2015 Mathematical Methods (CAS) Trial Exam 2 Solutions
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SECTION 1

1	2	3	4	5	6	7	8	9	10	11
D	E	C	E	C	B	B	A	D	E	B

12	13	14	15	16	17	18	19	20	21	22
E	C	E	E	D	D	A	E	C	D	E

Q1 $x = f(y)$ is the inverse of $y = f(x)$. If they do intersect, the intersecting point is at the line $y = x$. D

Q2 $2^x - 2^{\frac{x+b}{2}} + \frac{1}{2} = 0$, $\left(2^{\frac{x}{2}}\right)^2 - 2^{\frac{b}{2}}\left(2^{\frac{x}{2}}\right) + \frac{1}{2} = 0$

$\Delta = \left(-2^{\frac{b}{2}}\right)^2 - 4(1)\left(\frac{1}{2}\right) = 2^b - 2 > 0$ for two distinct solutions

$\therefore b > 1$ E

Q3 $g(x) < 0$ for $x \in (-2a, -a)$ and $g(-a)$ is undefined,

$\therefore f(g(x))$ is defined for $x \in R \setminus (-2a, -a]$ C

Q4 Before the transformations: gradient $= \frac{\text{rise}}{\text{run}} = \frac{m}{1}$

After the transformations: gradient $= \frac{2 \times m}{0.5 \times 1} = 4m$ E

Q5 $y = \log_e |2x|$, $\frac{dy}{dx} = \frac{1}{x} = \frac{1}{a}$ at $x = a$

$\therefore$ the gradient of the perpendicular at $x = a$ is $\frac{-1}{\frac{1}{a}} = -a$ C

Q6 Estimate the signed area of the regions bounded by the curve and the x-axis, then divide by 20. B

Q7 For a concave graph, points of inflection exist in pairs. B

Q8 $f(x) = \left(\frac{\sin(nx)}{2}\right)^2 = \frac{1}{8} - \frac{1}{8}\cos(2nx)$, $T = \frac{2\pi}{2n} = \frac{\pi}{n}$ A

Q9 There are $2n-1$ asymptotes inside interval $(0, 2\pi)$ D

Q10 $f(-x) + f(x) = 0$, $\therefore f(x)$ is an odd function. E

Q11 $\int_1^0 f(x-0.5)dx = 2$, $\therefore \int_0^1 f(x-0.5)dx = -2$

$f(1-2x)$ is the reflection (in the y-axis) and the dilation (by a factor of 0.5 in the x-direction) of $f(x-0.5)$,

$\therefore \int_{-0.5}^0 f(1-2x)dx = -1$ B

Q12 The local minimum of $f(x)$ is at $x > 0$. For the graph of $y = f(x)$ to cross the negative x-axis only once, the y-intercept must be at or above the origin, i.e. $f(0) = b - a \geq 0$, $\therefore b \geq a$.

Note: $a > \sqrt{1+3a} - 1$ for $a > 1$, $\therefore$ D is not the answer. E

Q13 $1 - 2^{ax+1} - 2 \times 2^{2ax} + 4 \times 2^{3ax} = 0$

$(1 - 2 \times 2^{ax}) - 2 \times 2^{2ax}(1 - 2 \times 2^{ax}) = 0$

$(1 - 2 \times 2^{ax})(1 - 2 \times 2^{2ax}) = 0$, $(1 - 2^{ax+1})(1 - 2^{2ax+1}) = 0$

$\therefore 2^{ax+1} = 1$ or $2^{2ax+1} = 1$

$\therefore x = -\frac{1}{a}$ or $x = -\frac{1}{2a}$ C

Q14 $\log_e\left(\frac{x}{b}\right) > \log_e(x + a - ab)$, $\therefore \frac{x}{b} > x + a - ab$ where $x > 0$,

$x + a - ab > 0$ and $b > a > 1$, $\therefore x > ab - a$

Also, from $\frac{x}{b} > x + a - ab$, $x < ab$

$\therefore (ab - a) < x < ab$ E

Q15 $y = a\log_e(bx)$ and $y = \frac{1}{b}e^{\frac{x}{a}}$ are inverse of each other.

Their common point is on the line $y = x$.

