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MATHS METHODS (CAS) 3 & 4 TRIAL EXAMINATION 1 SOLUTIONS 2015

Question 1 (5 marks)

a. $\frac{d}{dx}(4x^3 - x)^6 = 6(4x^3 - x)^5 \times (12x^2 - 1)$ (Chain rule)

(1 mark) – $6(4x^3 - x)^5$
(1 mark) – $(12x^2 - 1)$

b.

$$g(x) = \frac{\log_e(2x)}{1+2x}$$

$$g'(x) = \frac{(1+2x)\times\frac{2}{2x} - 2\log_e(2x)}{(1+2x)^2}$$ (quotient rule)

(1 mark) – correct numerator
(1 mark) – correct denominator

$$g'\left(\frac{1}{2}\right) = \frac{(1+1)\times 2 - 2\log_e(1)}{(1+1)^2}$$

$$= \frac{4 - 2\times 0}{2^2} \quad \text{since } \log_e(1) = 0$$

$$= 1$$

(1 mark)

Question 2 (2 marks)

$$\int \frac{2}{(3x-5)^4}dx$$

$$= \frac{2}{3\times -3}\times(3x-5)^{-3} + c$$

$$= \frac{-2}{9(3x-5)^3} + c$$

(1 mark) for $\frac{1}{(3x-5)^3}$
(1 mark) for $\frac{-2}{9}$

Question 3 (2 marks)

$$3^{2x} - 8 \times 3^{x} = 9$$
$$3^{2x} - 8 \times 3^{x} - 9 = 0$$

Let $a = 3^{x}$

$a^2 - 8a - 9 = 0$

$(a-9)(a+1) = 0$

$a = 9$ or $a = -1$ **(1 mark)**

Sub back

$3^{x} = 9$ or $3^{x} = -1$

$3^{x} = 3^{2}$ no real solution exists for x

$x = 2$

So $x = 2$.

(1 mark)

Question 4 (2 marks)

<u>Method 1</u>

$\log_5(x^3) + 2\log_5(x) = 15$

$\log_5(x^3 \times x^2) = 15$

$\log_5(x^5) = 15$

$5^{15} = x^5$ **(1 mark)**

$(x^5)^{\frac{1}{5}} = (5^{15})^{\frac{1}{5}}$

$x = 5^3$

$x = 125$

(1 mark)

<u>Method 2</u>

$\log_5(x^3) + 2\log_5(x) = 15$

$3\log_5(x) + 2\log_5(x) = 15$

$5\log_5(x) = 15$ **(1 mark)**

$\log_5(x) = 3$

$5^3 = x$

$x = 125$

(1 mark)

Question 5 (5 marks)

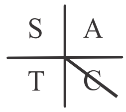
a. $f(x) = (x+2)\cos(x)$

$f'(x) = (x+2) \times {}^{-}\sin(x) + 1 \times \cos(x)$ (product rule)

So $f'(x) = -(x+2)\sin(x) + \cos(x)$

(1 mark) for first term
(1 mark) for second term

When $x = -\frac{\pi}{6}$,



$$f'\left(-\frac{\pi}{6}\right) = -\left(-\frac{\pi}{6}+2\right)\sin\left(-\frac{\pi}{6}\right) + \cos\left(-\frac{\pi}{6}\right)$$

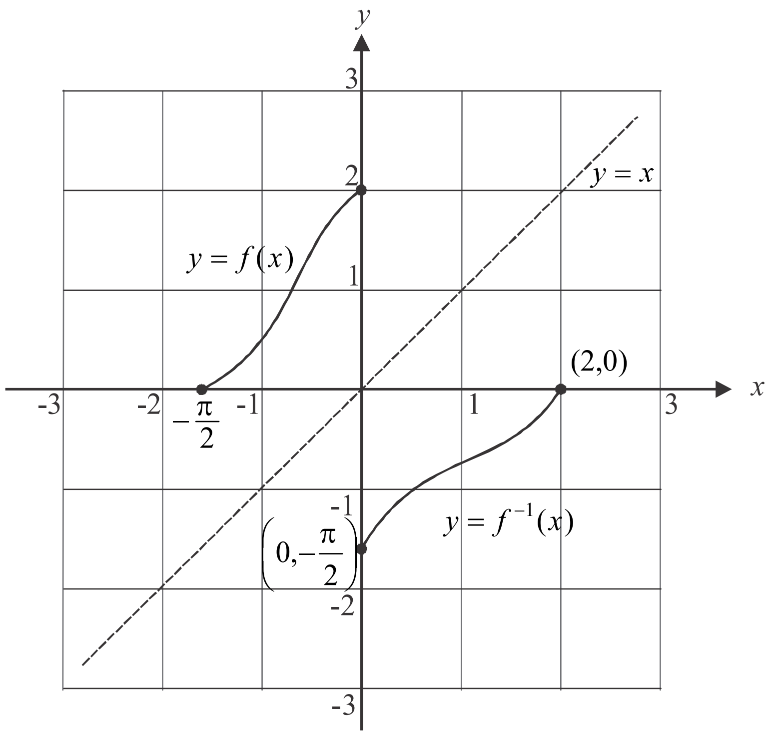
$$= \left(\frac{\pi}{6}-2\right) \times -\frac{1}{2} + \frac{\sqrt{3}}{2}$$

