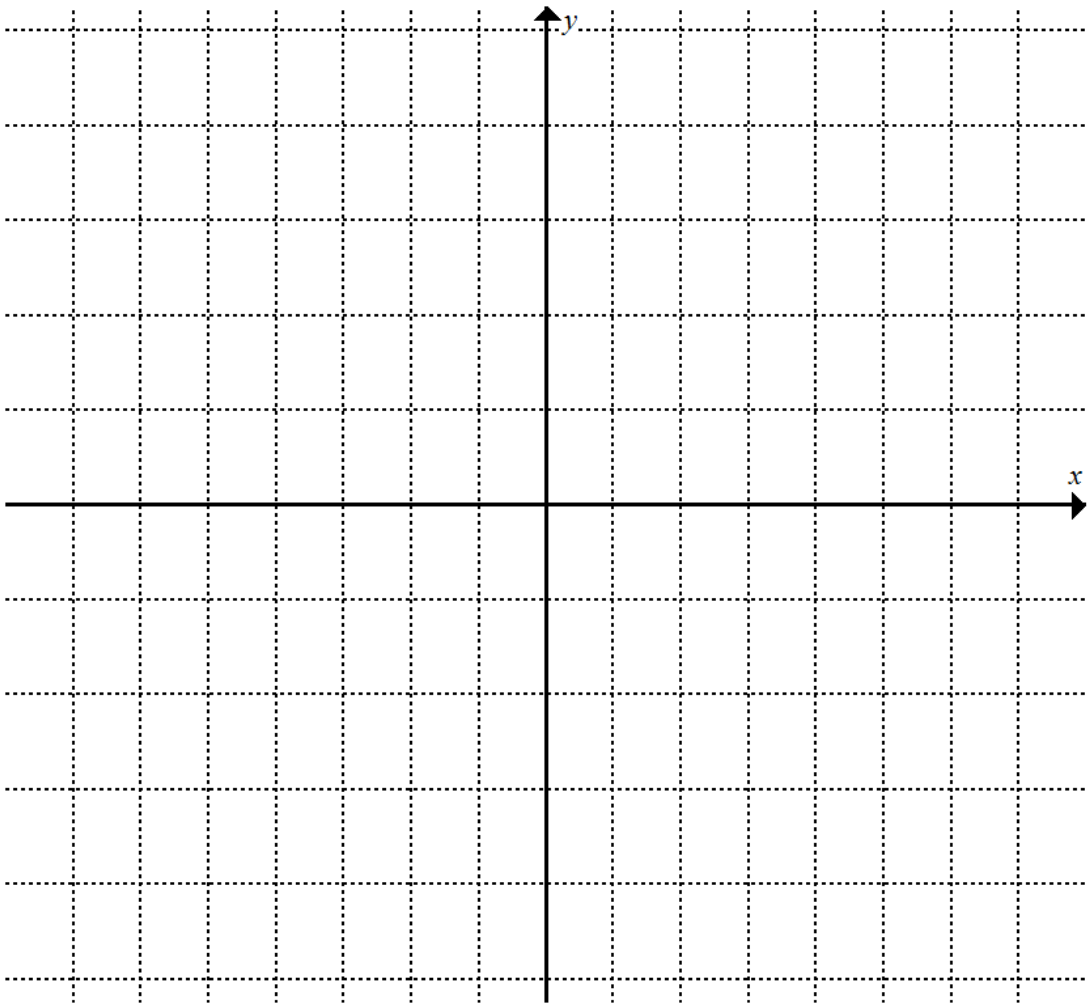
a i. $y = f(x) - g(x)$ can be expressed in the form $y = Bx + C$ where B and C are real constants. Find B and C in terms of a. 2 marks

a ii. Sketch the graph of $y = f(x) - g(x)$. Label the axis-intercept(s) in terms of a. 2 marks



b. The graph of $y = f(x) + g(x)$ has a turning point. Find the x-coordinate of the turning point. 2 marks

