A

Free Exam
for 2006-15 VCE study design

Engage Education Foundation

Units 3 and 4 Maths Methods (CAS): Exam 2

Practice Exam Solutions

Stop!

Don't look at these solutions until you have attempted the exam.

Any questions?

Check the Engage website for updated solutions, then email practiceexams@ee.org.au.

Section A – Multiple-choice questions

Question 1

The correct answer is C.

Substitute in the two y-values, and then find the matching domain.

$-1 = 4x - 1$
$0 = 4x$
$x = 0$

$11 = 4x - 1$
$12 = 4x$
$x = 3$

The domain value will include / exclude that point just as the corresponding value from the range does.

Question 2

The correct answer is B.

$kx - 4 = x^2 + 4x$
$x^2 + (4 - k)x + 4 = 0$

Use the discriminant and set it to 0 to find the value of k that will produce just one solution:

$D = 0 = (4 - k)^2 - 4 \times 4$
$0 = 16 - 8k + k^2 - 16$
$k^2 - 8k = 0$
$k(k - 8) = 0$
$k = 0$ and $k = 8$ will both satisfy the equation, but only $k = 0$ is an option.

Question 3

The correct answer is C.

$e^{6x} - 6e^{3x} + 5 = (e^{3x})^2 - 6e^{3x} + 5$

Let $a = e^{3x}$:

$a^2 - 6a + 5 = 0$
$(a - 5)(a - 1) = 0$
$a = 5$ and $a = 1$
$\therefore e^{3x} = 5$ and $e^{3x} = 1$

$\log_e e^{3x} = \log_e 5$
$3x = \log_e 5$
$x = \frac{1}{3}\log_e 5$

$x = 0$

Question 4
The correct answer is E.

Expand the matrix multiplication and addition (x' and y' represent the transformed coordinates, x and y are the original coordinates):

$$\begin{bmatrix} x' \\ y' \end{bmatrix} = \begin{bmatrix} 0 & 2 \\ 1 & 0 \end{bmatrix}\begin{bmatrix} x \\ y \end{bmatrix} + \begin{bmatrix} 1 \\ 2 \end{bmatrix}$$
$$\begin{bmatrix} x' \\ y' \end{bmatrix} = \begin{bmatrix} 2y+1 \\ x+2 \end{bmatrix}$$

Therefore:

$$x' = 2y + 1$$
$$x' - 1 = 2y$$
$$y = \frac{1}{2}(x' - 1)$$

$$y' = x + 2$$
$$x = y' - 2$$

Now substitute these into the given equation ($y = 4x^3$):

$$\frac{1}{2}(x' - 1) = 4(y' - 2)^3$$
$$\frac{1}{8}(x' - 1) = (y' - 2)^3$$
$$\frac{1}{2}(x' - 1)^{\frac{1}{3}} = y' - 2$$
$$y' = \frac{1}{2}(x' - 1)^{\frac{1}{3}} + 2$$

Question 5
The correct answer is E.

$\sin(0) = 0$ and $\sin(\pi) = 0$, so the average rate of change is 0.

Question 6
The correct answer is E.

Question 7
The correct answer is A.

$$\sin^2 x + \sqrt{3}\sin x = 0$$
$$\sin x\left(\sin x + \sqrt{3}\right) = 0$$
$$\sin x = 0$$

$$\sin x + \sqrt{3} = 0$$
$$\sin x = -\sqrt{3}$$

The only difficulty with this question is the notation. It is asking for the set that contains the same values as the solutions to the given equations, however, the sets themselves are given in terms of equations.

Question 8
The correct answer is B.

$$f(u) + f(2u) = \log_e u + \log_e 2u = \log_e(u \times 2u) = \log_e(2u^2) = f(2u^2)$$

Question 9
The correct answer is C.

The x asymptote comes from the fact that the denominator (bottom) of a fraction cannot be 0:
$x + 2 \neq 0$
$x \neq -2$

The y asymptote comes from the limit of the fraction as x approaches infinity:
$\lim_{x \to \infty} \frac{x-6}{x+2} = \lim_{x \to \infty} \frac{1-\frac{6}{x}}{1+\frac{2}{x}}$ (divide through by x)
$= \frac{1}{1} = 1$ (since $\lim_{x \to \infty} \frac{1}{x} = 0$)

Question 10
The correct answer is D.

