TEST 5

Functions, graphs, algebra and calculus
Technology-free end-of-year examination
Total marks: 30
Suggested writing time: 45 minutes

Specific instructions to students

- Answer all of the questions in the spaces provided.
- Show all workings in questions where more than one mark is available.
- An exact value must be provided in questions where a numerical answer is required, unless otherwise specified.

QUESTION 1

For the function $f(x) = \sqrt{x}$, find $f^{-1}(x)$.

Let $y = \sqrt{x}$.

Swap x and y to find the inverse.

$x = \sqrt{y} \quad \Rightarrow \quad y = x^2$

Domain of $f^{-1}(x)$ = range of $f(x) = [0, \infty)$

Answer: $f^{-1}: [0, \infty) \to \boldsymbol{R}, f^{-1}(x) = x^2$

2 marks

QUESTION 2

a For the function $f(x) = \frac{4}{x}$ with a maximal domain, find $f^{-1}(x)$.

Let $y = \frac{4}{x}$

Swap x and y to find the inverse.

$x = \frac{4}{y} \quad \Rightarrow \quad y = \frac{4}{x}$

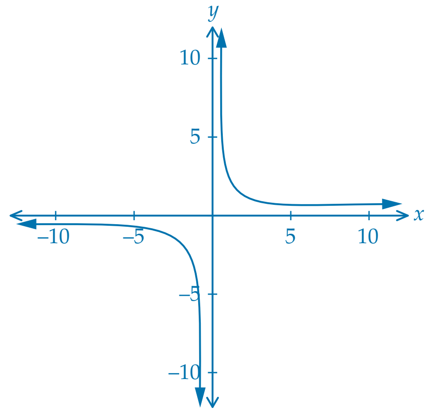
Domain of $f^{-1}(x)$ = range of $f(x) = \boldsymbol{R} \setminus \{0\}$

Answer: $f^{-1}: \boldsymbol{R} \setminus \{0\} \to \boldsymbol{R}, f^{-1}(x) = \frac{4}{x}$

2 marks

b Hence, sketch the graphs of $f(x)$ and $f^{-1}(x)$ on the same axes.

The graphs have asymptotes at $x = 0$ and $y = 0$.



2 marks
(Total: 4 marks)

QUESTION 3

If $f(x) = 3x - \frac{2}{x}$ and $g(x) = x + 2$, for $f(g(x))$ to exist, determine its rule and domain.

Test if $f(g(x))$ exists.

Test: $\text{ran}(g) \subseteq \text{dom}(f)$

$\boldsymbol{R} \not\subseteq \boldsymbol{R} \setminus \{0\}$, although it can be restricted to exist.

Require $\boldsymbol{R} \setminus \{0\} \subseteq \boldsymbol{R} \setminus \{0\}$

Restricting $\text{ran}(g)$ to $\boldsymbol{R} \setminus \{0\}$ restricts $\text{dom}(g)$ to $\boldsymbol{R} \setminus \{-2\}$.

Rule: $f(g(x)) = 3(x + 2) - \frac{2}{x+2}$

$= 3x + 6 - \frac{2}{x+2}$

Domain $f(g(x)) = \text{dom } g(x) = \boldsymbol{R} \setminus \{-2\}$.

(This domain can also be noticed from the graph of $f(g(x))$).

3 marks

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QUESTION 4

Given $f: [A, \infty) \to \boldsymbol{R}, f(x) = x^2 - 2x + 5$,

a state the least value of A such that $f(x)$ is a one-to-one function.

$f(x) = x^2 - 2x + 5$

Complete the square for turning point form.

$f(x) = x^2 - 2x + 1 - 1 + 5$

$= (x - 1)^2 + 4$

$A = 1$

So dom $f = [1, \infty)$.

1 mark

b Hence, find the rule of $f^{-1}(x)$.

Let $y = (x - 1)^2 + 4$

Swap x and y to find the inverse.

$x = (y - 1)^2 + 4$

$\Rightarrow y - 1 = \pm\sqrt{x - 4}$

$\Rightarrow y = \pm\sqrt{x - 4} + 1$

Select the upper branch.

$f^{-1}(x) = \sqrt{x - 4} + 1$

2 marks

c State the domain and range of $f^{-1}(x)$.

dom f^{-1} = range $f = [4, \infty)$

range f^{-1} = dom $f = [1, \infty)$

2 marks
(Total: 5 marks)

QUESTION 5

For the function $f(x) = \sqrt{x}$, find $f'(x)$.

$f(x) = \sqrt{x} = x^{\frac{1}{2}}$

$f'(x) = \frac{1}{2}x^{-\frac{1}{2}} = \frac{1}{2\sqrt{x}}$

2 marks

QUESTION 6

For $y = \frac{4}{x}$ find $\frac{dy}{dx}$.

$y = \frac{4}{x} = 4x^{-1}$

$\frac{dy}{dx} = -4x^{-2} = -\frac{4}{x^2}$

2 marks

QUESTION 7

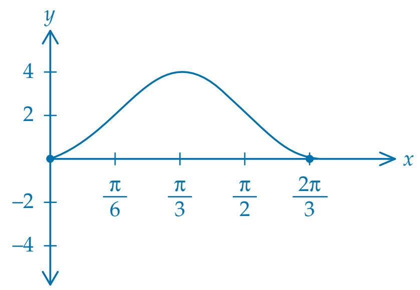
a For the graph of $y = 2\sin\left(3\left(x - \frac{\pi}{6}\right)\right) + 2$, state the amplitude and period.

Amp = 2

Period = $\frac{2\pi}{3}$

2 marks

b Hence, sketch the graph of $y = 2\sin\left(3\left(x - \frac{\pi}{6}\right)\right) + 2$, showing one complete cycle.



2 marks
(Total: 4 marks)

QUESTION 8

Solve for x in the equation $\tan(x) + 1 = 0$, where $x \in [-\pi, 2\pi]$.

$\tan(x) + 1 = 0 \quad \Rightarrow \quad \tan(x) = -1$

Reference angle = $\frac{\pi}{4}$

$x = -\frac{\pi}{4}, \pi - \frac{\pi}{4}, 2\pi - \frac{\pi}{4}$

$x = -\frac{\pi}{4}, \frac{3\pi}{4}, \frac{7\pi}{4}$

3 marks

QUESTION 9

For $y = x\left(x^2 - \frac{1}{x}\right)$ find $\frac{dy}{dx}$.

$y = x\left(x^2 - \frac{1}{x}\right) = x^3 - 1$

$\frac{dy}{dx} = 3x^2$

2 marks

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QUESTION 10

For $y = \sin(x)\left(x^2 - \frac{1}{x}\right)$,

a find $\frac{dy}{dx}$

$y = \sin(x)\left(x^2 - \frac{1}{x}\right)$

Using the product rule,

$\frac{dy}{dx} = \left(x^2 - \frac{1}{x}\right)\cos(x) + \sin(x)\left(2x + \frac{1}{x^2}\right)$

2 marks

b find $\frac{dy}{dx}$ at $x = \pi$

At $x = \pi$,

$\frac{dy}{dx} = \left(\pi^2 - \frac{1}{\pi}\right)\cos(\pi)$

$= \frac{1}{\pi} - \pi^2$

1 mark
(Total: 3 marks)

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