$\therefore \frac{dy}{dx} = \frac{a}{x} = 1$ and $\frac{dy}{dx} = \frac{1}{ab}e^{\frac{x}{a}} = 1$ at the common point

$\therefore x = a$ and $\therefore ab = e$ E

Q16 $2\sin\left(\frac{3\pi}{2} - kx\right) + \sqrt{3} = 0$, $\therefore -2\cos(kx) + \sqrt{3} = 0$

$\cos(kx) = \frac{\sqrt{3}}{2}$, $kx = 2n\pi \pm \frac{\pi}{6} = \left(\frac{12n \pm 1}{6}\right)\pi$

$\therefore x = \left(\frac{12n \pm 1}{6k}\right)\pi$ D

Q17 $\begin{bmatrix} -1 & 0 \\ 0 & a \end{bmatrix}\left(\begin{bmatrix} a \\ -1 \end{bmatrix} + \begin{bmatrix} -1 \\ 2 \end{bmatrix}\right) = \begin{bmatrix} 1-a \\ a \end{bmatrix}$ D

Q18 $a\sin 0 + a\sin\frac{\pi}{4} + a\sin\frac{\pi}{2} + a\sin\frac{3\pi}{4} + a\sin\pi = 1$

$\therefore a = \frac{1}{1+\sqrt{2}}$

$\therefore \Pr\left(X = \frac{\pi}{2}\right) = a\sin\frac{\pi}{4} = \frac{1}{1+\sqrt{2}} \times \frac{1}{\sqrt{2}} = \frac{1}{2+\sqrt{2}}$

$= \frac{2-\sqrt{2}}{2} = 1 - \frac{1}{\sqrt{2}}$ A

Q19 It is a binomial distribution of X with parameters $n = 5$ and $p = 0.9$.

$\mu = np = 4.5$ and $\sigma = \sqrt{np(1-p)} \approx 0.67$ E

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Q20 $\mu-2\sigma=1$ and $\mu+\sigma=4$, $\therefore\ \mu=3$ and $\sigma=1$

$\Pr\left(\mu-\frac{\sigma}{2}<X<\mu+\frac{3\sigma}{2}\right)=\Pr(2.5<X<4.5)\approx 0.6241$ C

Q21 $(3.5-2.5)p+(4.3-3.8)3p=1$, $\therefore\ p=0.4$

$\therefore\ (3.5-2.5)p=0.4<0.5$

$\therefore$ the median m is in $3.8\le x\le 4.3$

$\therefore\ (4.3-m)3\times 0.4=0.5$, $m\approx 3.88$ D

Q22

	A	A'	
B	x	$0.7-x$	0.7
B'	$0.5-x$	$x-0.2$	0.3
	0.5	0.5	1

where $x\ge 0$, $0.7-x\ge 0$, $0.5-x\ge 0$ and $x-0.2\ge 0$

$\therefore\ 0.2\le x\le 0.5$ E

SECTION 2

Q1a i $f(x)-g(x)=(x-a)\left(x-\frac{5}{2}\right)-\left(x^2-5x+\frac{25}{4}\right)$

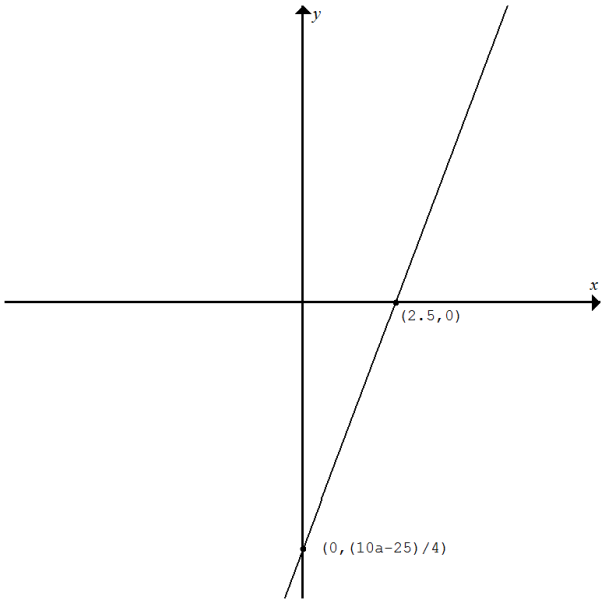
$=(x-a)\left(x-\frac{5}{2}\right)-\left(x-\frac{5}{2}\right)^2=\left(\frac{5}{2}-a\right)\left(x-\frac{5}{2}\right)$