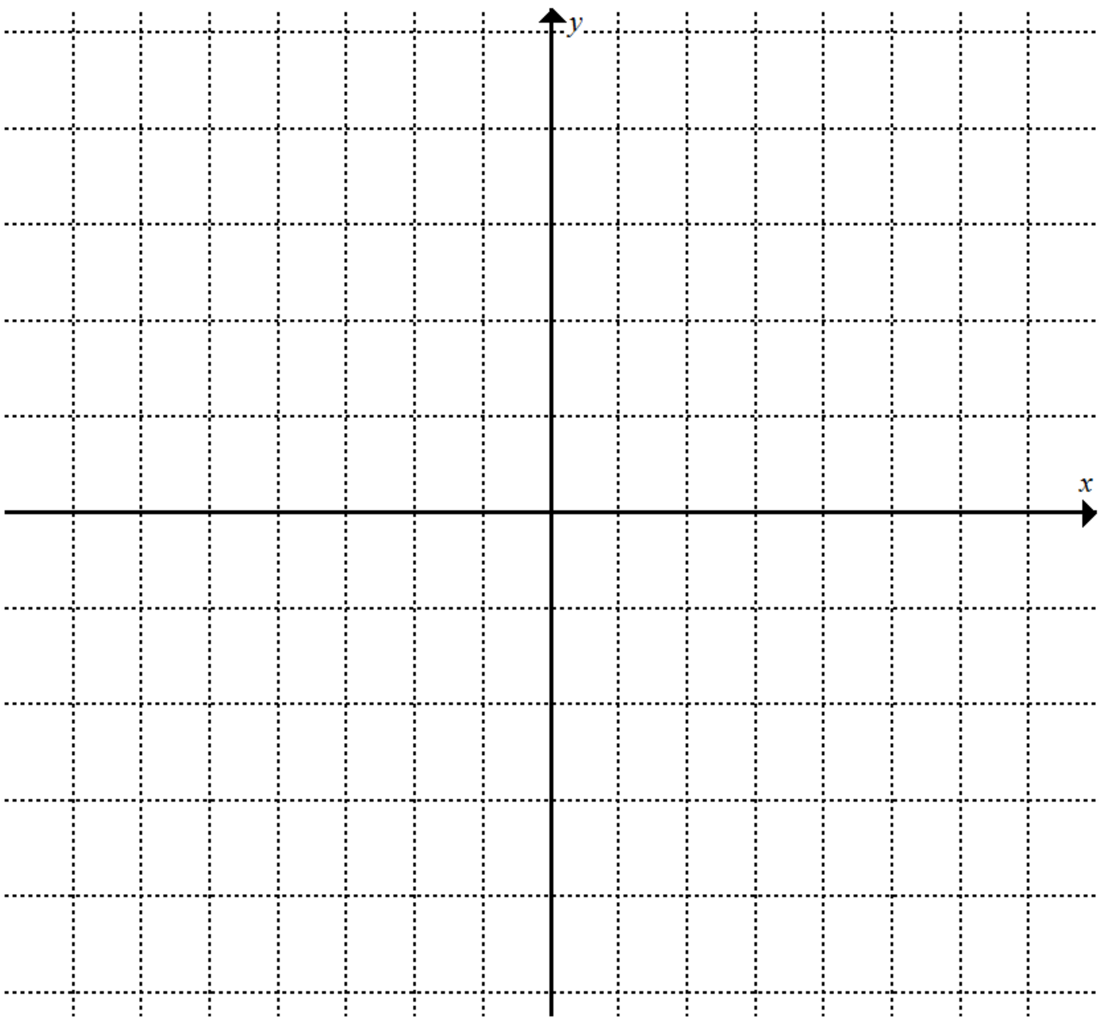
c. Sketch the graph of $y = |f(x)||g(x)|$. Label the axis-intercept(s) in terms of a. 4 marks



d i. $y=\frac{f(x)}{g(x)}$ can be expressed in the form $y=1+\frac{A}{x-\frac{5}{2}}$ where A is a real constant.