'Strictly positive' excludes 0.

Question 11
The correct answer is D.

$y' = 4e^{4x}\sin(3x) + 3e^{4x}\cos(3x)$
when $x = 0$: $y' = 4 \times 1 \times \sin(0) + 3 \times 1 \times \cos(0) = 0 + 3 = 3$

Question 12
The correct answer is D.

$\log_e x$ is defined only for $x > 0$.

Question 13
The correct answer is D.

This is an application of the chain rule.

Question 14
The correct answer is E.

$f'(x) = x^2 + 2x + 1 = (x+1)^2$, which has one x-intercept. Therefore there is only one stationary point.

Furthermore, $f'(x) \geq 0$ for all x, so f must always be stationary or increasing.

Hence, the sole stationary point is a stationary point of inflection.

Question 15
The correct answer is A.

$\int_1^4 (2f(x) - 4)dx = 2\int_1^4 f(x)dx - 4\int_1^4 1dx$ (by the properties of integrals)
$= 2 \times 2 - 4[x]_1^4 = 4 - 4(4-1) = 4 - 4 \times 3 = -8$

Question 16
The correct answer is C.

$\frac{1}{2}\int_{-1}^{1}(1-x^2)dx$ gives the average value of the function over $-1 \leq x \leq 1$.

$$= \frac{1}{2}\left(1 - \frac{1}{3} - \left(-1 + \frac{1}{3}\right)\right) = \frac{1}{2}\left(\frac{2}{3} - \left(-\frac{2}{3}\right)\right) = \frac{1}{2} \times \frac{4}{3} = \frac{2}{3}$$

Question 17
The correct answer is B.

$f(10x) = \log_e y = \log_e\big(2(10x)\big)$
$\log_e y = \log_e 20x$
$\therefore y = 20x$

Question 18
The correct answer is C.

$$\Pr(X < 1.5) = \int_0^{1.5} 0.5x dx = 0.5\left[\frac{x^2}{2}\right]_0^{1.5} = 0.5 \times (1.5)^2 \times \frac{1}{2} = \frac{9}{16}$$

Question 19
The correct answer is C.

The probability of getting a head is 0.5, and there are 11 trials. This is a binomial experiment, so:

$\mu = 11 \times 0.5 = 5.5$

Question 20
The correct answer is B.

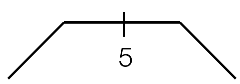
The only set of events that can possibly happen at the same time (ie. aren't mutually exclusive) is B. D only occurs when two dice is simultaneously rolled.

Question 21
The correct answer is A.

$\Pr(X < 11.5) = \Pr\left(Z < \frac{11.5-10}{3}\right)$ (use $z = \frac{x-\mu}{\sigma}$)
$= \Pr(Z < 0.5)$
$= \Pr(X > -0.5)$, by the symmetry properties of the normal distribution

Question 22
The correct answer is B.

If you are unsure about a question similar to this, draw a diagram such as the following, where the lines represent the slope of the curve around the point:



x	4	5	6
f'(x)	+ve	0	-ve

Section B

Marks allocated are indicated by a number in square brackets, for example, [1] indicates that the line is worth one mark. For any question worth more than one mark, students must show working to receive any marks for that question.

Question 1a

$f'(x) = -a\sin(x+b)$ [2]

Question 1b

$f\left(\frac{\pi}{2}\right) = 2-\sqrt{2} = a\cos\left(\frac{\pi}{2}+b\right)+c$ [1]

$f'\left(\frac{\pi}{2}\right) = 0 = -a\sin\left(\frac{\pi}{2}+b\right)$ [1]

$f\left(\frac{\pi}{4}\right) = 4-2\sqrt{2} = a\cos\left(\frac{\pi}{4}+b\right)+c$ [1]

Question 1c

Use second equation from 1b:

$a \neq 0$, since this would make the other equations false.