$=\left(\frac{5}{2}-a\right)x+\left(\frac{10a-25}{4}\right)$ $\therefore\ B=\frac{5}{2}-a$ and $C=\frac{10a-25}{4}$

Q1a ii $y=\left(\frac{5}{2}-a\right)x+\left(\frac{10a-25}{4}\right)$

x-intercept: Let $y=0$, $x=\frac{5}{2}$

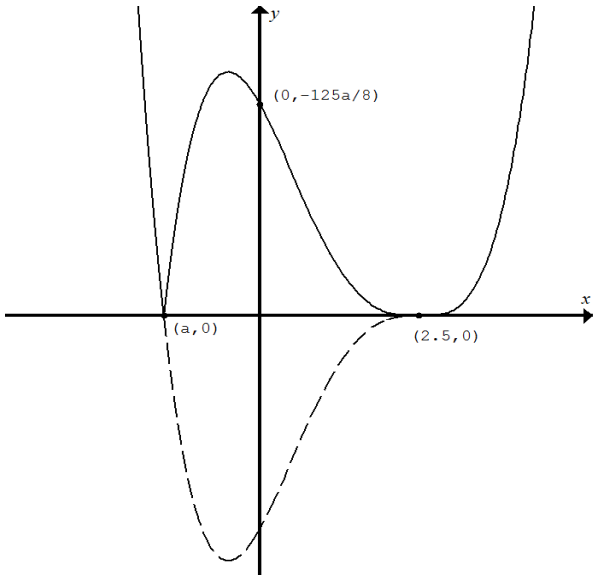
y-intercept: $y=C=\frac{10a-25}{4}$



Q1b $y=(x-a)\left(x-\frac{5}{2}\right)+\left(x-\frac{5}{2}\right)^2=2\left(x-\frac{5}{4}-\frac{a}{2}\right)\left(x-\frac{5}{2}\right)$

At the turning point, $x=\frac{1}{2}\left(\frac{5}{4}+\frac{a}{2}+\frac{5}{2}\right)=\frac{1}{2}\left(\frac{15}{4}+\frac{a}{2}\right)$

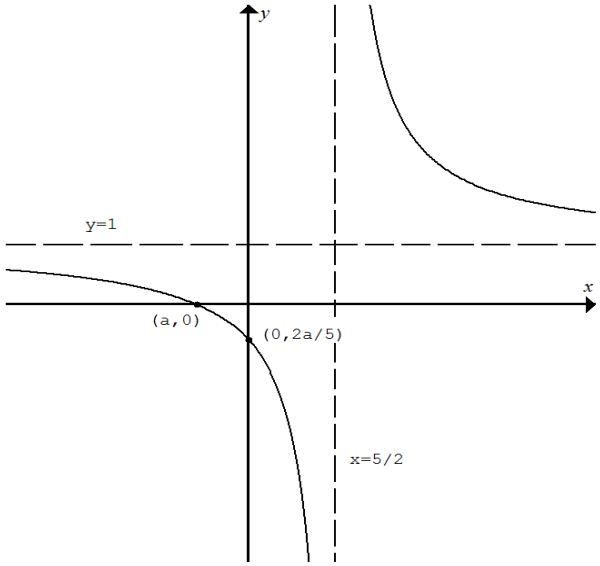
Q1c $y=|f(x)||g(x)|=|f(x)g(x)|=\left|(x-a)\left(x-\frac{5}{2}\right)^3\right|$



Q1d i $y=\frac{f(x)}{g(x)}=\frac{(x-a)\left(x-\frac{5}{2}\right)}{\left(x-\frac{5}{2}\right)^2}=\frac{x-a}{x-\frac{5}{2}}$

$=\frac{x-\frac{5}{2}+\frac{5}{2}-a}{x-\frac{5}{2}}=1+\frac{\frac{5}{2}-a}{x-\frac{5}{2}}$, $\therefore\ A=\frac{5}{2}-a$