Find A in terms of a. 2 marks

d ii. Sketch the graph of $y=\frac{f(x)}{g(x)}$. Show and label all asymptotes with their equations. Label the axis-intercept(s) in terms of a in simplest form. 4 marks



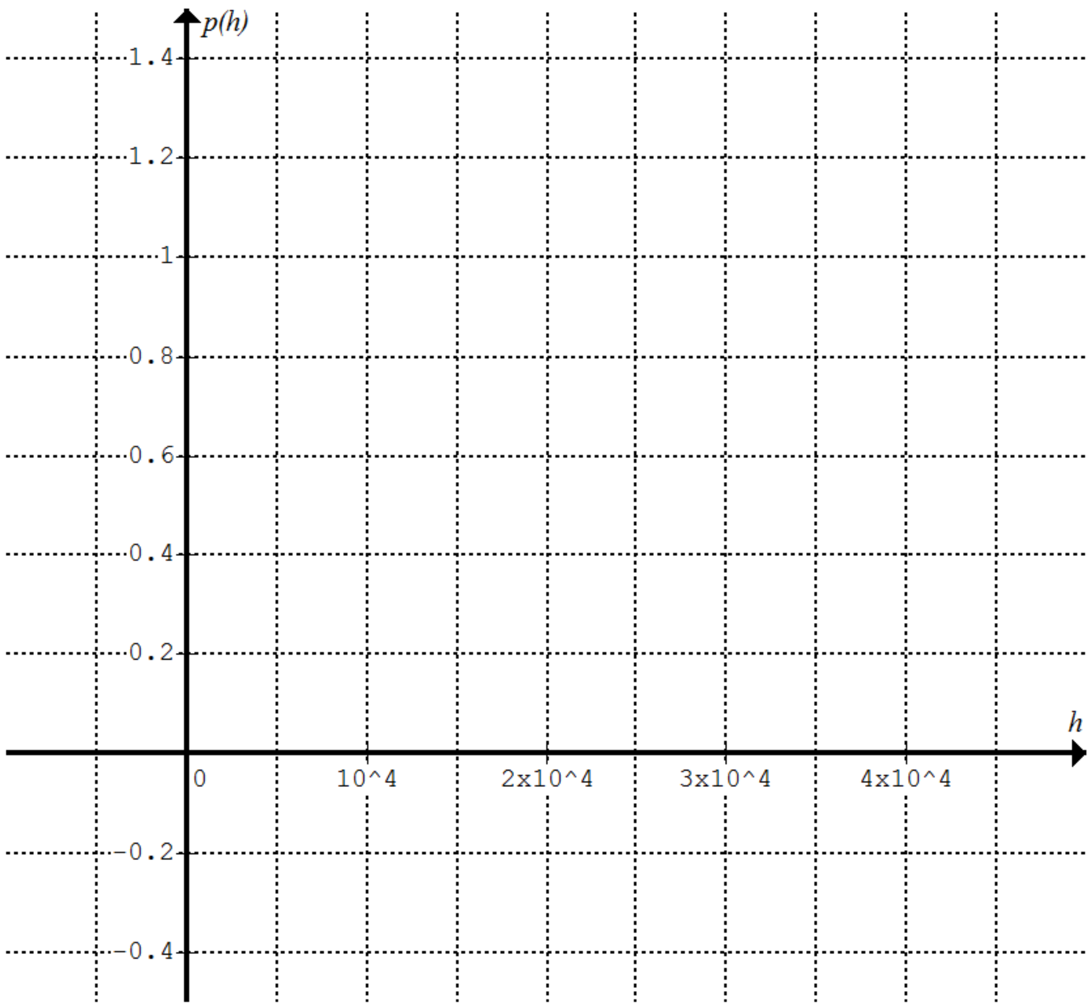
Question 2 Consider the air pressure $p(h)$, measured in atmosphere, at a height (altitude) of h metres above sea level.

To a high degree of accuracy, $p(h) = p_0 \times 10^{kh}$ for $h \geq 0$, where p_0 and k are real constants.