$\therefore \sin\left(\frac{\pi}{2}\right) = 0$

$\frac{\pi}{2}+b = n\pi, n \in \mathbb{Z}$

$b = n\pi - \frac{\pi}{2}$

$b = \frac{\pi}{2}$, since $0 \leq b < \pi$ [1]

Use first and third equations from 1b:

$2-\sqrt{2} = a\cos(\pi)+c = c-a$

$4-2\sqrt{2} = a\cos\left(\frac{3\pi}{4}\right)+c = c-\frac{a}{\sqrt{2}}$

Rearrange for c:

$c = 2-\sqrt{2}+a$

$c = 4-2\sqrt{2}+\frac{a}{\sqrt{2}} = 4-2\sqrt{2}+\frac{\sqrt{2}a}{2}$

Combine and solve for a:

$2-\sqrt{2}+a = 4-2\sqrt{2}+\frac{\sqrt{2}}{2}a$

$\left(1-\frac{\sqrt{2}}{2}\right)a = 2\sqrt{2}$

$(2-\sqrt{2})a = 4-2\sqrt{2}$

$a = \frac{4-2\sqrt{2}}{2-\sqrt{2}} = \frac{(4-2\sqrt{2})(2+\sqrt{2})}{(2-\sqrt{2})(2+\sqrt{2})} = \frac{4}{2}$

$a = 2$ [1]

Substitute in to solve for c:

$c = 2 - \sqrt{2} + 2 = 4 - \sqrt{2}$ [1]

Question 1d

$f(x) = 2\cos\left(x + \frac{\pi}{2}\right) + 4 - \sqrt{2}$ (or using values of a, b and c found in 1c)

$x = 2\cos\left(y + \frac{\pi}{2}\right) + 4 - \sqrt{2}$ [1]

$\frac{x - 4 + \sqrt{2}}{2} = \cos\left(y + \frac{\pi}{2}\right)$

$\cos^{-1}\left(\frac{x-4+\sqrt{2}}{2}\right) - \frac{\pi}{2} = y = f^{-1}(x)$ where x ∈ [2−$\sqrt{2}$,6−$\sqrt{2}$] [1]

Domain is required for full mark

Question 1e

$\left(f(x)\right)^{-1} = \frac{1}{2\cos\left(x+\frac{\pi}{2}\right)+4-\sqrt{2}}$ (or using values of a, b and c found in 1c) [1]

$2\cos\left(x + \frac{\pi}{2}\right) + 4 - \sqrt{2} \neq 0$ for all $x \in \mathbb{R}$ [1]

∴ domain = ℝ [1]

Question 1f

$2\cos\left(x + \frac{\pi}{2}\right) + 4 - \sqrt{2} = 4 - \sqrt{2}$ [1]

$2\cos\left(x + \frac{\pi}{2}\right) = 0$

$\cos\left(x + \frac{\pi}{2}\right) = 0$

$x + \frac{\pi}{2} = \frac{\pi}{2} + n\pi, n \in \mathbb{Z}$ [1]

$x = n\pi, n \in \mathbb{Z}$ [1]

Question 2a

Use Pythagoras' theorem:

$x^2 = 20^2 + 5^2$

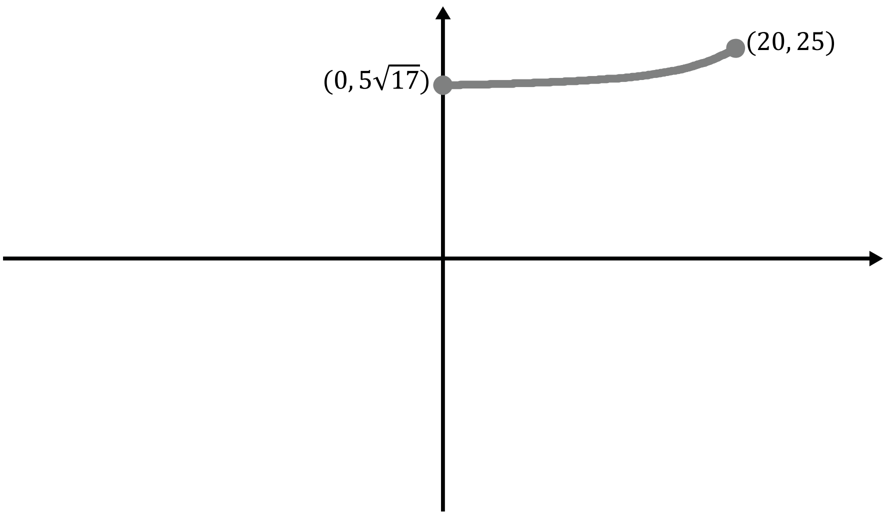
$x = \sqrt{425} = 5\sqrt{17}$ km [1]