$$= -\frac{\pi}{12} + 1 + \frac{\sqrt{3}}{2}$$

So the gradient of the graph at $x = -\frac{\pi}{6}$ is $-\frac{\pi}{12} + 1 + \frac{\sqrt{3}}{2}$.

(1 mark)

b.



The graph of f^{-1} is a reflection of the graph of *f* in the line $y = x$.

The endpoints of *f* are $\left(-\frac{\pi}{2}, 0\right)$ and $(0,2)$.

The endpoints of f^{-1} are therefore $\left(0, -\frac{\pi}{2}\right)$ and $(2,0)$.

(1 mark) – correct shape
(1 mark) – correct endpoints

Question 6 (3 marks)

Let (x', y') be an image point.

$$\begin{bmatrix} x' \\ y' \end{bmatrix} = \begin{bmatrix} \frac{1}{2} & 0 \\ 0 & -3 \end{bmatrix}\begin{bmatrix} x \\ y \end{bmatrix} + \begin{bmatrix} -2 \\ 1 \end{bmatrix}$$

$$= \begin{bmatrix} \frac{x}{2} \\ -3y \end{bmatrix} + \begin{bmatrix} -2 \\ 1 \end{bmatrix}$$

$$= \begin{bmatrix} \frac{x}{2} - 2 \\ -3y + 1 \end{bmatrix}$$

(1 mark)

So $x' = \frac{x}{2} - 2$ and $y' = -3y + 1$

$x' + 2 = \frac{x}{2}$ $\qquad$ $3y = 1 - y'$

$x = 2(x' + 2)$ $\qquad$ $y = \frac{1 - y'}{3}$

So $y = e^{\frac{x-4}{2}} - 1$

becomes $\frac{1 - y'}{3} = e^{\frac{2(x'+2)-4}{2}} - 1$

(1 mark)

$$\frac{1 - y'}{3} = e^{\frac{2x'+4-4}{2}} - 1$$

$$\frac{1 - y'}{3} = e^{x'} - 1$$

$$1 - y' = 3e^{x'} - 3$$

$$-y' = 3e^{x'} - 4$$

$$y' = 4 - 3e^{x'}$$

The image equation is $y = 4 - 3e^{x}$.
So $a = 4$ and $b = -3$.

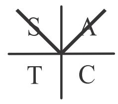
(1 mark)

Question 7 (5 marks)

a. The graphs will intersect when

$$f(x) = g(x)$$
$$2\sin\left(\frac{\pi x}{9}\right) = \sqrt{3}$$
$$\sin\left(\frac{\pi x}{9}\right) = \frac{\sqrt{3}}{2}$$

(1 mark)

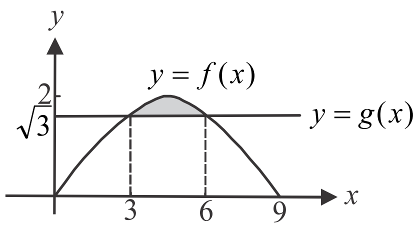


$$\frac{\pi x}{9} = \frac{\pi}{3}, \frac{2\pi}{3}, \frac{7\pi}{3}, ...$$
$$x = \frac{\pi}{3} \times \frac{9}{\pi}, \frac{2\pi}{3} \times \frac{9}{\pi}, \frac{7\pi}{3} \times \frac{9}{\pi}, ...$$
$$= 3,\ 6,\ 21, ...$$

Since f and g are each defined for $x \in [0,9]$, then $x = 3$ or 6.
Points of intersection are $\left(3, \sqrt{3}\right)$ and $\left(6, \sqrt{3}\right)$.