The air pressure is 1.00 atmosphere at sea level, and it is 0.287 atmosphere at 1.00×10^4 metres above sea level.

a. Show that $p_0 = 1.00$ and $k = -5.42 \times 10^{-5}$. 2 marks

b. Sketch the graph of air pressure $p(h)$ versus altitude h.
Show and label the typical points and features of the graph. 3 marks



c. Express h in terms of p, using base 10. 1 mark

d. Cabins in passenger jet planes are often set at an air pressure equivalent to that at 1600 metres above sea level. Determine the ratio of the cabin air pressure to the atmospheric air pressure at sea level, correct to 2 decimal places.

1 mark

e. Calculate the average rate of decrease in air pressure with increasing altitude from sea level to 1600 metres above sea level, correct to 2 decimal places.

1 mark

$p(h) = 1.00 \times 10^{-5.42 \times 10^{-5} h}$ for $h \geq 0$ can be expressed in the form $p(h) = e^{ch}$ where c is a real constant.

f. Compare he graph of $p(h) = e^{ch}$ with the graph of $p(h) = 1.00 \times 10^{-5.42 \times 10^{-5} h}$ in terms of shape, intercept(s) and/or asymptote(s). 1 mark

g. Show that $c=-1.25\times10^{-4}$ in $p(h)=e^{ch}$.

1 mark

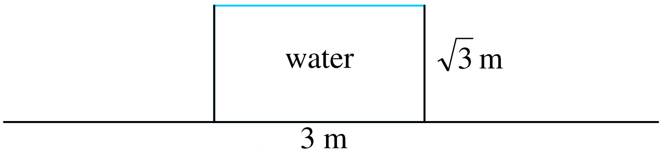
h. Determine the rate of decrease in atmospheric air pressure with increasing altitude at 1600 metres above sea level, correct to 2 decimal places.

1 mark

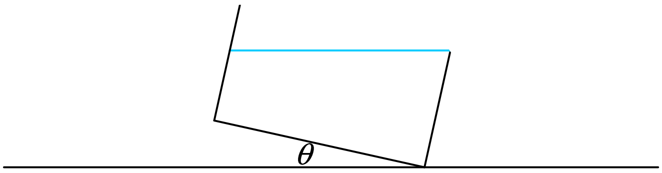
i. Show that the rate at which the air pressure changes with changing altitude is proportional to the air pressure. Write down the constant of proportionality if h is in metres and $p(h)$ is in atmospheres.

1 mark

Question 3 A square-base rectangular tank measuring 3 m by 3 m by $\sqrt{3}$ m is placed on horizontal ground. The tank is filled with water to its top edges.



Water starts to spill out when the tank begins to tilt. The tilt angle is $\theta°$ and $0 \le \theta < 90$.



The volume (m^3) of water remains in the tank is given by

$$V(\theta) = \begin{cases} \frac{27}{2}\left(\frac{2}{\sqrt{3}} - \tan\theta°\right), & 0 \le \theta < \alpha \\ f(\theta), & \alpha \le \theta < 90 \end{cases}$$

a. Find the value of α. 1 mark

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b. Show that the volume (m^3) of water remains in the tank is given by $\frac{27}{2}\left(\frac{2}{\sqrt{3}} - \tan\theta°\right)$ for $0 \le \theta < \alpha$.