Q1d ii



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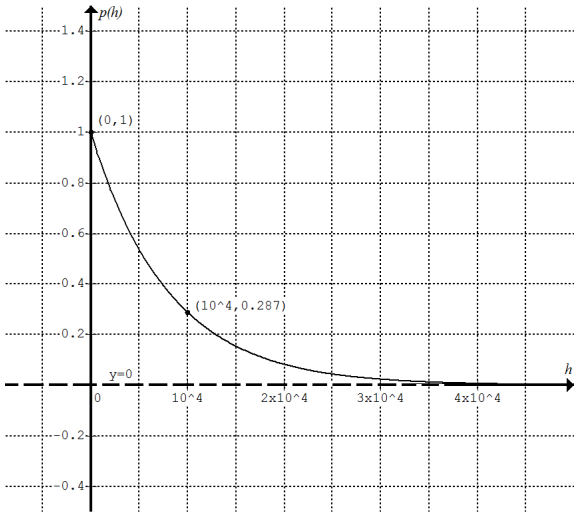
Q2a $p(h) = p_0 \times 10^{kh}$, $p(0) = p_0 \times 10^0 = 1.00$, $\therefore\ p_0 = 1.00$

$p\left(1.00 \times 10^4\right) = 1.00 \times 10^{\left(1.00 \times 10^4\right)k} = 0.287$

$\therefore\ 10^4 k = \log_{10} 0.287$, $\therefore\ k = -5.42 \times 10^{-5}$

Q2b



Q2c $p = p_0 \times 10^{kh}$, $10^{kh} = \dfrac{p}{p_0}$,

$h = \dfrac{1}{k} \log_{10} \dfrac{p}{p_0} \approx 1.84 \times 10^4 \log_{10} p$

Q2d $\dfrac{p_{\text{cabin}}}{p_0} = \dfrac{p(1600)}{p_0} = 10^{-5.42 \times 10^{-5} \times 1600} \approx 0.82$

Q2e Average rate of change $= \dfrac{\Delta p}{\Delta h} = \dfrac{0.82 - 1.00}{1600} \approx -1.13 \times 10^{-4}$

Average rate of decrease $\approx 1.13 \times 10^{-4}$ atmosphere per metre

Q2f The two graphs have the same shape, intercept and asymptote.

Q2g $p = 1.00 \times 10^{-5.42 \times 10^{-5} h} = e^{ch}$

$\therefore\ e^c = 10^{-5.42 \times 10^{-5}}$, $c = \log_e \left(10^{-5.42 \times 10^{-5}}\right) = -1.25 \times 10^{-4}$

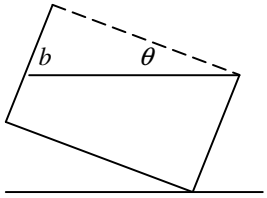
Q2h $\dfrac{dp}{dh} = -1.25 \times 10^{-4} \times e^{-1.25 \times 10^{-4} \times 1600} \approx -1.02 \times 10^{-4}$

Rate of decrease $= 1.02 \times 10^{-4}$ atmosphere per metre

Q2i $p = e^{ch}$, $\dfrac{dp}{dh} = ce^{ch} = cp$, $\therefore\ \dfrac{dp}{dh} \propto p$ and the constant of proportionality $c = -1.25 \times 10^{-4}$.

Q3a $\tan \alpha° = \dfrac{\sqrt{3}}{3}$, $\alpha = 30$

Q3b



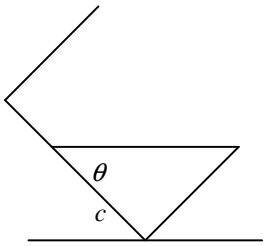
$b = 3 \tan \theta°$

Volume of water spilled out $= \dfrac{1}{2} \times 3 \times 3 \tan \theta° \times 3$

Volume of water remains $= 3 \times 3 \times \sqrt{3} - \dfrac{1}{2} \times 3 \times 3 \tan \theta° \times 3$