(1 mark)

b. Do a quick sketch.
For the graph of $y = f(x)$,
$\text{period} = 2\pi \div \frac{\pi}{9} = 18$
but $d_f = [0,9]$.



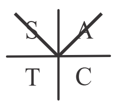
$$\text{Area} = \int_3^6 (f(x) - g(x))dx$$

(1 mark)

$$= \int_3^6 \left(2\sin\left(\frac{\pi x}{9}\right) - \sqrt{3}\right)dx$$
$$= \left[-2 \div \frac{\pi}{9}\cos\left(\frac{\pi x}{9}\right) - \sqrt{3}x\right]_3^6$$

(1 mark)

$$= \left[\frac{-18}{\pi}\cos\left(\frac{\pi x}{9}\right) - \sqrt{3}x\right]_3^6$$
$$= \left(\frac{-18}{\pi}\cos\left(\frac{2\pi}{3}\right) - \sqrt{3} \times 6\right) - \left(\frac{-18}{\pi}\cos\left(\frac{\pi}{3}\right) - \sqrt{3} \times 3\right)$$
$$= \frac{-18}{\pi} \times \frac{-1}{2} - 6\sqrt{3} + \frac{18}{\pi} \times \frac{1}{2} + 3\sqrt{3}$$



$$= \frac{9}{\pi} - 6\sqrt{3} + \frac{9}{\pi} + 3\sqrt{3}$$

$$\text{Area} = \frac{18}{\pi} - 3\sqrt{3} \text{ square units.}$$

(1 mark)

Question 8 (4 marks)

a. Since we have a probability density function,

$$\int_0^a \frac{1}{x+1}dx = 1$$ **(1 mark)**

$$\left[\log_e|x+1|\right]_0^a = 1$$

$$\log_e(a+1) - \log_e(1) = 1, \qquad a > 0$$

$$\log_e(a+1) = 1 \qquad \text{since } \log_e(1) = 0$$

$$e^1 = a+1$$

$$a = e - 1$$

(1 mark)

b. $$\Pr(X < m | X < 1)$$

$$= \frac{\Pr(X < m \cap X < 1)}{\Pr(X < 1)} \qquad \text{(conditional probability formula)}$$

$$= \frac{\Pr(X < m)}{\Pr(X < 1)}$$

(1 mark)

Now, $\Pr(X < m) = 0.5$ since m is the median.

$$\Pr(X < 1) = \int_0^1 \frac{1}{x+1}dx$$

$$= \left[\log_e|x+1|\right]_0^1$$

$$= \log_e(2) - \log_e(1)$$

$$= \log_e(2) \qquad \text{since } \log_e(1) = 0$$

So $\dfrac{\Pr(X < m)}{\Pr(X < 1)}$

$$= \frac{0.5}{\log_e(2)}$$

$$= \frac{1}{2\log_e(2)} \quad \text{or} \quad \frac{1}{\log_e(4)}$$

(1 mark)

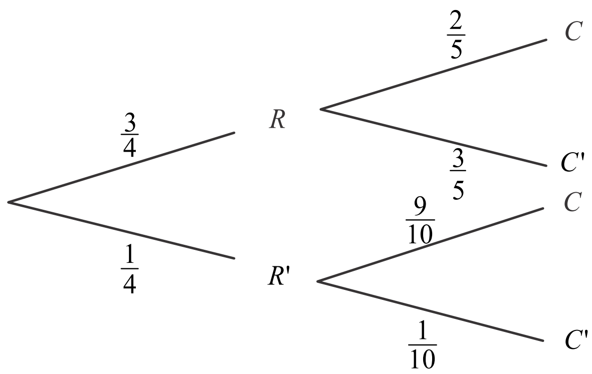
Question 9 (6 marks)

a. $\Pr(C, C') + \Pr(C', C)$ **(1 mark)**

$$= \frac{9}{10} \times \frac{1}{10} + \frac{1}{10} \times \frac{9}{10}$$
$$= \frac{9}{100} + \frac{9}{100}$$
$$= \frac{18}{100}$$
$$= \frac{9}{50}$$