3 marks

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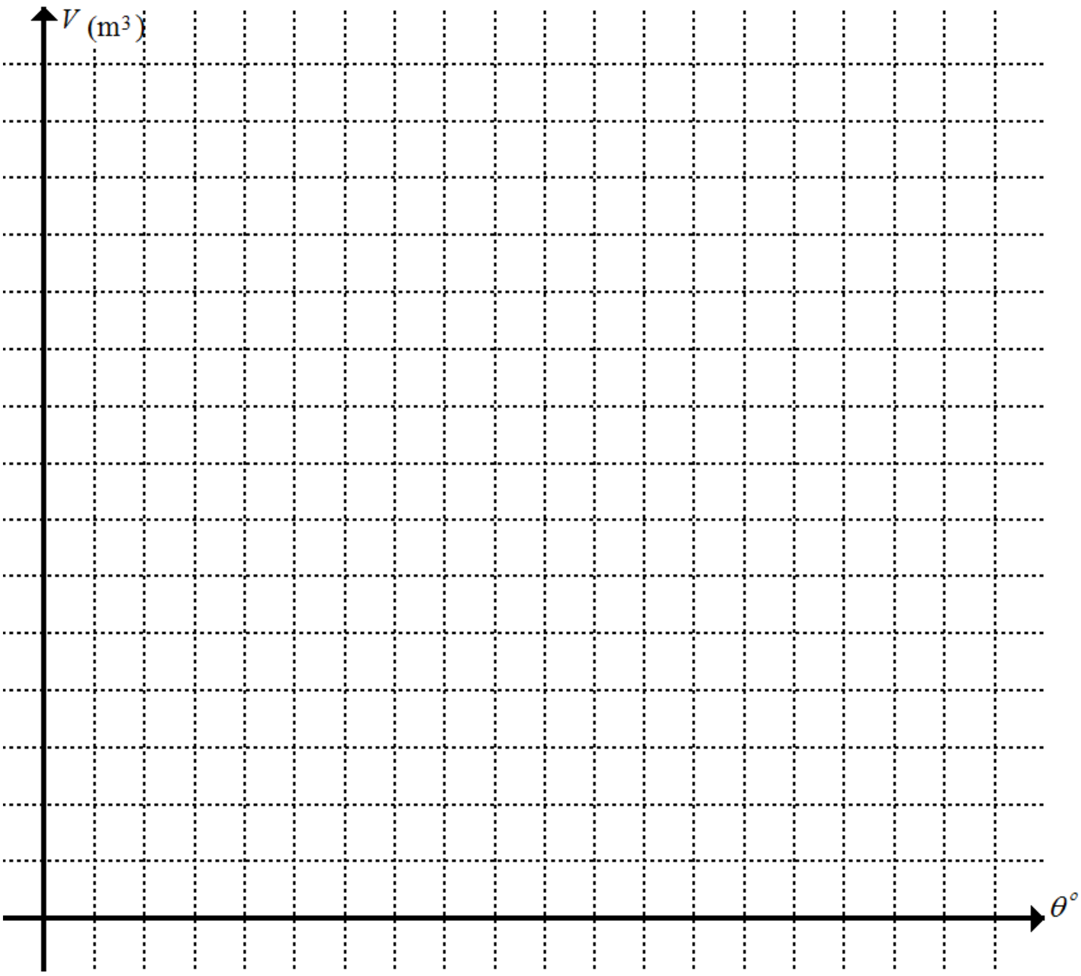
c. Find $f(\theta)$ for $\alpha \le \theta < 90$. 2 marks

d. Sketch the graph of V against θ for $0 \le \theta < 90$. Show the exact coordinates of the y-intercept.

3 marks



e. Find $\dfrac{dV}{d\theta}$ for $V = \dfrac{27}{2}\left(\dfrac{2}{\sqrt{3}} - \tan\theta°\right)$ and $V = f(\theta)$. 3 marks

f. Show that both derivatives in part e approach the same value when $\theta \to \alpha$. Find this exact value and its unit.