Question 2b

$L = \sqrt{5^2 + (20 - d)^2} + d$ [1]

$0 \leq d \leq 20$ [1]

Question 2c



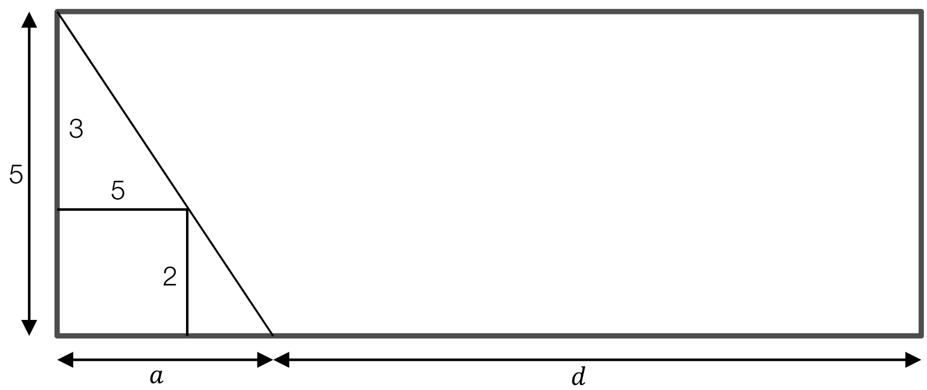
[1 mark for shape, 1 mark for each point, for a total of 3 marks]

Question 2d

If Marty can only walk 45 km in a day, that means each trip can be a maximum of 22.5 km. [1]

Solve $22.5 = \sqrt{25 + (20 - d)^2} + d$ by hand or using calculator to find $d = \frac{65}{4} = 16.25\text{ km.}$ [1]

Question 2e



Using similar triangles, $\frac{a}{5} = \frac{5}{3}$ [2 marks for diagram and explanation]

$3a = 25$
$a = \dfrac{25}{3}$
$d = 20 - a = \frac{35}{3}$ [1]

Substitute into equation for L to find $L = 21.38\text{ km.}$ [1]

Question 3a

Area given by model $= 2 \times 11.5 = 23\text{ m}^2$ [1]

$\text{error} = |\text{actual} - \text{predicted}| = |16.95 - 23| = 6.05$
$\%\text{ error} = \frac{\text{error}}{\text{actual}} \times 100 = \frac{6.05}{16.95} \times 100 = 36\%$ [1]

Question 3b

x-axis intercepts are 0 and 11.5.

$\therefore \text{ area } = \int_0^{11.5} -\frac{3}{10}(x-11.5)dx$ [1]

$= -\frac{3}{10}\left[\frac{x^2}{2} - 11.5x\right]_0^{11.5} = -\frac{3}{10}(0.5 \times 11.5^2 - 11.5^2) = \frac{3}{10} \times \frac{1}{2} \times 11.5^2 = \frac{1587}{80} = 19.84 \text{ m}^2$ [1]

Question 3c i

$x = 0$ is a solution (by inspection). So $(0, 0)$ is a point of intersection. [1]

$\frac{4}{5}\left(x^{\frac{5}{3}}\right) = 3\left(x^{\frac{3}{5}}\right)$ [1]

$\frac{4}{15} = x^{\left(\frac{3}{5}-\frac{5}{3}\right)}$

$\frac{4}{15} = x^{\frac{9-25}{15}}$

$\frac{4}{15} = x^{-\frac{16}{15}}$

$x^{\frac{16}{15}} = \frac{15}{4}$

$x = \left(\frac{15}{4}\right)^{\frac{15}{16}}$

At $x = \left(\frac{15}{4}\right)^{\frac{15}{16}}$, $y = 3 \times \left(\left(\frac{15}{4}\right)^{\frac{15}{16}}\right)^{\frac{3}{5}} = 3 \times \left(\frac{15}{4}\right)^{\frac{9}{16}}$. So the other point of intersection is $\left(\left(\frac{15}{4}\right)^{\frac{15}{16}}, 3 \times \left(\frac{15}{4}\right)^{\frac{9}{16}}\right)$ [1]

