TEST 9

Functions, graphs, algebra, calculus, probability and statistics
Technology-free end-of-year examination
Total marks: 35
Suggested writing time: 55 minutes

Specific instructions to students

- Answer all of the questions in the spaces provided.
- Show all workings in questions where more than one mark is available.
- An exact value must be provided in questions where a numerical answer is required, unless otherwise specified.

QUESTION 1

Pr(A) and Pr(B) are independent events. If $\Pr(A \cap B) = 0.06$ and $\Pr(B) = 0.2$, what is Pr(A)?

A and B are **independent** if $\Pr(A \cap B) = \Pr(A) \times \Pr(B)$.

$\Rightarrow 0.06 = \Pr(A) \times 0.2$

$\Pr(A) = \frac{0.06}{0.2} = 0.3$

1 mark

QUESTION 2

a Two independent events A and B have probabilities respectively of 0.2 and 0.9. Find $\Pr(A \cup B)$.

Use the addition formula $\Pr(A \cup B) = \Pr(A) + \Pr(B) - \Pr(A \cap B)$.

$\Pr(A \cup B) = 0.2 + 0.9 - \Pr(A \cap B)$ where

$\Pr(A \cap B) = 0.2 \times 0.9 = 0.18$ (independent)

$\Pr(A \cup B) = 0.2 + 0.9 - 0.18$

$= 1.1 - 0.18$

$= 0.92$

2 marks

b Two events C and D have probabilities respectively of 0.1 and 0.2. If $\Pr(D \mid C) = 0.3$, find $\Pr(C \cap D)$.

Conditional probability

$\Pr(D \mid C) = \frac{\Pr(D \cap C)}{\Pr(C)}$

$0.3 = \frac{\Pr(D \cap C)}{0.1}$

$\Pr(D \cap C) = \Pr(C \cap D) = 0.3 \times 0.1 = 0.03$

2 marks
(Total: 4 marks)

QUESTION 3

Differentiate the function $f(x) = \frac{1}{\sqrt{1-x^2}}$.

$f(x) = \frac{1}{\sqrt{1-x^2}} = (1-x^2)^{-\frac{1}{2}}$

Using the chain rule,

$f'(x) = -\frac{1}{2}(1-x^2)^{-\frac{3}{2}} \times (-2x)$

$= \frac{x}{(1-x^2)^{\frac{3}{2}}}$

2 marks

QUESTION 4

a Let $f(x) = \sin(3x)\cos(3x)$. Find $f'(x)$.

$f(x) = \sin(3x)\cos(3x)$.

Using the product rule,

$f'(x) = \cos(3x) \times 3\cos(3x) + \sin(3x) \times [-3\sin(3x)]$

$= 3\cos^2(3x) - 3\sin^2(3x)$

2 marks

b Hence find $f'\left(\frac{\pi}{12}\right)$.

$f'(x) = 3\cos^2(3x) - 3\sin^2(3x)$

$f'\left(\frac{\pi}{12}\right) = 3\cos^2\left(\frac{\pi}{4}\right) - 3\sin^2\left(\frac{\pi}{4}\right)$

$= \frac{3}{2} - \frac{3}{2} = 0$

1 mark
(Total: 3 marks)

QUESTION 5

Find an anti-derivative with respect to x of $\cos(4-2x)$.

$\int \cos(4-2x)\,dx = -\frac{1}{2}\sin(4-2x)$

1 mark

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QUESTION 6

Consider the following functions.

$f: D \to \mathbf{R}, f(x) = \log_e(\log_e(x))$ and

$g: (b, \infty) \to \mathbf{R}, g(x) = \frac{1}{4}x$

a For the maximal domain of $f(x)$, find D.

For $f(x) = \log_e(\log_e(x))$ to exist, we need,

ran (inner) $\subseteq$ dom (outer).

$\mathbf{R} \not\subset (0, \infty)$

$f(x)$ will exist if we restrict range inner, $\mathbf{R}$, to $(0, \infty)$.

So dom (inner) = $(1, \infty)$.

dom $f(x)$ = dom (inner) = $(1, \infty)$

$D = (1, \infty)$

2 marks

b If $f(g(x))$ is defined over the domain of g, find the smallest possible value of b.

Domain of $g = (b, \infty)$.

For $f(g(x))$ to exist, test ran $(g) \subseteq$ dom (f).

$(\frac{1}{4}b, \infty) \subseteq (1, \infty)$

So $\frac{1}{4}b = 1$

$b = 4$

So the least value of $b = 4$.

2 marks

(Total: 4 marks)

QUESTION 7

A probability density function is defined by

$$f(x) = \begin{cases} k\sin(x), & 0 \le x \le \pi \\ 0, & \text{otherwise} \end{cases}$$

a Find the value of k.