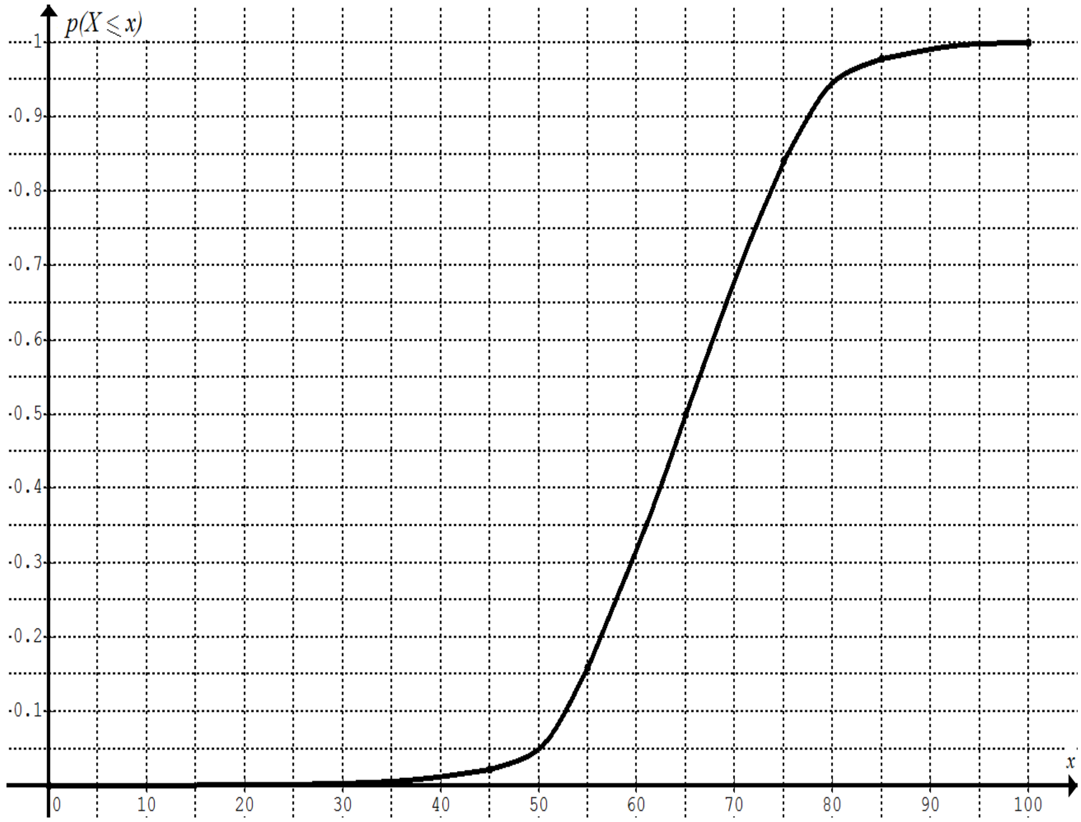
2 marks

The tilt angle $\theta°$ increases at $0.2°$ per second.

g. Find the exact rate of decrease of the volume of water remaining in the tank in m^3 per second when $\theta = 30$.

2 marks

Question 4 The scores X on an examination are normally distributed.
Let x be the value of X, and $p(X \le x)$ be the proportion of students with $X \le x$.
The graph of $p(X \le x)$ versus x is shown below.



a. **Use the graph above** to find the mean μ and the standard deviation σ of X, correct to the nearest whole number. 2 marks

b. A student is selected randomly from the group attempted the examination.
Find the probability of selecting a student scoring between 60 and 70, correct to 2 decimal places. 1 mark

The examiner decides to assign letter grades according to the following scheme:

SCORES	GRADE
Less than $(\mu - 1.5\sigma)$	E
$(\mu - 1.5\sigma)$ to $(\mu - 0.5\sigma)$	D
$(\mu - 0.5\sigma)$ to $(\mu + 0.5\sigma)$	C
$(\mu + 0.5\sigma)$ to $(\mu + 1.5\sigma)$	B
$(\mu + 1.5\sigma)$ and above	A

c. Find the percentage of students scoring grade D or E, correct to the nearest whole number. 1 mark

d. Hence find the percentage of students scoring grade C or above. 1 mark

Three students are selected randomly one by one from the group (very large group) attempted the examination. You may assume the probability distribution is binomial in the following parts of the question.

e. Find the probability that the first student selected has a B grade, correct to 2 decimal places.

1 mark

f. Find the probability that at least one of the three students selected has a B grade, correct to 2 decimal places.

2 marks

g. Find the mean number of students scoring grade B or above, correct to 2 decimal places.

1 mark

h. Find the standard deviation of the number of students scoring grade B or below, correct to 2 decimal places.

2 marks

i. Assume that all three selected students scored higher than D grade.
Find the probability that two of them have a B grade, correct to 2 decimal places. 3 marks

End of exam 2