Question 3c ii

$\text{area} = \int_0^{\left(\frac{15}{4}\right)^{\frac{15}{16}}} \left(3\left(x^{\frac{3}{5}}\right) - \left(\frac{4}{5}\right)\left(x^{\frac{5}{3}}\right)\right)dx$ [1]

$= 8.43 \text{ m}^2$ [1]

Question 3d

$F = \int_0^{a} \left(3\left(x^{\frac{3}{5}}\right) - \left(\frac{4}{5}\right)\left(x^{\frac{5}{3}}\right)\right)dx$ [2]

Question 3e

Use calculator to solve $\int_0^{a} \left(3\left(x^{\frac{3}{5}}\right) - \left(\frac{4}{5}\right)\left(x^{\frac{5}{3}}\right)\right)dx = 5$ [1]

$a = 2.42 \text{ m}$ [1]

Question 4a

$\text{Pr(Clare wins 7 games in a row)} = (0.52)^7 = 0.0103$ [1]

Question 4b

Let X = the number of games out of 10 won by Clare

$\Pr(X \geq 7) = \Pr(X = 7) + \Pr(X = 8) + \Pr(X = 9) + \Pr(X = 10)$ [1]

$= \binom{10}{7}(0.52)^7(0.48)^3 + \binom{10}{8}(0.52)^8(0.48)^2 + \binom{10}{9}(0.52)^9(0.48)^1 + \binom{10}{10}(0.52)^{10}$

$= 0.262$ [1]

Question 4c

$\text{Pr(Sally wins next 2 points)} = 0.61^2 = 0.3721$ [1]

Question 4d

$\Pr(\text{Sally wins 2 of next 3 points}) = \Pr(\text{SSC}) + \Pr(\text{SCS}) + \Pr(CSS)$, where C represents Clare winning, and S represents Sally winning [1]

$= 0.61^2 \times 0.39 + 0.61 \times 0.39 \times 0.51 + 0.39 \times 0.51 \times 0.61$ [1]
$= 0.3878$ [1]

Question 4e

$T = \begin{bmatrix} 0.61 & 0.51 \\ 0.39 & 0.49 \end{bmatrix}, S_0 = \begin{bmatrix} 1 \\ 0 \end{bmatrix}$[1]

$S_{12} = T^{12} \times S_0 = \begin{bmatrix} 0.61 & 0.51 \\ 0.39 & 0.49 \end{bmatrix}^{12} \times \begin{bmatrix} 1 \\ 0 \end{bmatrix} = \begin{bmatrix} 0.5667 \\ 0.4333 \end{bmatrix}$ [1]

Therefore, $\Pr\left(\text{Clare wins } 12^{\text{th}} \text{ point}\right) = 0.4333$ [1]

Question 4f

$S_\infty = \begin{bmatrix} 0.5667 \\ 0.4333 \end{bmatrix}$

Therefore, Clare wins 43.33% of points in the long term.

Question 4g

Want $\int_{-\infty}^{\infty} f(n)dn = 1$ [1]

$\lim\limits_{k\to\infty} \int_0^k ae^{-\frac{n}{100}}dn = 1$
$a \lim\limits_{k\to\infty} \left[-100e^{-\frac{n}{100}}\right]_0^k = 1$
$a \lim\limits_{k\to\infty} \left(-100e^{-\frac{k}{100}} + 100\right) = 1$
$a \lim\limits_{k\to\infty} \left(100 - \frac{100}{e^{\frac{k}{100}}}\right) = 1$
$100a = 1$
$a = \frac{1}{100}$ [1]

Question 4h

$\mu = \lim\limits_{k\to\infty} \int_0^k \frac{n}{100} e^{-\frac{n}{100}}dn$ [1]

$= \frac{1}{100} \lim\limits_{k\to\infty} \int_0^k ne^{-\frac{n}{100}}dn = 100$ [1]

Question 4i

$\Pr(N > 50) = \lim\limits_{k\to\infty} \int_{50}^k \frac{1}{100} e^{-\frac{n}{100}}dn$ [1]

$= 0.6065$ [2]