For a PDF,

$\int_0^{\pi} k\sin(x)\,dx = 1$

$-k\,[\cos(x)]_0^{\pi} = 1$

$-k\,(\cos(\pi) - \cos(0)) = 1$

$-k\,(-1 - 1) = 1$

$k = \frac{1}{2}$

2 marks

b Hence, find $\Pr\left(X < \frac{\pi}{3}\right)$.

$$f(x) = \begin{cases} \frac{1}{2}\sin(x), & 0 \le x \le \pi \\ 0, & \text{otherwise} \end{cases}$$

$$\Pr\left(X < \frac{\pi}{3}\right) = \int_0^{\frac{\pi}{3}} \frac{1}{2}\sin(x)\,dx$$

$$= -\frac{1}{2}[\cos(x)]_0^{\frac{\pi}{3}}$$

$$= -\frac{1}{2}\left[\cos\left(\frac{\pi}{3}\right) - \cos(0)\right]$$

$$= -\frac{1}{2}\left(\frac{1}{2} - 1\right)$$

$$= \frac{1}{4}$$

2 marks

(Total: 4 marks)

QUESTION 8

Solve the equation $\sin^2(2x) = \cos^2(2x)$ for $x \in [0, \pi]$.

$\sin^2(2x) = \cos^2(2x) \Rightarrow \tan^2(2x) = 1$

$\Rightarrow \tan(2x) = \pm 1$

$$2x = \frac{\pi}{4}, \pi - \frac{\pi}{4}, \pi + \frac{\pi}{4}, 2\pi - \frac{\pi}{4}$$

$$= \frac{\pi}{4}, \frac{3\pi}{4}, \frac{5\pi}{4}, \frac{7\pi}{4}$$

$$x = \frac{\pi}{8}, \frac{3\pi}{8}, \frac{5\pi}{8}, \frac{7\pi}{8}$$

2 marks

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QUESTION 9

The probability of scoring each goal in netball practice is independent and Claire finds that the probability of her scoring a goal is 0.8.

a State an expression for the probability that, in 10 shots, Claire scores exactly 2 goals.

$\text{Bi}(n, p) = \text{Bi}(10, 0.8)$

$\Pr(X = 2) = {}^{10}C_2 \times (0.2)^8 \times (0.8)^2$

2 marks

b Find the probability that in 4 shots, Claire scores at least 1 goal.

$\text{Bi}(n, p) = \text{Bi}(4, 0.8)$

$$\begin{aligned}\Pr(X \geq 1) &= 1 - \Pr(X = 0)\\ &= 1 - {}^4C_0 (0.2)^4 (0.8)^0\\ &= 1 - (0.2)^4\\ &= 1 - 0.0016\\ &= 0.9984\end{aligned}$$

2 marks
(Total: 4 marks)

QUESTION 10

Solve the following equations for x.

a $2(x + 1)^2 - 7 = 0$

$2(x + 1)^2 - 7 = 0 \Rightarrow x + 1 = \pm\sqrt{\frac{7}{2}}$

$x = -1 \pm \sqrt{\frac{7}{2}}$

2 marks

b $5(x - 1)^3 = 320$

$5(x - 1)^3 = 320 \Rightarrow (x - 1)^3 = 64$

$\Rightarrow x - 1 = 4$

$x = 5$

2 marks

c $\log_{10}(x + 5) = 2$

$\log_{10}(x + 5) = 2 \Leftrightarrow 10^2 = x + 5$

$\therefore x = 10^2 - 5$

$x = 95$

2 marks

d $\sqrt{3}\tan(3x) = -1, x \in [0, \pi]$

$\sqrt{3}\tan(3x) = -1$

$\tan(3x) = -\frac{1}{\sqrt{3}}$

Reference angle $= \frac{\pi}{6}$

$\Rightarrow 3x = \pi - \frac{\pi}{6}, 2\pi - \frac{\pi}{6}, 3\pi - \frac{\pi}{6}$

$= \frac{5\pi}{6}, \frac{11\pi}{6}, \frac{17\pi}{6}$

$x = \frac{5\pi}{18}, \frac{11\pi}{18}, \frac{17\pi}{18}$

2 marks

e $\log_{10}(x + 5) + \log_{10}(x) = 1$

$\log_{10}(x + 5) + \log_{10}(x) = 1$

$\Rightarrow \log_{10}(x(x + 5)) = 1$

$\Rightarrow \log_{10}(x^2 + 5x) = 1$

$\Leftrightarrow x^2 + 5x = 10^1$

$x^2 + 5x - 10 = 0$

$\Rightarrow x = \frac{-5 \pm \sqrt{25 + 40}}{2}$

Testing in the original equation, we reject the negative value.

Answer is $x = \frac{-5 + \sqrt{65}}{2}$

2 marks
(Total: 10 marks)

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