(1 mark)

b. **i.**



(1 mark)

$$\Pr(C) = \Pr(R, C) + \Pr(R', C)$$
$$= \frac{3}{4} \times \frac{2}{5} + \frac{1}{4} \times \frac{9}{10}$$
$$= \frac{6}{20} + \frac{9}{40}$$
$$= \frac{21}{40}$$

(1 mark)

ii. $\Pr(R \mid C') = \dfrac{\Pr(R \cap C')}{\Pr(C')}$ (conditional probability formula) **(1 mark)**

$$= \frac{\frac{3}{4} \times \frac{3}{5}}{1 - \Pr(C)}$$ (using the tree diagram)

$$= \frac{9}{20} \div \left(1 - \frac{21}{40}\right)$$ (from part i.)

$$= \frac{9}{20} \div \frac{19}{40}$$
$$= \frac{9}{20} \times \frac{40}{19}$$
$$= \frac{18}{19}$$

(1 mark)

Question 10 (6 marks)

a. **i.**

$$f(x)=\sqrt{u-x}$$
$$=(u-x)^{\frac{1}{2}}$$
$$f'(x)=\frac{1}{2}(u-x)^{-\frac{1}{2}}\times -1$$
$$=\frac{-1}{2(u-x)^{\frac{1}{2}}}$$
$$=\frac{-1}{2\sqrt{u-x}}$$

When the gradient of the tangent equals -1,

$$\frac{-1}{2\sqrt{u-x}}=-1$$

(1 mark)

$$1=2\sqrt{u-x}$$
$$\frac{1}{2}=\sqrt{u-x}$$
$$\left(\frac{1}{2}\right)^2=u-x$$
$$x=u-\frac{1}{4}$$

So $f(x)=\sqrt{u-\left(u-\frac{1}{4}\right)}$

$$=\sqrt{\frac{1}{4}}$$
$$=\frac{1}{2}$$

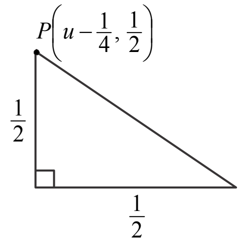
The y-coordinate of P is $\frac{1}{2}$ when the gradient of the tangent is -1.

(1 mark)

ii. Method 1

Since the gradient of the tangent is -1, and since the y coordinate of P is $\frac{1}{2}$,

then the horizontal distance will be $\frac{1}{2}$ unit.



(1 mark)

Method 2

Equation of tangent

$$y-\frac{1}{2}=-1\left(x-\left(u-\frac{1}{4}\right)\right)$$

$$y=-x+u-\frac{1}{4}+\frac{1}{2}$$

$$y=-x+u+\frac{1}{4}$$

When $y=0,\ x=u+\frac{1}{4}$

So the tangent intersects the x-axis at $\left(u+\frac{1}{4},0\right)$ and P is $\left(u-\frac{1}{4},\frac{1}{2}\right)$.

So the horizontal distance is $u+\frac{1}{4}-\left(u-\frac{1}{4}\right)=\frac{1}{2}$ unit.

(1 mark)

b. Let area of rectangle $OMQN$ be A

$$A=OM\times MQ$$
$$=x\times f(x)$$
$$=x\sqrt{u-x}$$
$$=x(u-x)^{\frac{1}{2}}$$

$$\frac{dA}{dx}=1\times\sqrt{u-x}+x\times\frac{1}{2}(u-x)^{-\frac{1}{2}}\times -1$$

$$=\sqrt{u-x}-\frac{x}{2\sqrt{u-x}}$$ **(1 mark)**

$\frac{dA}{dx}=0$ for max.

$$\sqrt{u-x}-\frac{x}{2\sqrt{u-x}}=0$$
$$\sqrt{u-x}=\frac{x}{2\sqrt{u-x}}$$
$$x=2(u-x)$$
$$x=2u-2x$$
$$3x=2u$$
$$x=\frac{2u}{3}$$ **(1 mark)**

Substitute

$x=\frac{2u}{3}$ into $A=x\sqrt{u-x}$

$$A=\frac{2u}{3}\times\sqrt{u-\frac{2u}{3}}$$
$$=2\times\frac{u}{3}\times\sqrt{\frac{u}{3}}$$
$$=2\left(\frac{u}{3}\right)^{\frac{3}{2}}$$

Maximum area is $2\left(\frac{u}{3}\right)^{\frac{3}{2}}$ square units.

(1 mark)