$= \dfrac{27}{2}\left(\dfrac{2}{\sqrt{3}} - \tan \theta°\right)$

Q3c

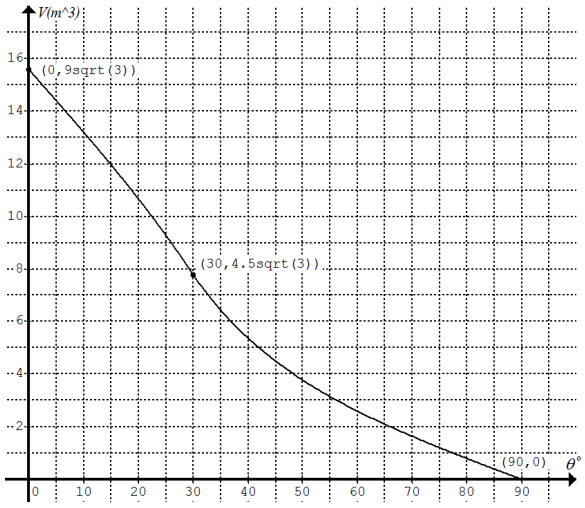


$c = \dfrac{\sqrt{3}}{\tan \theta°}$

Volume of water remains

$= f(\theta) = \dfrac{1}{2} \times \dfrac{\sqrt{3}}{\tan \theta°} \times \sqrt{3} \times 3 = \dfrac{9}{2 \tan \theta°}$

Q3d



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Q3e $V = \frac{27}{2}\left(\frac{2}{\sqrt{3}} - \tan\theta°\right) = \frac{27}{2}\left(\frac{2}{\sqrt{3}} - \tan\frac{\pi\theta}{180}\right)$

$\frac{dV}{d\theta} = -\frac{27}{2} \times \frac{\pi}{180}\sec^2\frac{\pi\theta}{180} = -\frac{3\pi}{40}\sec^2\theta° = -\frac{3\pi}{40\cos^2\theta°}$

$V = \frac{9}{2\tan\theta°} = \frac{9}{2}\left(\tan\frac{\pi\theta}{180}\right)^{-1}$

$\frac{dV}{d\theta} = -\frac{\pi}{180} \times \frac{9}{2\tan^2\frac{\pi\theta}{180}} \times \sec^2\frac{\pi\theta}{180} = -\frac{\pi\sec^2\theta°}{40\tan^2\theta°} = -\frac{\pi}{40\sin^2\theta°}$

Q3f When $\theta \to 30$, $-\frac{3\pi}{40\cos^2\theta°} = -\frac{\pi}{10}$, $-\frac{\pi}{40\sin^2\theta°} = -\frac{\pi}{10}$

.: both derivatives approach the same value $-\frac{\pi}{10}$ m^3 per degree

Q3g $\frac{dV}{dt} = \frac{dV}{d\theta} \times \frac{d\theta}{dt} = -\frac{\pi}{10} \times 0.2 = -\frac{\pi}{50}$

Rate of decrease $= \frac{\pi}{50}\ \text{m}^3$ per second

Q4a $\mu = 65$, $\sigma = 10$

Note: Some of the following answers may differ in the second decimal place because of using CAS ($\mu = 65$, $\sigma = 10$) or working from the given graph.

Q4b $\Pr(60 < X < 70) \approx 0.37$ from graph

Q4c $0.31 = 31\%$ from graph

Q4d $100\% - 31\% = 69\%$

Q4e $\Pr(\text{B grade}) = \Pr(70 < X < 80) \approx 0.26$ from graph

Q4f

$\Pr(\text{at least one B grade}) = 1 - \Pr(\text{none}) \approx 1 - (1 - 0.26)^3 \approx 0.59$

Q4g $\Pr(\text{B grade or above}) \approx 1 - 0.68 = 0.32$

Mean number of students $= np = 3 \times 0.32 = 0.96$

Q4h $\Pr(\text{B grade or below}) \approx 0.94$

$\sigma = \sqrt{np(1-p)} = \sqrt{3 \times 0.94 \times 0.06} \approx 0.41$

Q4i $\Pr(\text{B grade} \mid \text{higher than D grade}) = \frac{0.26}{0.68} \approx 0.382$

$\Pr(X = 2) = {}^3C_2 \times 0.382^2 \times (1 - 0.382) \approx 0.27$

Please inform mathline@itute.com re conceptual, mathematical and/or typing errors

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